Quantization of Lie bialgebras via the formality of the operad of little disks

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Abstract. We give a proof of Etingof-Kazhdan theorem on quantization of Lie bialgebras based on the formality of the chain operad of little disks and show that the Grotendieck-Teichmüller group acts non-trivially on the corresponding quantization functors.

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1. Introduction

1.1.

In this paper we give a proof of Etingof-Kazhdan theorem on quantization of Lie bialgebras based on the formality of the chain operad of little disks.

Any known construction of such a formality involves multiple zeta values; in particular there is no canonic way to establish such a formality over $\mathbb{Q}$. For example, in the construction from [8] one needs to choose an associator over $\mathbb{Q}$. In [9] it is shown that different associators produce homotopically non-equivalent formalities of chain operad of little disks. Each of these formalities, in turn, produces a certain quantization procedure of Lie bialgebras and we prove that these procedures are not isomorphic. This can be considered as a step in studying the action of Grotendieck-Teichmüller group on quantization functors originated in [5].

1.2. Idea of the construction of quantization

1.2.1. Let $\mathfrak{g}$ be a Lie bialgebra with bracket $[,]$ and cobracket δ. Call $\mathfrak{g}$ *conilpotent* if for any $x \in \mathfrak{g}$ there exists an N such that any N-fold iteration of δ applied to x produces zero.

1.2.2. Let H be a Hopf algebra with product $\cdot$, coproduct Δ, unit 1, and counit ε (we do not assume that the antipode exists). Let $\Delta'(x) = \Delta(x) - 1 \otimes x - x \otimes 1$. Call H *conilpotent* if for any x such that $\varepsilon(x) = 0$ there exists an N such that any N-fold iteration of Δ' applied to x produces zero.

1.2.3. Note that in any conilpotent H there exists an antipode map and it is uniquely defined.

1.2.4. We are going to construct a functor Q from the category of conilpotent Lie bialgebras to the category of conilpotent Hopf algebras

1.2.5. Let $\mathfrak{g}$ be a conilpotent Lie bialgebra. Let $C^\bullet(\mathfrak{g})$ be its cochain complex with respect to the cobracket. This means that $C^\bullet(\mathfrak{g}) = S(\mathfrak{g}[-1])$ is a free graded commutative algebra equipped with a differential D defined on the space of generators $\mathfrak{g}[-1] \subset S(\mathfrak{g}[-1])$ by the cobracket $\delta : \mathfrak{g}[-1] \to S^2(\mathfrak{g}[-1])$.

$C^\bullet(\mathfrak{g})$ has a structure of Gerstenhaber algebra so that the bracket on $\mathfrak{g}[-1] \subset C^\bullet(\mathfrak{g})$ is defined by the bracket on $\mathfrak{g}$.

1.2.6. Let e_2 be the operad of Gerstenhaber algebras and $\mathbf{hoe}_2$ a free resolution of e_2 Let M be the operad of brace structures described in 6.10-6.10.3 and let $\mathbf{ho}\mathscr{M} \to M$ be a free resolution We have a quasi-isomorphism $\mathbf{ho}\mathscr{M} \to e_2$ ([6],[8]). Therefore, there is a way to construct an M-algebra out of an e_2-algebra. Denote this way by W (it is a functor from the category of Gerstenhaber algebras to the category of M-algebras (= brace algebras). Thus, $W(C^\bullet(\mathfrak{g}))$ is a brace algebra.

1.2.7. Remark 1. It is exactly on this step that the associators or integrals are being used.

Remark 2. One of the steps of the proof of the formality of M in [6] is linking M with the operad of singular chains of the operad of little disks (this step is purely "combinatorial",— it does not use transcendental methods). Thus, the formality of M follows from the formality of the operad of singular chains of the operad of little disks.

1.2.8. In 6.10.3 it is explained that there exists a simple way H to construct a conilpotent Hopf algebra out of a brace algebra. Thus $h(\mathfrak{g}) = HWC^\bullet(\mathfrak{g})$ is a differential graded Hopf algebra. We prove that

Proposition 1.1. *the cohomology $H^i(h(\mathfrak{g}))$ vanishes for all $i \neq 0$.*

Set $Q(\mathfrak{g}) = H^0(h(\mathfrak{g}))$. We prove that Q is an equivalence of categories.

1.3. Universal language

Algebraic structures of Lie bialgebra and of Hopf algebra cannot be described in the language of operads. One has to consider a certain generalization of operad called PROP (5).

A PROP P is a symmetric monoidal category with certain properties. A *P-algebra* in a symmetric monoidal category C is the same as a symmetric monoidal map $P \to C$.

If we have a definition of a certain algebraic structure, then the PROP P describing this structure is defined as a unique up to an isomorphism symmetric monoidal category P such that the notion of P-algebra coincides with the definition of our algebraic structure.

1.3.1. Let $\mathscr{LBA}$ be the PROP of Lie bialgebras and $\mathscr{BA}$ be the PROP of Hopf algebras. Thus, a Lie bialgebra in the category of complexes can be described as a symmetric monoidal functor $\mathscr{LBA} \to$ **complexes**. Let $\mathscr{LBA}\vee$ be the category of all functors (without symmetric monoidal structure) $\mathscr{LBA} \to$ **complexes**. We can introduce a symmetric monoidal structure on $\mathscr{LBA}\vee$ (see 3.9). For a symmetric monoidal functor $F : \mathscr{LBA} \to$ **complexes**, its symmetric monoidal structure defines a commutative algebra structure on F as an element of symmetric monoidal category $\mathscr{LBA}\vee$.

Remark Note that a commutative algebra structure on $F \in \mathscr{LBA}\vee$ corresponds to a *weak* symmetric monoidal structure on F; for a commutative algebra F in $\mathscr{LBA}\vee$ to be a strong symmetric monoidal functor $\mathscr{LBA} \to$ **complexes**, one has to impose an additional condition on F.

Similar statements are true for the PROP $\mathscr{BA}$.

How to describe conilpotent Lie bialgebras and conilpotent Hopf algebras in this language? They correspond to *conilpotent* functors $\mathscr{LBA} \to$ **complexes** and $\mathscr{BA} \to$ **complexes** with symmetric monoidal structure.

Let S (resp L) be the full subcategory of all functors in $\mathscr{BA}\vee$ (resp. $\mathscr{LBA}\vee$) which are quasi-isomorphic to conilpotent functors $\mathscr{BA} \to$ **complexes** (resp. $\mathscr{LBA} \to$ **complexes**) (we do not assume any symmetric monoidal structure on these functors). These categories have symmetric monoidal structures defined by the restriction from $\mathscr{LBA}\vee$ and $\mathscr{BA}\vee$. Let $L_0 \subset L$ be the full subcategory consisting of conilpotent functors $\mathscr{LBA} \to$ **vect**, let $S_0 \subset S$ be the similar thing.

The constructions of 1.2.4 can be rewritten in terms of a symmetric monoidal functor $\mathbf{Q} : L \to S$ so that given a conilpotent Lie bialgebra $\mathfrak{g} \in L$, $\mathbf{Q}(\mathfrak{g}) \in S$ is a conilpotent Hopf algebra.

We prove that $\mathbf{Q}$ induces an equivalence of certain 'good' subcategories $L' \subset L$ and $S' \subset S$.

1.3.2. To prove Proposition 1.1, one introduces t-structures on L' and S' and shows that Q is a t-functor. (More precisely, first one takes the categories DL' (resp. DS') with the same objects as in L' (resp. S') but $\hom_{DL'}(X,Y) = H^0 \hom_{L'}(X,Y)$ (similarly for DS') and endows them with triangular structure. See [1] for the theory of triangular categories and t-structures. In this paper we have tried to give all the necessary definitions.)

This implies that $\mathbf{Q}$ induces an isomorphism of the cores of L' and S' which are L_0 and S_0, which implies Proposition 1.1 and the fact that Q is an equivalence of categories.

1.3.3. Let Pe_2 be the PROP generated by the operad of Gerstenhaber algebras and PM be the PROP generated by the operad of brace algebras. Let $Pe_2\vee$ be the symmetric monoidal category of functors $Pe_2 \to$ **complexes** and $PM\vee$ be the same thing for PM. Then the map Q decomposes as a sequence of maps

$$\mathbf{Q} : L \to Pe_2\vee \to PM\vee \to S. \tag{1.1}$$

We define certain 'good' subcategories of each of the categories involved, endow them with t-structures and show that all the arrows are t-functors and symmetric monoidal homotopy equivalences.

Also we construct the inverse functor $DQ : S \to L$ by inverting each of the arrows in (1.1).

1.3.4. In the last part of the paper we investigate how the functors Q and DQ depend on a choice of associator by means of which the formality $\mathbf{hoe}_2 \to M$ is constructed.

We know that the variety of associators is a torsor over the Grotendieck-Teichmüller group. In [9] it is shown that the graded Lie algebra $\mathfrak{g}rt$ acts on an appropriate resolution $\mathbf{hoe}_2$ of e_2. Let t be a formal infinitesimal parameter, then any $x \in \mathfrak{g}rt$ produces a family of associators A_t^x and a family $F_t^x : P\mathbf{hoe}_2^\frown \to P\mathbf{hoe}_2^\frown[[t]]$ such that $F_t^x = \mathrm{Id} \pmod t$. Let $f_A : \mathbf{hoe}_2 \to M$ be the formality constructed by means of A. and let $Q_t : L_0 \to S_0[[t]]$ be the family of equivalences constructed using A_t. The functor $Q = Q_0$ produces the constant functor $Q' : L_0 \xrightarrow{Q} S_0 \to S_0[[t]]$. We prove that there is no equivalence of Q_t and Q. Without loss of generality we can assume that this equivalence is equal to identity modulo t. This statement easily reduces to non-existence of an isomorphism J_t between the map $T_{F_t^x}$ induced by F_t^x on the core of $Pe_2^\frown$ and Id , such that $J_t = \mathrm{Id} \mod t$.

Next, we use the theorem which states that the assumption that the automorphisms of the core induced by F_t^x and Id are isomorphic implies that F_t^x is *deformationally equivalent* to Id . Then we use section 4 to show that this implies that $f_t : \mathbf{hoe}_2 \to \mathbf{hoe}_2[[t]]$ produces zero in the deformational complex of $\mathbf{hoe}_2$ which contradicts to the result of [9].

1.4. Content of the paper

The first four sections of the paper are preparatory. They explain some terminology that is not commonly used and include some simple propositions to be used in the sequel.

The first part contains a review of differential graded symmetric monoidal categories. We assume that in each such a category a class of morphisms called quasi-isomorphisms is chosen so that it behaves nicely with respect to tensor product. We discuss the notion of a *weak* symmetric monoidal functor F between two such categories. Weakness means that the maps $F(X) \otimes F(Y) \to F(X \otimes Y)$ of the symmetric monoidal structure on F are not obliged to be isomorphisms. If, though, all these maps are quasi-isomorphisms, then we call F *essentially strong*.

Then we discuss the notion of homotopy equivalence of two symmetric monoidal categories and of homotopy equivalence of two weak symmetric monoidal functors between two fixed symmetric monoidal categories. We do not use the usual term "weak homotopy equivalence" as the word "weak" is used with symmetric monoidal functors.

After that we discuss the notion of deformational homotopy equivalence, which will be only used to prove nonequivalence of different quantizations.

The second part deals with how to define the class of quasi-isomorphisms in a dg category C. Proposition 3.1 provides us with such a recipe. It has a drawback that the class of quasi-isomorphisms defined in this way is very narrow. For example, let A be a usual ring and let C be the category of dg A-modules. Let M and M' be A-modules and let $p : M' \to M$ be a free resolution. Most likely, p is not a quasi-isomorphism. Although, if A is semi-simple, the definition works fine.

How can we fix this problem? First, let us analyze how the notion of quasi-isomorphism is usually defined (say for C being the category of dg A-modules). We take the forgetful functor F from our category to some semi-simple category **triv** and call an arrow f in our category quasi-isomorphic if such is $F(f)$ according to Definition-Proposition 3.1.

But now our quasi-isomorphisms in C do not satisfy Proposition 3.1, which makes it impossible to work with them. In many cases this situation can be corrected by taking a subcategory $\mathbf{Cof}_F C \subset C$ within which any quasi-isomorphism satisfies Proposition 3.1. An object of $\mathbf{Cof}_F(C)$ is called F-cofibrant. In the example $C = A-$ mod any complex of projective A-modules bounded from the right hand side is in $\mathbf{Cof}_F(C)$. Also, the total of any bounded complex (see 3.0.9) of F-cofibrant objects is cofibrant. Unfortunately, it may happen that there are not enough cofibrant objects.

After this is done we restrict ourselves to the so-called saturated categories. The key property of them is that they are (informally speaking) closed with respect to taking bounded complexes of its objects. For any dg category C the category $C\vee$ of all functors $C \to$ **complexes** is such and we only study such categories. Any functor $F : D \to C$ induces a functor $F^{-1} : D\vee \to C\vee$. We show that the category $C^{\frown} := \mathbf{Cof}_{F^{-1}} C\vee$ possesses a variety of good properties.

Next, we study the case when C is symmetric monoidal. We see that in this case $C^{\frown}$ has a natural symmetric monoidal structure.

After that , we discuss t-structures on such a $C^{\frown}$. It is shown that under certain restrictions on C we can define them.

Finally, we study the action of deformationally equivalent weak symmetric monoidal functors on certain operads (see section 4). This section is only needed for studying the Grotendieck-Teichmüller group action.

In the fifth part the theory developed above is applied to the case when C is a PROP after which we discuss a notion of generalized maps of PROPs. They formalize the situation when we have a recipe how given a P-algebra we may construct a Q-algebra (as for example in 1.2.5.) One of the variants of such a definition is standard and can be found say in [4]. This variant is not completely satisfactory for us because it is difficult to incorporate the notion of conilpotent algebra in it (for this one has to consider topologic PROPs) Thus, we develop another variant of such a definition and show how to pass from one of them to another. There is a problem with our definition because in general given a P-algebra it produces only a *weak* Q-algebra. Thus we need to impose some restrictions to insure that we always obtain a usual(=strong) Q-algebra. That's how we arrive at the definition of a strong generalized map of PROPs.

After all this preparation we follow the plan announced in 1.3-1.3.4. First we define the rightmost arrow in (1.1) and a quasi-inverse to it and show that both of them are homotopy equivalences and then we do the same for the remaining arrows in (1.1) in order from the right to the left.

We conclude with studying the action of Grotendieck-Teichmüller group on quantizations.

2. Symmetric monoidal categories.

Let C,D be *dg* symmetric monoidal categories such that in each of them a class of morphisms called quasi-isomorphisms is chosen and the tensor product of quasi-isomorphisms is a quasi-isomorphism.

Let $F : C \to D$ be a functor preserving quasi-isomorphisms; such functors are called *exact*. *A weak symmetric monoidal structure* on F is a natural transformation of functors $e_{XY} : F(X) \otimes F(Y) \to F(X \otimes Y)$ which is commutative and associative and is not necessarily an isomorphism. *A strong symmetric monoidal structure* on F is a weak structure in which all e are isomorphisms; *an essentially strong* symmetric monoidal structure is a weak symmetric monoidal structure in which all e_{XY} are quasi-isomorphisms.

2.0.1. F is called a homotopy equivalence if it has an essentially strong symmetric monoidal structure , $F : \hom(X,Y) \to \hom(F(X),F(Y))$ is a quasi-isomorphism for any X,Y and any $Z \in D$ is quasi-isomorphic to some $F(X)$.

2.0.2. Let $F,G : C \to D$ be weak symmetric monoidal functors. A homotopy equivalence $F \to G$ is a collection of a homotopy equivalences $I : C' \to C$ and $J : D \to D'$ and a natural symmetric monoidal transformation $Q : JFI \to JGI$ such that all Q_X are quasi-isomorphisms.

2.0.3. F,G are called homotopy equivalent if they can be connected by a chain of homotopy equivalences.

2.0.4. Let $F,G : C \to D[[t]]$ be functors such that $F = G \bmod t$; for definition of $D[[t]]$ see 4.1.2. A *deformational* homotopy equivalence between F and G is a collection of homotopy equivalences $I : C' \to C$ and $J : D \to D'$ and a natural transformation $Q : JFI \to JGI$ such that all Q_X are quasi-isomorphisms and $Q = \mathrm{Id} \bmod t$.

2.0.5. Call F,G deformationally homotopy equivalent if they are connected by a chain of deformational homotopy equivalences.

3. dg-categories

3.0.6. Let now S be any dg category. Let $f : X \to Y$ be a morphism in S.

Proposition 3.1. *The following properties are equivalent:*

1) For any Z, $f^* : \hom(Y,Z) \to \hom(X,Z)$ is a quasi-isomorphism;

2) For any Z, $f_* : \hom(Z,X) \to \hom(Z,Y)$ is a quasi-isomorphism;

3) There exists a map $g : Y \to X$ such that both $fg - \mathrm{Id}_Y = du$ and $gf - \mathrm{Id}_X = dv$ for some u,v.

Proof. 1) $\to$ 3). Take $Z = Y$. Then by 1), there exists $g : Z \to X$ such that $f_*g = fg$ is homologous to Id_Y. We thus have $fg - Id_Y = du$. Consider now $f_*(gf - Id_X) = fgf - f = d(uf)$. Since f_* is a quasi-isomorphism, $gf - Id_X$ is homologous to 0, QED.

3) $\to$ 1) Note that both $g_* f_*$ and $f_* g_*$ induce identities on cohomology.

2) $\to$ 3) and 3) $\to$ 2) is proved in the same way as above. □

Any f satisfying one of the conditions of the Proposition is called *natural quasi-isomorphism.*

3.0.7. Let $F : C \to D$ be a functor of dg-categories. Call a morphism m in C an F-quasi-isomorphism if $F(m)$ is a natural quasi-isomorphism. For an object X of C denote $h_X(Y) = \hom_C(X,Y)$; $h_X : C \to$ **complexes** is a functor. Call X F-cofibrant if h_X maps F-quasi-isomorphisms to quasi-isomorphisms of complexes.

3.0.8. Let $\mathbf{Cof}_F(C)$ be the full subcategory of F-cofibrant objects in C. In $\mathbf{Cof}_F(C)$ any F-quasi-isomorphism is natural and vice versa in virtue of Proposition 3.1,3.

3.0.9. An N-bounded complex in C is a collection of objects X_n, $n \geq N$ and maps $d_{nm} : X_n \to X_m[m-n+1], n > m$ such that $\sum_m d_{nm} d_{mk} = 0$ for any n,k. We refer to such a complex as $(X_\bullet, d)$ or simply $X_\bullet$. A bounded complex is the same as an N-bounded complex for some N.

For $Y \in C$ set $h_X(Y)_k = \hom(X_k, Y)$. We see that $H = h_X(Y)_\bullet$, $H^{ij} = (h_{X_j}(Y))^i$ is a complex of complexes, i.e. a bicomplex and we can take its total complex; denote it by $h_X(Y)$, $h_X(Y)^N = \prod_i H^{i,N-i}$. $h_X : C \to$ **complexes** is a functor. If h_X is representable, denote by $|X_\bullet|$ the representing object.

Call C *saturated* if $|X_\bullet|$ exists for any non-positive X.

3.0.10. Let Com C be the dg category of bounded complexes in C. One can show that for any saturated C, there exists a functor $|| : \mathrm{Com}\, C \to C$ such that $|X|$ represents h_X for any X.

3.0.11. Categories C^+ and C^{++} Let C be an arbitrary dg category. Let C^+ be the category an object X of which is a pair consisting of a family $X_i \in C, i \in I$ indexed by a set I_X. and a family $n_i^X, i \in I_X$ of integers. Set

$$\hom(X,Y) = \prod_{j \in I_X} \oplus_{i \in I_Y} \hom(X_j, Y_i)[n_i^Y - n_j^X].$$

Let C^{++} be the category of bounded complexes in C^+. It is clear that C^{++} is saturated; if C is saturated, then we have a natural functor $\iota : C^{++} \to C$.

Assume C is symmetric monoidal. Then C^+ and C^{++} are also naturally symmetric monoidal and ι has a strong symmetric monoidal structure.

3.1. Triangulated category structure

3.1.1. Let C be saturated. For $X \in C$ and an integer n set $X'_k = 0$ if $k \neq n$; $X'_n = X$; $d_r = 0$ for any r. $X'_\bullet$ is then a bounded complex. Set $X[n] = |X'|$. We have canonic isomorphisms $X[0] \cong X$ and $X[n][m] \cong X[n+m]$.

3.1.2. Let $f : X \to Y$ be a map in C. Set $Z_1 = X$; $Z_0 = Y$; $Z_n = 0$ for $n \neq 0,1$; set $d_{0,-1} : Z_0 \to Z_{-1}$ to be equal to f. Call Cone $f := |Z|$.

3.1.3. We have a natural sequence of maps $\cdots \to X \xrightarrow{f} Y \to \text{Cone } f \to X[1] \to Y[1] \cdots$. Call this sequence *a special distinguished triangle.*

3.1.4. Let DC be the category with the same objects as in C but now $\hom_{DC}(X,Y) = H^0(\hom_C(X,Y))$. A distinguished triangle in DC is any triangle isomorphic to the image in DC of a special distinguished triangle. This endows DC with a structure of triangulated category.

3.2. Compatible morphisms

3.2.1. Let C,D be saturated categories and $F : C \to D$ a functor. Let $(X_\bullet, d)$ be a bounded complex in C. Then $(F(X_\bullet), F(d))$ is a complex in D. We have a natural map $F' : h_X(Y) \to h_{F(X)}(F(Y))$. Let Id $\in h_X(|X|)$. Then $F'(Id)$ defines a map $F'' : |F(X)| \to F(|X|)$. Call F *compatible* if F'' is an isomorphism for any X.

3.2.2. Assume that C,D are saturated and fix a compatible map of dg categories $F : C \to D$. Assume that F has a left adjoint G.

Lemma 3.2. *G is compatible.*

3.3. Standard bar resolution

The material of this subsection is standard and can be found in e.g. [2].

3.3.1. Denote $S = GF$. The adjointness implies that we have maps of functors $\varepsilon : S \to I$ and $\Delta : I \to I \circ I$ such that Δ is associative and ε is a counit. In other words, S is a monad. This allows us to construct a simplicial object $\Sigma_\bullet$ in the category of endofunctors of C as follows. Set $\Sigma_i = S^{\circ(i+1)}$. Define the face map $d_i : \Sigma_n \to \Sigma_{n-1}$ by

$$S^{n+1} \cong S^{\circ n-i} \circ \mathbf{S} \circ S^{(i-1)} \xrightarrow{\varepsilon} S^{\circ n-i} \circ S^{(i-1)} \to S^n,$$

where ε is applied to the bold S (it is the same S as usual S; it is made bold to show where ε is applied.) The i-th degeneration $s_i : \Sigma_{n-1} \to \Sigma_n$, $0 \le i \le n-1$ is application of δ to the i-th S from the right hand side of $S^{\circ n}$. The map $\varepsilon : \Sigma \to \text{Id}$ is an augmentation. Let $\Sigma' := C_\bullet(\Sigma_\bullet)$ be the chain complex of the simplicial object $\Sigma_\bullet$. The augmentation induces a map $\iota : \Sigma' \to \text{Id}$; let Σ'' be the augmented complex.

3.3.2. Let $e : \mathrm{Id}_D \to FG$ be the canonical map Define maps $\gamma_i : F\Sigma''_i \to \Sigma''_{i+1}$ by $\gamma_i : FS^{i+1} \to FGFS^{i+1} \cong \Sigma''_{i+1}$. Let $\gamma = \sum \gamma_i$. It is well known that $d\gamma + \gamma d = \mathrm{Id}$. Therefore, $F\Sigma''(X)$ is contractible for any X and $F(\iota)$ is a universal quasi-isomorphism. Let $\Sigma(X) = |\Sigma'(X)|$. We have the induced map $p : \Sigma(X) \to X$ which is an F-quasi-isomorphism.

3.3.3. $\Sigma(X)$ is cofibrant for any X.

3.3.4. Let $\mathbf{Cof}_F(C)$ be the full subcategory of cofibrant objects of C.

Proposition 3.3. $\mathbf{Cof}_F$ *is saturated.*

Proof. One needs to check that total of a bounded complex of cofibrant objects is cofibrant which is immediate. □

The two notions of F-quasi-isomorphism and natural quasi-isomorphism coincide in $\mathbf{Cof}_F(C)$ and for any object X in C there exists an object $\Sigma(X)$ which is F-quasi-isomorphic to X

3.3.5. Assume a symmetric monoidal structure given on C and D such that F has a weak symmetric structure and preserves $\mathbf{Cof}_F(C)$.

3.4. Symmetric monoidal structure on Σ

Consider the composition

$$X \otimes Y \to FGX \otimes FGY \to F(GX \otimes GY)$$

and assume that the adjoint map $m : G(X \otimes Y) \to G(X \otimes Y)$ is an isomorphism. Then G and hence GF are symmetric monoidal functors and the transformations $FG \to \mathrm{Id}$ $\mathrm{Id} \to GF$ both preserve the tensor structures. Therefore, we have a morphism of simplicial objects in C: $\Sigma_\bullet(X) \otimes \Sigma_\bullet(Y) \to \Sigma_\bullet(X \otimes Y)$. Since the chain functor has a symmetric monoidal structure, we have the induced map $\Sigma(X) \otimes \Sigma(Y) \to \Sigma(X \otimes Y)$ which determines a tensor structure on Σ.

3.5. Categories $C\vee$

Let C be a small dg category. Denote by $C\vee$ the dg category of functors $C \to$ **complexes**. We have the Ioneda inclusion $i : C^{op} \to C\vee$ sending $X \in C$ to $i(X) : C \to$ **complexes**, where $i(X)(Y) = \hom(X,Y)$.

3.5.1. $C\vee$ is saturated.

3.6. Functoriality

Let $F : C \to D$ be a functor of small dg categories. The composition with F defines a functor $F^{-1} : D\vee \to C\vee$ so that $F^{-1}A(X) = A(F(X))$.

Proposition 3.4. *F^{-1} has the left and the right adjoint.*

Proof. Let us construct the left adjoint functor $F_!$. For $X \in C$ and $Y \in D$ denote

$$h(X,Y) = \hom_D(F(X),Y); \tag{3.2}$$

h is a functor $C^{op} \otimes D \to$ **complexes**. For $A \in C\vee$ set $F_!A = A \otimes_C h$. It is clear that $\hom_D(F_!A, B) \cong \hom(A, F^{-1}B)$.

Define also the right adjoint functor F_* by $F_*(A) = \hom_C(h,A)$. □

3.6.1. Let **triv** be a small k-linear category such that the abelian category of functors **triv** $\to$ **vect** is semi-simple. Let C be a small category equipped with a map s : **triv** $\to C$. Call an s-quasi-isomorphism in $C\vee$ simply a quasi-isomorphism.

3.6.2. Denote by $C^\frown = \mathbf{Cof}_s C\vee$ the full subcategory consisting of cofibrant objects. Let $F : C \to D$ be a functor. An Fs-quasi-isomorphism in D will be called quasi-isomorphism.

Proposition 3.5. *The functor F^{-1} maps quasi-isomorphisms to quasi-isomorphisms. The functor $F_!$ maps $C^\frown$ to $D^\frown$ and it maps any quasi-isomorphism in $C^\frown$ to a quasi-isomorphism in $D^\frown$.*

3.7.

Let $F : C \to (D^{op})\vee$ be a map of categories. Define $F^\# : D^\frown \to C\vee$ by $F^\#(U)(X) = U \otimes_D F(X)$. One sees that $\Sigma(F^\#)$ is compatible.

3.8. t-structures

3.8.1. Let C be a saturated dg-category. A *t-structure on* C is a pair of full subcategories $\mathscr{D}^{\geq 0}$, $\mathscr{D}^{\leq 0}$ of C satisfying certain properties; to formulate them introduce a notation

3.8.2. Set $\mathscr{D}^{\geq n} \subset C^\frown$ to be the full subcategory of objects U such that $U[n] \in \mathscr{D}^{\geq 0}$. Set $\mathscr{D}^{\leq n} \subset C$ to be the full subcategory of objects U such that $U[n] \in \mathscr{D}^{\leq 0}$.

3.8.3. Axioms of t-structure

1 $\mathscr{D}^{\leq 0} \subset \mathscr{D}^{\leq 1}$;

2 $\mathscr{D}^{\geq 1} \subset \mathscr{D}^{\geq 0}$;

3 For any $X \in \mathscr{D}^{\leq 0}$ and $Y \in \mathscr{D}^{\geq 0}$, $H^0 \hom(X,Y) = 0$;

4 For any X in C there exists a distinguished triangle $\cdots \to A \to X \to B \to \cdots$ with $A \in \mathscr{D}^{\leq -1}$ and $B \in \mathscr{D}^{\geq 0}$.

3.8.4. It follows that in 4. the objects A and B are defined by X uniquely up to a quasi-isomorphism. Furthermore there exist functors $\tau_{\leq n}$ and $\tau_{\geq n}$ so that $\tau_{\leq n}X \cong (\tau_{\leq 0}X[-n])[n]$ and $\tau_{\geq n}X \cong (\tau_{\leq 0}X[n])[-n]$; and there are natural transformations $\tau_{\leq -1} \to \mathrm{Id} \to \tau_{\geq 0}$ so that

$$\to \tau_{\leq -1}X \to X \to \tau_{\geq 0}X \to$$

is a distinguished triangle for any X.

3.8.5. Example Assume that the spaces of homomorphisms for any two objects in C lie totally in the zeroth degree. Set $\mathscr{D}^{\geq 0} \subset C^{\frown}$ to consist of all functors U such that $H^{<0}U(X) = 0$ and $\mathscr{D}^{\leq 0}$ to include all U such that $H^{>0}U(X) = 0$ for any X.

3.8.6. Example Assume that C is a dg category such that for any $X, Y \in C$ $\hom_C(X,Y)^i = 0$ for $i < 0$ and $d_0 : C(X,Y)^0 \to C(X,Y)^1$ is equal to 0. Assume that **triv** is the category with the same objects as in C but $\hom_{\mathbf{triv}}(X,Y) = 0$ if $X \neq 0$ and $\hom_{\mathbf{triv}}(X,X) = k\mathrm{Id}$ and let $s : \mathbf{triv} \to C$ to be the inclusion. Define a category B with the same set of objects as in C in which $\hom_B(X,Y) = \hom_C(X,Y)^0$. We have a map $p : C \to B$ and $i : B \to C$. Furthermore, s takes values in B so that we have a map $s' : \mathbf{triv} \to B$. Call a map $f \in C\vee$ (resp. $g \in B\vee$) a quasi-isomorphism if such is $s^{-1}f$ (resp. $(s')^{-1}g$).

3.8.7. Let $\mathscr{D}^{\geq 0} \subset C^{\frown}$ be the full subcategory of objects U such that $H^i(U(X)) = 0$, for any $i < 0$ and any X.

3.8.8. Let $\mathscr{D}^{\leq 0} \subset C^{\frown}$ be the full subcategory of objects V such that $H^i((p_!V)(X)) = 0$ for any $i > 0$ and any X.

3.8.9.

Proposition 3.6. *Assume that $i_! : \mathscr{B}^{\vee} \to C\vee$ maps quasi-isomorphisms to quasi-isomorphisms. Then the pair $\mathscr{D}^{\geq 0}$ and $\mathscr{D}^{\leq 0}$ described above is a t-structure.*

Proof. 1,2 are clear.

3. Let $X' = \Sigma(X)$. It suffices to prove the statement for $\hom(X',Y)$. Set $(F^iY)^j = Y^j$ if $j \geq i$ $(F^iY)^j = 0$ otherwise. Note that

$$\hom(X,Y) \to \mathrm{liminv}\ \hom(X',Y/F^iY)$$

is an isomorphism of complexes and the maps $p_i : \hom(X, Y/F^{i+1}Y) \to \hom(X, Y/F^iY)$ are componentwise surjective. Therefore, it suffices to check the statement for $\hom(X', F^iY/F^{i+1}Y)$, $i > 0$ and for $\hom(X', Y/F^1Y)$.

Note that $F^iY/F^{i+1}Y = p^{-1}Z$ for some Z (because all morphisms from $\operatorname{Ker} p$ act trivially), and the statement follows from the adjointness axioms; in the second case, we have a quasi-isomorphism in $C\vee$ $Y/F^1Y \to H^0(Y/F^1Y) = Y'$ and again $Y' = p^{-1}Z'$ for some Z'.

4. We can write that X' is the functor $||$ applied to a complex

$$\cdots \to s_! A_n \stackrel{D}{\to} \cdots .$$

One can assume without loss of generality that all A_n are complexes with zero differential. Let $U_n = s_!^! A_n$. Therefore, $s_! A_n \cong i_! U_n$. Then each U_n has zero differential. In what follows we say that $X \in C\vee$ is in $D^{\leq 0}$ if so is ΣX (or any other cofibrant object quasi-isomorphic to X). We see that if $0 \to U \to V \to W \to 0$ is a sequence of objects from $C\vee$ such that $0 \to U(X) \to V(X) \to W(X) \to 0$ is an exact sequence of complexes for any $X \in C$ and $U, W \in D^{<0}$, then so is V.

For $U \in B\vee$ being a complex with zero differential and $X \in B$ set $U_{<n}(X) = \tau_{<n} U(X)$, where for a complex K we set

$$\tau_{<n} K^i = 0,\ i \geq n;$$
$$\tau_{<n} K^i = K^i\ i \leq n-1.$$

Set

$$V_n = i_! \tau_{<n} U_n + D i_! \tau_{<n+1} U_{n+1} \subset i_! U_n.$$

It is clear that $V_\bullet \subset U_\bullet$ is a subcomplex and that $H^{<0}(U_\bullet/V_\bullet) = 0$. Show that $|\Sigma(V)|$ is in $\mathscr{D}^{<0}$. Set $F^iV_n = V_n,\ n < i$; $F^iV_i = i_! \tau_{<i} U_i$; $F^iV_n = 0,\ n > i$. It suffices to check that $|\Sigma(F^{i+1}V/F^iV)|$ is in $\mathscr{D}^{<0}$. Let $G = F^{i+1}V/F^iV$. Note that

$$G_n = 0,\ n \neq i, i+1;$$
$$G_i = (D i_! \tau_{<i+1} U_{i+1})/(D i_! \tau_{<i+1} U_{i+1} \cup \tau_{<i} U_i);$$
$$G_{i+1} = i_! \tau_{<i+1} U_{i+1}.$$

Note that $D(i_! \tau_{<k} U_{i+1}) \subset i_! \tau_{<k+1} U_i$ (because $\hom_C(X,Y)^{<0} = 0$). Let $K_{i+1} = i_! \tau_{<i} U_{i+1} \subset G_{i+1}$; $K_n = 0,\ n \neq i+1$. The above remark shows that K is a subcomplex of G. Since $K \in \mathscr{D}^{<0}$, it suffices to prove that $G/K \in \mathscr{D}^{<0}$.

Consider the following complex.

$$D' : i_!(\tau_{<i+1} U_{i+1}/\tau_{<i} U_{i+1})[i+1] \to i_!(\tau_{<i+1} U_i/\tau_{<i} U_i)[i],$$

where D' is induced by D. Then G/K is a subcomplex of this complex. We have $\tau_{<i+1} U/\tau_{<i} U = U^i[-i]$ for any complex with zero differential. Therefore, G/K is isomorphic to the total of $D' : i_! U^i_{i+1} \to D'(i_! U^i_{i+1})$ which is quasi-isomorphic to $\operatorname{Ker} D'[1]$.

One sees that the map $D' : i_! U^i_{i+1} \to i_! U^i_i$ is induced by a certain map $\delta : U^i_{i+1} \to U^i_{i+1}$. Because of the exactness of $i_!$ on $B'\vee$, this implies that $\operatorname{Ker} D'[1] \cong i_! \operatorname{Ker} \delta[-1]$, hence is in $\mathscr{D}^{\leq 0}$. □

3.8.10. let C, D be as above. A compatible functor $F : D^\frown \to C^\frown$ is called a t-functor if it maps $\mathscr{D}^{\geq 0}$ (resp. $\mathscr{D}^{\leq 0}$) to the corresponding subcategories in $C^\frown$.

3.8.11. Let $F : C \to D$ be a homotopy equivalence of dg-categories and $F(\mathscr{D}^{\geq 0}) \subset \mathscr{D}^{\geq 0}$ Then F is a t-functor.

3.9. Symmetric monoidal structures

3.9.1. Assume C is symmetric monoidal. Define the tensor product on $C\vee$ as follows. Let $X, Y \in C\vee$ and let $m : C \otimes C \to C$ be the tensor product. Set $X \otimes Y = m_!(X \boxtimes Y)$

Proposition 3.7. *Let $f : C \to D$ be a symmetric monoidal functor. Then $f_!$ has a symmetric monoidal structure and f^{-1} has an induced weak symmetric monoidal structure.*

3.9.2. Assume that **triv** is symmetric monoidal and $s :$ **triv** $\to C$ is symmetric monoidal map.

Proposition 3.8. *$C^\frown$ is preserved by the tensor product on $C\vee$.*

Proposition 3.9. *Assume C is as in 3.8.6. Then $\mathscr{D}^{\leq n} \otimes \mathscr{D}^{\leq m} \subset \mathscr{D}^{\leq n+m}$.*

3.9.3. For A, B in T_C set $A \otimes_T B = \tau_{\geq 0}(A \otimes B)$. The above result implies that $\otimes_T$ is a tensor product in T_C

3.9.4. Let $F : C^\frown \to D^\frown$ be a symmetric monoidal functor with a weak symmetric structure on it. Then $F_T : T_C \to T_D$ has an induced weak symmetric monoidal structure.

3.9.5. Assume that a (symmetric monoidal) category C is such that $\hom(X, Y)$ are always in degree 0. Then T_C as a (symmetric monoidal) category is equivalent to the category of functors $C \to$ **vect**.

4. Full operads and weak symmetric monoidal functors.

4.1. Definitions

4.1.1. Let C be a symmetric monoidal dg category and $V \in C$. For a finite set X denote $\mathscr{O}_V(X) = \hom_C(V^{\otimes X}, V)$. We see that $\mathscr{O}_V$ is a dg operad.

4.1.2. Denote by $C[[t]]$ the category with the same objects as in C but $\hom_{C[[t]]}(X,Y) = \hom_C(X,Y)[[t]]$, where t is a formal parameter. We have natural functors $i : C \to C[[t]]$ and $\bmod t : C[[t]] \to C$.

4.1.3. We will call natural quasi-isomorphisms in C simply "quasi-isomorphisms".

4.1.4. Let $F : C \to D$ be a weak symmetric monoidal functor. We have an induced map $F_{\cdot} : \mathscr{O}_V \to \mathscr{O}_{F(V)}$. Similarly, for $F_t : C \to D[[t]]$, we have an induced map $F_{t.} : \mathscr{O}_V \to \mathscr{O}_{F_t(V)}[[t]]$.

4.1.5. Let $\mathscr{O}$ be an arbitrary dg-operad and let $X \in \mathscr{O}([1])$ be an invertible element. It defines a map $\mathrm{Ad}_X : \mathscr{O} \to \mathscr{O}$ such that for $f \in \mathscr{O}([n])$ $\mathrm{Ad}_X f = X^{-1} \circ f \circ X^{\otimes n}$.

4.1.6. Let $\mathscr{O}_1$, $\mathscr{O}_2$ be arbitrary dg operads. Assume a map of operads $F_t : \mathscr{O}_1 \to \mathscr{O}_2[[t]]$ is given. Denote by $F_0 = i(F_t \bmod t) : \mathscr{O}_1 \to \mathscr{O}_2[[t]]$ to be the term of zeroth order in t. F_t is called *deformationally trivial* if there exists a quasi-isomorphism $q : \mathscr{O}_1' \to \mathscr{O}_2$, an $X_t \in \mathscr{O}_{F(V)}([1])[[t]]$, $X_t = \mathrm{Id} \pmod{\mathrm{t}}$ and an extension $\gamma : \mathscr{O}_1' \to \mathscr{O}_2[[t,dt]]$ of

$$\mathrm{Ad}_{X_t} F_0 q : \mathscr{O}_1' \to \mathscr{O}_2[[t]],$$

where for an arbitrary operad $\mathscr{O}$ we define the operad $\mathscr{O}[t,dt]$ to be $\mathscr{O} \otimes \mathbf{k}[t,dt]$, where $\mathbf{k}[t,dt]$ is a commutative dg ring freely generated by the variables t, $\deg t = 0$ and dt, $\deg t = 1$, the differential is defined by $dt = dt$.

4.1.7.

Proposition 4.1. *Let $q : O_1' \to \mathscr{O}_1$ and $p : \mathscr{O}_2 \to \mathscr{O}_2'$ be quasi-isomorphisms and let $F_t : \mathscr{O}_1 \to \mathscr{O}_2[[t]]$. Then F_t is deformationally trivial if and only if such is pF_tq*

4.2.

Let $F_t : C \to C[[t]]$ be a weak symmetric functor. We have a map $F_{t.} : \mathscr{O}_V \to \mathscr{O}_{F(V)}[[t]]$. Assume we have another functor G_t having the same properties as F_t and a natural symmetric monoidal transformation $a : G_t \to F_t$ which is a homotopy equivalence.

Proposition 4.2. *$F_{t.}$ is a homotopically trivial deformation if and only if $G_{t.}$ is such.*

We may assume that C and D are saturated (if not, replace them with C^{++} and D^{++}).

Let $A = F(V)$, $B = G(V)$. We have a quasi-isomorphism $\alpha : A \to B$ in $D[[t]]$ induced by the natural transformation. It follows that the diagram

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{F_{t.}} & A \\ \Big\downarrow \alpha^{\otimes n} & & \Big\downarrow \alpha \\ B^{\otimes n} & \xrightarrow{G_{t.}} & B \end{array} \tag{4.3}$$

commutes.

Let $h : P \to Q$ be any map in C. Let s, ds be variables such that degree of s is 0 and degree of ds is 1. Let $Q[s]$ be the direct sum of countably many copies of Q (each copy denoted by Qs^k, $k \geq 0$) and $Q[s]ds$ be the direct sum of countably many copies of $Q[-1]$ (each copy denoted by $Qs^k ds$, $k \geq 0$.

Set $D_k : Qs^k \to Qs^{k-1}ds$ to be the map equal to $k\mathrm{Id}$; The maps D_k define a map

$$D : Q[s] \to Q[s]ds[1]$$

of degree 0. Set $Q[s, ds] = \mathrm{Cone}\, D$. Consider the map $e : Q[s, ds] \to Q[ds]$ which is evaluation at $s = 0$. This map splits in C; therefore the colimit of the diagram

$$P \to Q[ds] \leftarrow Q[s, ds]$$

exists. Denote it Ch We have canonic quasi-isomorphic maps $p_1 : Ch \to P$ and $p_2 : Ch \to Q$, p_1 being the canonic projection of the colimit onto P and p_2 being the evaluation at $s = 1$. The following important property holds:

Lemma 4.3. *For any $X \in C$ the maps of complexes* $\hom(X, Ch) \stackrel{p_{1*}}{\to} \hom(X, P)$ *and* $\hom(X, Ch) \stackrel{p_{2*}}{\to} \hom(X, Q)$ *are componentwise surjective and quasi-isomorphic.*

let $h : P \to Q$ and $g : P' \to Q'$ be quasi-isomorphisms. We have maps

$$D_1 : \hom(Ch, Cg) \oplus \hom(P, P') \to \hom(Ch, P'),$$

and

$$D_2 : \hom(Ch, Cg) \oplus \hom(Q, Q') \to \hom(Ch, Q')$$

given by $D_i(a \oplus b) = p_{i*}a - p_i^* b$. It follows that D_i are componentwise surjective. Denote $\hom_i(Ch, Cg) = \mathrm{Ker}\, D_i$. The natural maps $\hom_i(Cf, Cg) \to \hom(Ch, Cg)$, $\hom_1(Ch, Cg) \to \hom(P, P')$, and $\hom_2(Ch, Cg) \to \hom(Q, Q')$ are quasi-isomorphisms.

Also we have a map

$$Ch \otimes Ch' \to C(h \otimes h'). \tag{4.4}$$

which is a quasi-isomorphism.

The commutativity of (4.3) means that we have a commutative diagram

$$\begin{tikzcd}
\hom(V^n,V) \arrow[r] \arrow[d] & \hom_1(C(\alpha)^{\otimes n},C(\alpha)) \arrow[r, "\approx"] \arrow[d, "\approx"] & \hom(A^{\otimes n},A)[[t]] \\
\hom_2(C(\alpha)^{\otimes n},C(\alpha)) \arrow[r, "\approx"] \arrow[d, "\approx"] & \hom(C(\alpha)^{\otimes n},C(\alpha)) & \\
\hom(B^{\otimes n},B)[[t]]) & &
\end{tikzcd}$$

Note that each term of this diagram forms an operad and all the arrows are maps of operads. The composition of the two arrows in the upper row is equal to the map $F_{t.}$; the composition of the two arrows in the leftmost column is equal to $G_{t.}$.

Now the statement follows from 4.1. $\square$

4.2.1. Let now $I : C' \to C$ and $J : D \to D'$ be homotopy equivalences of symmetric monoidal categories. Let $V' \in C'$ be an element such that $I(V')$ is quasi-isomorphic to V.

Proposition 4.4. *the deformation $(JF_tI)_{.} : \mathcal{O}'_V \to \mathcal{O}_{JF_tI(V')}[[t]]$ is trivial if and only if such is $F_{t.}$.*

Proof.

Lemma 4.5. *Let $f : W \to V$ be a quasi-isomorphism in C. and $H_t : C \to D[[t]]$ a weak symmetric monoidal functor. Then the deformation $H^V_{t.} : \mathcal{O}_V \to O_{F_t(V)}[[t]]$ is trivial if and only if $H^W_{t.} : \mathcal{O}_W \to \mathcal{O}_{F_t(W)}[[t]]$ is trivial.*

Proof. First consider the case when f is a retraction, i.e. there exist $g : V \to W$ such that $fg = \mathrm{Id}$. Let $P = gf$, $P^2 = P$. We have a map $q : \mathcal{O}_V \to \mathcal{O}_W$ such that $\phi : V^{\otimes n} \to V$ goes into $g\phi f^{\otimes n} : W^{\otimes n} \to W$.

Similarly, we have a map $q_t : \mathcal{O}_{H_t(V)} \to \mathcal{O}_{H_t(W)}[[t]]$ such that $\phi : H_t(V)^{\otimes n} \to V$ goes into $H_t(g)\phi H_t(f)^{\otimes n}$:.

Sublemma 4.6. *The deformation q_t is trivial*

Proof. First, show that There exists a family $U_t \in \hom(H_t(W),H_t(W))[[t]]$ such that $U_t = \mathrm{Id} \bmod t$ and $H_t(F) = U_tF_0$ for some $F_0 \in \hom(H_t(W),H_t(W))$. Such a U_t can be given by the formula

$$U_t = \mathrm{Id} + H_t(F)H_0(G) - H_0(FG).$$

Now, $\mathrm{Ad}_{U_t} q_t = \mathrm{Id}$, whence the statement. $\square$

The composition $\mathcal{O}_V \to \mathcal{O}_W \to \mathcal{O}_{H_t(W)}$ is equal to $\mathcal{O}_V \to \mathcal{O}_{H_t(V)} \to \mathcal{O}_{H_t(W)}$, whence the statement for f being a retraction.

If f is arbitrary, then there exists $W' \in C$ and quasi-isomorphic retractions $W' \to V$ and $W' \to W$, whence the statement in general case. $\square$

Now our statement reduces to the following one: the deformation $F_{t.}: \mathscr{O}'_V \to \mathscr{O}_{IF_tJV'}$ is trivial if and only if such is $\mathscr{O}_{JV'} \to \mathscr{O}_{F_tJV'}$. But $F_{t.}$ decomposes as $\mathscr{O}_{V'} \to \mathscr{O}_{JV'} \to \mathscr{O}_{F_tJV'} \to \mathscr{O}_{IF_tJV'}$, whence the statement. □

4.2.2. The two previous subsections imply the following result: Let $F_t, G_t : C \to D[[t]]$ be deformationally homotopically equivalent. Then for any $V \in C$ the triviality of the deformation $F_{t.}$ implies the triviality of the deformation $G_{t.}$.

5. PROP's

Let **setf** be the category of finite sets.

5.0.3. Trivial prop Consider the groupoid **iso** of finite sets and their isomorphisms. It has an obvious symmetric monoidal structure in which $X \otimes Y = X \sqcup Y$. Let **triv** be the **k**-span of **iso** A dg PROP is a symmetric monoidal dg category P with its objects being finite sets equipped with a morphism of symmetric monoidal categories $s : \mathbf{triv} \to P$ which induces the identity on the classes of objects. It is clear that any PROP is a small category.

5.0.4. Example Let V be an object of an arbitrary symmetric monoidal dg category $\mathscr{A}$. We thus have for any finite set X the corresponding tensor product $\otimes_X V$. For $X = \{1, \ldots, n\}$,

$$\otimes_X V \cong V^{\otimes n}.$$

Define a PROP P_V in which

$$\hom_{P_V}(X,Y) = \hom_{\mathscr{A}}(\otimes_X V, \otimes_Y V).$$

the tensor structure and the map s are straightforward. The PROP P_V is universal among all PROP's P equipped with a symmetric monoidal map of categories $f : P \to \mathscr{A}$ such that $f([1]) = V$.

5.0.5. P-algebras In the notation of a previous section, a P-algebra structure on V is a map of PROPS $P \to P_V$.

5.1. Weak P-algebras

A weak P-algebra in C is a weak symmetric monoidal map $P \to C$.

5.1.1. Let $C = Q^{op}\vee$ Then any weak P-algebra in C is defined by a weak symmetric monoidal map $A : P \to Q\vee$ which is the same as a map $A' : P \otimes Q^{op} \to$ **complexes** or the same as a weak symmetric monoidal map $A'' : Q^{op} \to P\vee$.

5.1.2. Let $s^{-1} : P\vee \to \mathbf{triv}\vee$. A is a strong symmetric monoidal map if and only if such is $s^{-1}A''$

5.2. Generalized maps of PROPs

5.2.1. Weak generalized maps A weak generalized map $P \to Q$ is by definition a weak symmetric monoidal map $F : Q\vee \to P\vee$. For any usual map of PROPs $f : P \to Q$, f^{-1} is a weak generalized map.

5.2.2. Strong generalized map Let $s_Q^{-1} : Q\vee \to \mathbf{triv}\vee$ and $s_P^{-1} : P\vee \to \mathbf{triv}\vee$. A weak generalized map $F : Q\vee \to P\vee$ is called *strong* if for any symmetric monoidal category C and any weak symmetric monoidal map $A : C \to Q\vee$ such that $s_Q^{-1}A$ is strong symmetric monoidal so is $s_P^{-1}FA$.

5.2.3. Homotopic versions A weak map of PROPs $P \to Q$ is by definition a weak symmetric monoidal map $F : Q^\frown \to P^\frown$. For any usual map of PROPs $f : P \to Q$, Σf^{-1} is a weak generalized map. A weak map $F : Q\vee \to P\vee$ is called strong if for any symmetric monoidal category C and any weak symmetric monoidal map $A : C \to Q\vee$ such that $s_Q^{-1}A$ is essentially strong symmetric monoidal so is $s_P^{-1}FA$.

5.2.4. Let $F : P \to Q^{op}\vee$ be a symmetric monoidal map. Consider the map $F\# : Q\vee \to P\vee$ defined in 3.7. Define a symmetric monoidal structure on $F\#$. it suffices to construct maps $F\#(U)(X) \otimes F\#(V)(Y) \to F\#(U \otimes V)(X \otimes Y)$ for each X,Y,U,V. We have the following sequence of natural maps

$$F\#(U)(X) \otimes F\#(V)(Y) \cong F(X) \otimes_Q U \otimes F(X) \otimes_Q V \to F(X \otimes Y) \otimes_Q (U \otimes V).$$

5.2.5. If F is an essentially strong symmetric monoidal map, then $F\#$ is essentially strong.

5.2.6. Let $F : P \to Q^{op}\vee$ be a strong symmetric monoidal map. We have a natural extension $F' : P^{op}\vee \to Q^{op}\vee$.

5.2.7.

Proposition 5.1. *Assume that $F([1])$ is quasi-isomorphic to $h_{[1]}$ in $Q^{op}\vee$, where $h_{[1]}(X) =$* $\hom_Q([1],X)$*, and that the induced map $F_* : P \to P_{f[1]}$ is a quasi-isomorphism. Then* $\Sigma(F\#) : Q^\frown :\to P^\frown$ *is a quasi-isomorphism of PROPs.*

6. Conilpotent Hopf algebras

6.1. Prop of bialgebras

6.1.1. Let C be a symmetric monoidal category with unit. We say that $X \in C$ has a structure of bialgebra if C has structure of associative algebra with unit and a coassociative coalgebra with counit such that the coproduct map $X \to X \otimes X$ and the counit map $X \to \mathbf{1}$ are morphisms of unital associative algebras.

6.1.2. Say that $m \in C$ has a $\mathscr{BA}$-structure if $X := \mathbf{1} \oplus m$ exists and we have a bialgebra structure on X with the natural inclusion of $\mathbf{1}$ being the unit and the natural projection onto $\mathbf{1}$ being the counit.

6.1.3. There exists a PROP $\mathscr{BA}$ of BA-algebras. $\mathscr{BA}$ is uniquely specified by the condition that $\mathscr{BA}$-structures on $m \in C$ are in 1-1 correspondence with the maps of PROPs $\mathscr{BA} \to P_m$.

6.1.4. A $\mathscr{BA}$-structure on m implies structures of associative algebra without unit and of a coassociative algebra without unit on m. Let **assoc** (resp. **coass**) be the PROPs describing associative algebra (resp. coassociative algebra) structure. We thus have maps of PROPs $i : \mathbf{assoc} \to \mathscr{BA}$ and $j : \mathbf{coass} \to \mathscr{BA}$. The composition defines a map ϕ_Z : $\mathbf{assoc}(Y,Z) \otimes_{S_Y} \mathbf{coass}(X,Y) \to \mathscr{BA}(X,Z)$. Therefore, we have a map

$$\phi = \sum \phi_n : \bigoplus_{n=|X|+1}^{\infty} Ass([n],Z) \otimes_{S_n} Coass(X,[n]) \to \mathscr{BA}(X,Z)$$

Proposition 6.1. *ϕ is an isomorphism*

6.1.5.

Corollary 6.2. *The tensor product on $\mathscr{BA}^{\vee}$ is exact.*

Proof. We see that $i^{-1}(X \otimes Y) \cong i^{-1}(X) \otimes i^{-1}(Y)$. Therefore it suffices to show that the tensor product in $\mathbf{assoc}\vee$ is exact. Let A,B be finite sets. Call a map $h : A \to B$ in **assoc** simple if it is a composition of several products (no linear compositions are allowed). To specify such a map is the same as to prescribe a surjective map of sets $h' : A \to B$ and a total order on each $f^{-1}a$. Let now $A = A_1 \sqcup A_2$. Call h' irreducible if for any $a \in A$ and any neighboring $x,y \in f^{-1}a$ (i.e. there are no elements in $f^{-1}a$ between x and y), either $x \in A_1$,$y \in A_2$ ore vice versa. Any simple map $h : A_1 \sqcup A_2 \to B$ in **assoc** uniquely splits as a product $h_1 h_2$, where $h_2 : A_1 \sqcup A_2 \to B_1 \sqcup B_2$ maps A_i to B_i and $h_1 : B_1 \sqcup B_2 \to B$ is irreducible.

Let now $X,Y \in \mathbf{assoc}\vee$. We have

$$X \otimes Y([k]) = \oplus X([m]) \otimes X([n]) \otimes_{S_m \otimes S_n} \mathbf{Irred}\,([m] \sqcup [n],[k]),$$

which is manifestly exact. $\square$

6.2. conilpotent functors

A functor $F : \mathscr{BA} \to$ **complexes** is called conilpotent if for any $n > 0$ and any $x \in F([n])$ there exists an m such that for any $m' \geq m$

$$j(\mathbf{coass}([n],[m])x = 0.$$

A $\mathscr{BA}$-algebra X in **complexes** is called conilpotent if such is X as a functor $\mathscr{BA} \to$ **complexes**. Denote by S the full subcategory of $\mathscr{BA}\vee$ consisting of conilpotent functors. Denote by $I : S \to \mathscr{BA}\vee$ the obvious inclusion.

6.2.1. Denote by $S' \subset \mathscr{BA}^{\frown}$ the full subcategory of objects quasi-isomorphic in $\mathscr{BA}^{\vee}$ to elements from S.

6.3. Study of S'

6.3.1. Let $\mathscr{BA}_n$ be the subcategory of $\mathscr{BA}^{\vee}$ consisting of objects X such that $X([m]) = 0$ for all $m > n$. We have $\mathscr{BA}_n \subset S$.

Proposition 6.3. *Any object X from S admits an increasing filtration $X_0 \subset X_1 \subset X_2 \subset \cdots = X$ such that $X_n \in \mathscr{BA}_n$.*

Proof. Set X_n to be the subobject generated by all $x \in X([k])$ such that $j(\mathbf{coass}([k],[n]))x = 0$. $\square$

6.3.2.

Proposition 6.4. *Let $\otimes$ be the tensor product in $\mathscr{BA}\vee$. Then*

1. $\mathscr{BA}_n \otimes \mathscr{BA}_m \subset \mathscr{BA}_{n+m}$;
2. $S \otimes S \subset S$ *and*
3. $S' \otimes S' \subset S'$.

Proof. Since the tensor product exact, 3. follows from 2. In virtue of Proposition 6.3, 2. follows from 1. Finally, 1. can be checked directly. $\square$

6.3.3. Denote by $S'_n \subset S'$ the full subcategory of objects quasi-isomorphic to objects from $\mathscr{BA}_n$. We have $S'_n \otimes S'_m \subset S'_{n+m}$.

Proposition 6.5. *Any object X from S' is quasi-isomorphic to an object X' admitting an increasing exhausting filtration $X'_0 \subset X'_1 \subset \cdots$ such that $X'_k \in S'_k$ and all the quotients X'_{k+1}/X'_k are cofibrant.*

Proof. We have that X is quasi-isomorphic to an object $Y \in S$. Let $Y_0 \subset Y_1 \subset \cdots$ be a filtration as in 6.3.1. Set $X' = \Sigma X$ and $X'_n = \Sigma X_n$. □

6.4. Functors between $\mathscr{BA}_n$ for different n.

6.4.1. Let C be a category whose objects are finite sets. Let $C_{\leq n}$ be its full subcategory consisting of sets with at most n elements. Let $i_n^C : C_{\leq n} \to C$ be the inclusion.

6.4.2. Let $\mathfrak{I}_n \subset \mathscr{BA}$ be the double-sided deal generated by all $\hom([p],[q])$ with $p,q \geq n$, $p < q$. Let $\mathscr{BA}'_n = \mathscr{BA}/I_n$, $p_n : \mathscr{BA} \to \mathscr{BA}'_n$ be the natural projection and $S_n = (BA'_n)_{\leq n}$. We have: $S_n(p,q) \cong \oplus_{r=2}^n \mathbf{coass}(p,r) \otimes \mathbf{assoc}(r,q)$ We have natural maps $p'_{nm} : \mathscr{BA}'_n \to \mathscr{BA}'_m$; $p_{nm} : S_n \to (\mathscr{BA}'_m)_{\leq n}$; $i_{mn} : S_m \to (\mathscr{BA}'_m)_{\leq n}$, where $m \leq n$ and $i_{mn} = i_m^{(\mathscr{BA}'_m)_{\leq n}}$.

6.4.3. We have $\mathscr{BA}_n \cong S_n\vee$ and the inclusion $I_n : \mathscr{BA}_n \to \mathscr{BA}\vee$ is equivalent to $p_n^{-1}(i_n^{\mathscr{BA}})_! \cong p_n^{-1}(i_n^{\mathscr{BA}})_*$.

6.4.4. The functor $T_n = (i_n^{\mathscr{BA}})^{-1} p_{n!}$ is the left adjoint to I_n.

6.4.5. Similarly, we have maps $I_{mn} : \mathscr{BA}_m \to \mathscr{BA}_n$, $m < n$ given by $I_{mn} = p_{nm}^{-1}(i_{mn})_! = p_{nm}^{-1}(i_{mn})_*$. Therefore, I_{mn} has a left adjoint $T_m = i_{mn}^{-1}(p_{nm})_!$; $T_m : \mathscr{BA}_n \to \mathscr{BA}_m$.

6.4.6. A similar notation can be introduced for the category **coass**. Let $C_n = \mathbf{coass}_{\leq n}$ and $i_n : C_n\vee \to \mathbf{coass}\vee$ be the functor of the extension by zero. The map j defines maps $j_n : C_n \to S_n$. We have the left adjoint functor $T_n : \mathbf{coass}\vee \to C_n\vee$ for the functor i_n. We have a natural quasi-isomorphism $T_n j_! K \to j_{n!} T_n K$ for any $K \in \mathbf{coass}^\frown$

6.4.7. Define the symmetric monoidal structure on $S_n\vee$ by setting for $U,V \in \mathscr{BA}_n$, $U \otimes V = T_n(\mu_{nn})_!(U \boxtimes V)$ The functor $T_n : \mathscr{BA} \to \mathscr{BA}_n$ has then a symmetric monoidal structure.

6.5. Properties of T_n and I_n

6.5.1.

Proposition 6.6. *Let $A \in S_n^\frown$. Then the natural map $m_n : T_n \Sigma I_n A \to T_n I_n A \cong I_n A$ is a quasi-isomorphism in $\mathscr{BA}_n$.*

Proof. One sees that any object from $S_n^\frown$ is quasi-isomorphic to bounded complex of objects of the form $j_{n!}K$, where $K \in C_n^\frown$. The problem then reduces to $A = j_{n!}K$. We have a quasi-isomorphism $j_!\Sigma_{\mathbf{coass}} i_n K \to \Sigma_{\mathscr{B}\mathscr{A}} j_{n!}K$. The composition

$$T_n j_! \Sigma_{\mathbf{coass}} i_n K \to T_n \Sigma_{\mathscr{B}\mathscr{A}} j_{n!}K \to j_{n!}K$$

is equal to

$$T_n j_! \Sigma_{\mathbf{coass}} i_n K \to j_{n!} T_n \Sigma_{\mathbf{coass}} i_n K \to j_{n!}K,$$

and the problem is reduced to showing that

$$T_n \Sigma_{\mathbf{coass}} i_n K \to K$$

is a quasi-isomorphism, which follows from the isomorphism $T_n \Sigma_{\mathbf{coass}} i_n K \to \Sigma_{C_n} K$. $\square$

6.5.2.

Proposition 6.7. *Let $X \in S'_n$. Then the natural map $X \to I_n T_n X$ is a quasi-isomorphism in $\mathscr{B}\mathscr{A}\vee$*

Proof. We have a quasi-isomorphism $f : X \to X'$ in $\mathscr{B}\mathscr{A}\vee$ for some $X' \in \mathscr{B}\mathscr{A}_n$. and the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\approx"] & I_n T_n X \arrow[d] & I_n T_n \Sigma X \arrow[l, "\approx"'] \arrow[d, "\approx"] \\ X' \arrow[r, "\approx"] & I_n T_n X' & I_n T_n \Sigma X', \arrow[l, "\approx"'] \end{tikzcd} \tag{6.5}$$

where $\approx$ means “quasi-isomor-phism”, therefore all the remaining arrows are quasi-isomorphisms. $\square$

6.5.3.

Proposition 6.8. *Let $A, B \in S_n^\frown$. Then* $\hom_{S_n^\frown}(A,B) \to \hom_{S'}(\Sigma I_n A, \Sigma I_n B)$ *is quasi-isomorphism.*

Proof. We have

$$\hom_{S'}(\Sigma I_n A, \Sigma I_n B) \approx \hom_{BA\vee}(\Sigma I_n A, I_n B) \cong \hom_{\mathscr{B}\mathscr{A}_n}(T_n \Sigma I_n A, B).$$

Since we know that $T_n \Sigma I_n A \to T_n I_n A \leftarrow A$ are quasi-isomorphisms, the statement follows. $\square$

Proposition 6.9. *Let $X \in \mathscr{B}\mathscr{A}^\frown$ and $Y \in S'_n$. Then for any $N \geq n$ the map* $\hom(X,Y) \to \hom_{S_n^\frown}(T_n X, T_n Y)$ *is a quasi-isomorphism.*

Proof.
We have a chain of quasi-isomorphisms $\hom(X,Y) \to \hom(X, I_N T_N Y) \to \hom(T_N X, T_N Y)$. $\square$

6.6. S' is saturated

6.6.1.

Proposition 6.10. *Any object X in $\mathscr{BA}^\frown$ admitting the filtration as in 6.3 is in S'.*

Proof. Without loss of generality we can assume that the filtration on X is such that each X_i/X_{i-1} is cofibrant (i.e. the filtration is as in 6.5).

Let $A_i = \mathrm{Ker}\, T_i|_{X_i}$. We have $A_i \subset X_i$ and $A_{i+1} \cap X_i \subset A_i$. Let

$$Z_n = X_n / \sum_{i=0}^{n} A_i.$$

and $Z = X / \sum_{i=0}^{\infty} A_i$. It is clear that $Z_n, Z \in S$, We have obvious maps $Z_n \to Z_{n+1}$ which are inclusions and $Z = \mathrm{limdir}\, Z_n$. We have $Z_{n+1}/Z_n \cong T_{n+1}X_{n+1}$. Also we have an obvious projection $p : X \to Z$ such that $p(X_n) \subset Z_n$ and the associated graded map is isomorphic to the collection of maps $X_{n+1}/X_n \to T_{n+1}(X_{n+1}/X_n)$. which are quasi-isomorphisms. □

6.6.2.

Proposition 6.11. *Let I be any totally ordered set and let $X \in \mathscr{BA}$ and $X_i, i \subset I$ be a filtration such that $X_i / \bigcup_{j<i} X_j$ belongs to $S_n(i)$ for some n_i. Then $X \in S'$*

Proof. Same as above. □

6.6.3.

Proposition 6.12. *$S' \subset \mathscr{BA}^\frown$ is a saturated subcategory.*

Proof. Let $X_\bullet$ be a bounded complex with $X_i \in S'$. One can replace it with a termwise quasi-isomorphic bounded complex $Y_\bullet$ such that each Y_n admits a filtration 6.5. Denote this filtration by $G_0Y_n \subset G_1Y_n \subset G_2Y_n \subset \ldots$. Denote

$$F_{ij}Y_k = 0, \quad k > i;$$
$$F_{ij}Y_i = G_jK_i$$
$$F_{ij}Y_k = Y_k, \quad k < i.$$

It is a filtration with respect to the lexicographic order on the set of pairs of integers (ij), $i \geq n$, $j \geq 0$. Now apply the previous proposition. □

6.7. t-structures

The PROP $\mathscr{BA}$ is concentrated in the zeroth degree, therefore the category $\mathscr{BA}^\frown$ has the standard t-structure.

6.7.1. We have $D^{\geq 0}\otimes D^{\geq 0}\subset D^{\geq 0}$

Proof. Follows from 6.2 □

Proposition 6.13. *The t-structure on $\mathscr{BA}^\frown$ defines a t-structure on S'*

Proof. It suffices to check that the operators $\tau_{\geq 0}$ and $\tau_{\leq 0}$ preserve S'. Let $H\in S'$, $G\in S$ and $f:H\to G$ a quasi-isomorphism in $\mathscr{BA}^\vee$ Then we have an induced quasi-isomorphism $\tau_{\geq 0}H\to\tau_{\geq 0}G$ in $\mathscr{BA}^\vee$, Where $\tau_{\geq 0}G$ is by definition a quotient of G such that $\tau_{\geq 0}G(X)=\tau_{\geq 0}(G(X))$ for any finite set X with the induced action of $\mathscr{BA}$. We see that $\tau_{\geq 0}G\in S$, therefore $\tau_{\geq 0}H\in S'$. The proof for $\tau_{\leq 0}$ is similar. □

6.7.2. Let S_0 be the category of conilpotent functors $\mathscr{BA}\to\mathbf{vect}$. The functor of taking H^0: $H^0:T_{S'}\to S_0$ is equivalence of symmetric monoidal categories.

6.8.

Proposition 6.14. *Let $F':\mathscr{BA}^\frown\to BA^\frown$ be a weak symmetric monoidal t-functor preserving S'. Let $F:S'\to S'$ be the restriction and a symmetric monoidal isomorphism $T_F\to\mathrm{Id}$ is given. Then F and Id are homotopy equivalent.*

The proof will occupy the rest of the section.

6.8.1. Categories C^+ and C^{++} For their definitions see 3.0.11.

6.8.2. Let $C\subset\mathscr{BA}^\frown$ be the full subcategory of finitely generated cofibrant objects concentrated in the degree 0. It is a symmetric monoidal subcategory. Therefore, so are C^+,C^{++}. We have a symmetric monoidal compatible equivalence $\iota:C^{++}\to\mathscr{BA}^\frown$

6.8.3. We may assume that there exist a compatible weak monoidal exact functor $G:C^{++}\to C^{++}$ and an exact functor $G':\mathscr{BA}^\vee\to\mathscr{BA}^\frown$ such that there exist homotopy equivalences $G\iota\to\iota F$ and $\iota G\to G'\iota$.

6.8.4. Let $C^n\subset C^{++}$ be the full subcategory of objects X such that $\Sigma\iota X\in S_n^\frown$

Lemma 6.15. *G preserves C^n for any n.*

6.8.5. Let $C^\infty=\cup_n C^n$. We have a functor $\kappa:C^{\infty++}\to C^{++}$ Let $\mathbf{SC}\subset C^{++}$ be the full subcategory formed by the image of κ. $\mathbf{SC}$ is a symmetric monoidal subcategory such that $\iota(\mathbf{SC})\cong S'$. G preserves $\mathbf{SC}$ and our problem reduces to showing that $G|_{\mathbf{SC}}$ is homotopy equivalent to Id.

6.8.6. Category P_∞ Let P_n be the full subcategory of $S_n^\frown$ consisting of objects which are quasi-isomorphic to cofibrant finitely generated objects in $S_n^\frown$ concentrated in degree 0. We have functors $T_m : P_n \to P_m$, $n > m$ which are restrictions of $T_m : \mathcal{BA}_n \to \mathcal{BA}_m$. Let P_∞ be the category whose objects are collections $R_n \in P_n$, $n = 1, 2, \ldots$ equipped with surjective quasi-isomorphisms $p_n : T_n R_{n+1} \to R_n$. Let $R, R' \in S_\infty$. We have a natural map of complexes

$$D : \prod_k \hom_{S_k}(R_k, R'_k) \to \prod_k \hom_{S_k}(T_k R_{k+1}, R'_k). \tag{6.6}$$

defined as follows. Let $f_k : R_k \to R'_k$, $k = 1, 2, \ldots$. Then

$$D(f_k|_{k\geq 1}) = (\phi_l|_{l\geq 1}),$$

where $\phi_l : T_l R_{l+1} \to R'_l$ is given by

$$\phi_l = p_l f_{l+1} - f_l p_{l+1}.$$

Lemma 6.16. *D is surjective.*

Proof. It suffices to prove that the composition map

$$\hom(R_{k+1}, R"_{k+1}) \to \hom(T_k R_{k+1}, T_k R'_{k+1}) \xrightarrow{p_k} \hom(T_k R_{k+1}, R'_k),$$

is surjective. The second map is surjective because $T_k R'_{k+1}$ is cofibrant and T_k is surjective. The first map is isomorphic to the map

$$\hom(R_{k+1}, R'_{k+1}) \to \hom(R_{k+1}, I_{k,k+1} T_k R_{k+1})$$

induced by the canonic map $R'_{k+1} \to I_{k,k+1} T_k R_{k+1}$ which is surjective. Since R_{k+1} is cofibrant, the first map is also surjective. $\square$

Set

$$\hom_{P_\infty}(R, R') = \operatorname{Ker} D.$$

Define the symmetric monoidal structure on P_∞ by setting $(X \otimes Y)_i = X_i \otimes Y_i$, where the tensor product in $S_i^\frown$ is defined in 6.4.7.

6.8.7. Categories $P_{\leq 0}$ and P_0 Set

$$\hom_{P_{\leq 0}}(X, Y) = \tau_{\leq 0} \hom_{P_\infty}(X, Y);$$
$$\hom_{P_0}(X, Y) = H^0 \hom_{P_\infty}(X, Y);$$

The categories $P_{\leq 0}$ and P_0 have an induced symmetric monoidal structure and we have symmetric monoidal functors $P_0 \leftarrow P_{\leq 0} \to P_\infty$ which are symmetric monoidal equivalences.

6.8.8. The functor G' extends naturally to P_∞, $P_{\leq 0}$ and P_0. Namely, we set $T_{G'}(X)_i = T_i G'(X_i)$. $T_{G'}$ has a clear symmetric monoidal structure. denote by $T^\infty_{G'}$ the action of $T_{G'}$ on P_∞, by $T^{\leq 0}_{G'}$ the action of $T_{G'}$ on $P_{\leq 0}$, and by $T^0_{G'}$ the action of $T_{G'}$ on P_0.

6.8.9. it is immediate that $T^0_G \cong \mathrm{Id}$. Therefore T^∞_G and $T^{\leq 0}_G$ are homotopy equivalent to identity.

6.8.10. The collection of functors $T'_n : C \to \mathscr{B}\mathscr{A}^\frown \xrightarrow{T_n} S^\frown_n$ defines a functor $T' : C \to P_\infty$ by $T'(X)_n = T'_n X$.

6.8.11. Let U_∞ (resp. $U_{\leq 0}, U_0$) be P^{++}_∞, (resp. $P^{++}_{\leq 0}, P^{++}_0$). We have a functor $T : C^{++} \to U$ which extends T'. The functor T'_G extends to each of $U_\infty, U_{\leq 0}, U_0$ and is homotopy equivalent to identity.

6.8.12.

Lemma 6.17. *Let* $X, Y \in$ **SC** . *Then* $T : \hom(X,Y) \to \hom(TX,TY)$ *is a quasi-isomorphism.*

Proof. Actually, the statement is true for any X in C^{++}. Let $X_\bullet$ be a bounded complex such that each $X^k = \oplus_i X_{ik}[n_i]$ and $X_{ik} \in C_{r_{ik}}$. Then $\hom(X,Y) = \hom(X^\bullet, Y) = (\prod(X_{ik}[n_i], Y), d + D_X)$ and

$$\hom(TX,TY) = (\prod \hom(TX_{ik}[n_i], TY), d + D'_X),$$

where D_X, D'_X are determined by the differential on $X_\bullet$. It follows from the boundedness of $X_\bullet$ that it suffices to check that $\hom(X_{ik}[n_i], Y) \to \hom(TX_{ik}[n_i], TY)$ is a quasi-isomorphism for all X_{ik}. In other words, it suffices to check the statement for $X \in C$, which we assume from now on. Similarly, Let $Y = Y_\bullet$, where $Y_k = \oplus Y_{ik}[n_{ik}]$, where $Y_i k \in C^{m_{ik}}$ Then we have $\hom(X,Y) = (\oplus_{ik} \hom(X, Y_{ik}[n_k]), d + D_Y)$ and same for $\hom(TX,TY)$. We see again that it suffices to check that the statement for $X \in C$ and $Y \in C^k$. It follows from the fact that $\hom(TX,TY)$ is quasi-isomorphic to the complex (6.6) and that $\hom(X,Y) \to \hom(T^n X, T^n Y)$ is a quasi-isomorphism for all $n \geq k$ (see 6.9). □

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6.8.13. We have a natural transformation $\mu : TG \to T_{G'}T$ which is defined as follows. $TG(X)_k = T_k G(X)$ and $T_{G'}T(X)_k = T_k G' T_k X$ and we have maps $T_k G(X) \to T_k G'(X) \to T_k G'(T_k X)$.

6.8.14.

Proposition 6.18. *For any* $X \in$ **SC** , μ_X *is a quasi-isomorphism.*

Proof. If $X \in C_k$, then the statement follows from the quasi-isomorphism $X \to T_k X$. The statement now follows easily. □

6.8.15. Let $A \subset U$ be the full subcategory of objects quasi-isomorphic to objects from the image of T. Then $T_{G'}$ preserves A and $T_{G'}|_A$ is homotopy equivalent to G. Since $T_{G'}$ is homotopy equivalent to Id , so is $T_{G'}|_A$, whence the statement.

6.8.16. Let $F_t' : \mathscr{BA}^\frown \to \mathscr{BA}^\frown[[t]]$ be a weak symmetric monoidal functor mapping $S' \to S'[[t]]$ and let $F : S' \to S'[[t]]$ be the restriction. Assume that $F_t = F_0(\mathrm{mod}\, t)$ and that $F_0 \cong \mathrm{Id}$. Let $I : S' \to S'[[t]]$ and $i : T_{S'} \to T_{S'}[[t]]$ be natural inclusions. The following proposition is proved in the same way as the previous one.

Proposition 6.19. *Assume that there exists an isomorphism of functors $\mu : T_{F_t} \to iT_{F_0}$ such that $\mu = \mathrm{Id}\ (mod\, t)$. Then F_t is deformationally homotopically equivalent to iF_0.*

6.9. Universal conilpotent $\mathscr{BA}$-algebras

6.9.1. Let P be a PROP and let $A : \mathscr{BA} \to P\vee$ be a functor. Any such an A determines a functor $A' : \mathscr{BA} \otimes P^{op} \to$ **complexes**. Call A *conilpotent* if for any n, k and any $x \in A'([n] \otimes [k])$ there exists an m such that for any $m' > m$ and $t \in j(\mathbf{coass}([n],[m'])$ we have $(t \otimes \mathrm{Id}_{[k]})x = 0$.

6.9.2.

Proposition 6.20. *If A is conilpotent, then $A\# : P\vee \to \mathscr{BA}\vee$ takes only values in S.*

Corollary 6.21. $\Sigma(A\#) : P^\frown \to S'$ *is an exact weak symmetric monoidal functor.*

6.9.3. Call a functor $F : P^\frown \to S'$ essentially strong if such is the composition $P^\frown \to S' \to \mathscr{BA}^\frown$ (see 5.2.3). We see that $\Sigma(A\#)$ is essentially strong whenever A is essentially strong. This follows from 5.2.5.

6.10. Operad B_∞

Let C be a symmetric monoidal additive category and let C' be the symmetric monoidal category of complexes in C. Let $n \in C'$ and assume $m = \oplus_{i=1}^{\infty} T^i n = Tn$ exists in C. m has a structure of cofree coassociative algebra. Call a $\mathscr{B}_\infty$-structure on n a dg-$\mathscr{BA}$-structure on m such that the induced coassociative algebra structure is the cofree one and the restriction of the differential on $n \subset Tn$ is equal to the existing differential on n.

Such a structure is uniquely determined by the corestriction of the differential $d_r : Tn \to Tn \to n$ and of the product $m_r : Tn \otimes Tn \to Tn \to n$. Certain identities should be satisfied. One sees that these identities are described by an operad denoted by B'_∞. The B'_∞-structure on the shifted space $n[1]$ is called $\mathscr{B}_\infty$-structure.

6.10.1. Let $P\mathscr{B}_\infty$ be the PROP generated by $\mathscr{B}_\infty$. Since any $\mathscr{B}_\infty$-algebra structure on n determines a BA-structure on $T(n[1])$, we have a map $H' : \mathscr{B}\mathscr{A} \to P\mathscr{B}_\infty{}^\frown$.

6.10.2. Since the constructed bialgebra Tn is manifestly conilpotent, we have $H'\#$: $P\mathscr{B}_\infty\vee \to S \subset \mathscr{B}\mathscr{A}\vee$ (see 3.7).

6.10.3. Brace structures A $\mathscr{B}_\infty$-structure on n is called *brace structure* if

$$d_r|_{T^{\geq 2}(n[1])\otimes T(n[1])} = 0$$

and $m_r(T^{\geq 3}(n[1])) = 0$. Thus, a brace structure on n is described by a degree zero map $m : n \otimes n \to n$ and by a collection of "braces" $b_m : n \otimes T^m n \to n$, $m \geq 1$ of degree $-m$. The operation b_k has the following notation:

$$b_k(n_0; n_1, \dots, n_k) = n_0\{n_1, \dots, n_k\}.$$

Brace structures are described by the operad M. We have a map $p : \mathscr{B}_\infty \to M$ and the composition $H = p^\frown \circ H' : \mathscr{B}\mathscr{A} \to M^\frown$. We have the corresponding functor $H^\# \; P\mathscr{M}\vee \to S \subset \mathscr{B}\mathscr{A}\vee$

6.10.4. It is known that there exists a (non-canonic) quasi-isomorphism $u : \mathbf{hoe}_2 \to M$.

6.10.5. Let n be a $\mathscr{B}\mathscr{A}$-algebra. Define a brace structure on $B(n) = T(n[1])$ as follows. Let $m_2 : Tn[1] \otimes Tn[1]$ be the associative product and let

$$n\{n_1, n_2, \dots, n_k\} = \Delta_k(n) \times (n_1 \otimes \cdots \otimes n_k),$$

where Δ denotes the k-fold coproduct in the bialgebra $\mathbf{1} \oplus n$ and $\times : T^k n \otimes T^k n \to T^k n$ is the component-wise product. We thus have a map $B : M \to (\mathscr{B}\mathscr{A}^{op})^\frown$. We have a functor $B^\# : \mathscr{B}\mathscr{A}\vee \to P_M\vee$.

Denote $\Gamma = BH : \mathscr{B}\mathscr{A} \to \mathscr{B}\mathscr{A}^\frown$. We have $\Gamma^\# = H^\# B^\# : \mathscr{B}\mathscr{A}\vee \to S$.

6.10.6. Let δ_k be the k-fold coproduct on m. Observe that $\delta_2(x) = \Delta_2 x - 1 \otimes x - x \otimes 1$. Let X be a conilpotent $\mathscr{B}\mathscr{A}$-coalgebra. Define the map $Z : X \to \Gamma^\#(X)$ as follows. The inclusion $i : X[1] \to T(X[1])$ defines the map $Ti : T(X[1][-1]) \to T(T(X[1])[-1])$. We have also the map $\delta = \sum_k \delta_k : X \to TX$. Set $Z = (Ti)\delta$.

6.10.7. Z is a quasi-isomorphism.

6.10.8. Thus we have a symmetric monoidal transformation $Z : \mathrm{Id}_S \to \Gamma^\# I$.

6.10.9. Set

$$\Delta : S' \to \mathscr{B}\mathscr{A}^\frown \xrightarrow{\Gamma} S \xrightarrow{\Sigma} S'.$$

6.10.10.

Proposition 6.22. *The functor Δ is a t-functor*

Proof. Indeed, for any $X \in S'$ there exists a quasi-isomorphic $X' \in S$. We have $\Delta(X) \approx \Gamma(X') \overset{Z}{\approx} X' \approx X$, where $\approx$ means "quasi-isomorphic". Therefore, Δ preserves any full subcategory closed with respect to quasi-isomorphisms, in particular $\mathscr{D}^{\geq 0}$ and $\mathscr{D}^{\leq 0}$. $\square$

6.10.11. The transformation Z establishes equivalence $T_\Delta \cong \mathrm{Id}$. Therefore, by 6.8 Δ is equivalent to Id .

6.11.

Prove that the inverse composition $L = \Sigma B^\# H^\#$ is an equivalence of symmetric monoidal categories. We have that L is homotopy equivalent to $BH^\#$. According to 5.2.6 it suffices to check that $BH([1])$ is quasi-isomorphic to $[1]$ (which can be done straightforwardly) and that the induced map $P_{[1]} \to P_{BH([1]}$ is a quasi-isomorphism. Note that since $[1] \approx BH([1])$, $P_{BH[1]} \approx P_[1]$. Since $P_{[1]} \cong PM$ and the cohomology of $PM \approx Pe_2$ are generated by $H^\bullet \hom_{PM}([2],[1]) \cong M([2]) \approx e_2([2])$, it suffices to check that the induced map

$$H^\bullet(BH) : H^\bullet \hom_{PM}([2],[1]) \to H^\bullet \hom(BH([1]) \otimes BH([1]), BH([1]))$$

is an isomorphism, which also can be done straightforwardly.

6.11.1. Corollary Functors $H' = \Sigma H\#$ and $B' : \Sigma B\#$ are homotopy equivalences of symmetric monoidal categories.

6.11.2. Let $M' = M\{-1\}$ be the shift of M. Let $\mathscr{D}^{\geq 0}_{M'} \subset (PM')^\frown$ be the full subcategory of objects U such that $H^{<0}U(X) = 0$ for any X. We have the shift maps $\mathbf{sh} : PM' \to PM$ and $\mathbf{sh}' : PM \to PM'$. Let $H'' = H'\mathbf{sh}$ and $B'' = \mathbf{sh}'B'$.

Lemma 6.23. *$H''(\mathscr{D}^{\geq 0}_{M}) \subset H''(\mathscr{D}^{\geq 0})$ $B''(\mathscr{D}^{\geq 0}) \subset H''(\mathscr{D}^{\geq 0}_{M})$. Therefore $\mathscr{D}^{\geq 0}_{M'}$ is the category of objects quasi-isomorphic to objects from $B'(\mathscr{D}^{\geq 0})$*

Proof. Straightforward. $\square$

Set $\mathscr{D}^{\leq 0}_{M'}$ to be the full subcategory of objects quasi-isomorphic to $B'(\mathscr{D}^{\leq 0})$. Then the pair $(\mathscr{D}^{\geq 0}_{M'}, \mathscr{D}^{\leq 0}_{M'})$ is a t-structure.

6.11.3. Fix a quasi-isomorphism $u : \mathbf{hoe}_2 \to \mathscr{M}$. Let $e_2' = e_2\{-1\}$ We have a chain of equivalences

$$Q_u : (Pe_2')^\frown \to (Phe_2')^\frown \to PM' \to S'. \tag{6.7}$$

Also we have a t-structure on Pe_2' from 3.8.9.

Proposition 6.24. *All arrows in 6.7 are t-functors.*

Corollary 6.25. T''_H, T''_B, T_{Q_u} *are equivalences of symmetric monoidal categories.*

6.12.

Corollary 6.26. *The symmetric monoidal category* S_0 *is equivalent to the symmetric monoidal category* T_M.

6.13. Lie bialgebras and Etingof-Kazhdan quantization

6.13.1. Let $\mathscr{LBA}$ be the PROP of Lie bialgebras, **lie** be the PROP of Lie algebras and **colie** be the PROP of Lie coalgebras. We have maps $i : \mathbf{lie} \to \mathscr{LBA}$ and $j : \mathbf{colie} \to \mathscr{LBA}$.

Proposition 6.27. *The natural map* $\phi : \oplus_{n \geq |Y|} \mathbf{lie}([n], X) \otimes_{S_n} \mathbf{colie}([Y], [n])$ *is an isomorphism.*

6.13.2. A functor $U \in \mathscr{LBA}^\vee$ is called *conilpotent* if for any finite set X and any $x \in U(X)$ there exists an N such that for all $n > N$, $j(\mathbf{colie}(X, [n]))x = 0$. Let L be the full subcategory of conilpotent functors and $L' \subset \mathscr{LBA}^\frown$ be the full subcategory of objects quasi-isomorphic in $\mathscr{LBA}^\vee$ to conilpotent functors.

6.13.3. The category $\mathscr{LBA}^\frown$ has a canonic T-structure which defines a T-structure on L'.

6.13.4. Let L_0 be the category of conilpotent functors $\mathscr{LBA} \to \mathbf{vect}$. We have a symmetric monoidal equivalence $T_{L'} \cong L_0$ given by H^0.

6.13.5. Let A be a Lie bialgebra in a symmetric monoidal category C closed with respect to limits. Then $C^\bullet(A)$ is a Gerstenhaber algebra in the category of complexes in C (see 1.2.5). This construction defines a functor $B : Pe_2 \to \mathscr{LBA}^\frown$ and the corresponding strong generalized map of PROPs $B^\#$.

6.13.6. Let X be a Gerstenhaber algebra in a symmetric monoidal category C closed with respect to limits. Let $\mathbf{lie}(n)$ be the n-th space of the operad of Lie algebras and let $\mathbf{lie}^*(n)$ be its linear dual. For any $Y \in C$ Set $\mathbf{CL}(Y) = \oplus_{n=1}^\infty \mathbf{lie}^*(n) \otimes_{S_n} Y^{\otimes n}$. $\mathbf{CL}(Y)$ is naturally a Lie coalgebra in C. The commutative product on X defines a differential on $\mathbf{CL}(X[1])$ so that its corestriction $\mathbf{CL}(X[1]) \to \mathbf{CL}(X[1]) \to X[1]$

does not vanish only on the summand $\mathbf{lie}(2)^* \otimes_{S_2} (X[1])^{\otimes 2} \cong S^2X[2]$ and is defined by the commutative product $S^2X \to X$. The bracket on X Defines a bracket $\mathbf{CL}(X[1]) \otimes \mathbf{CL}(X[1]) \to \mathbf{CL}(X[1])$ such that the corestriction

$$\mathbf{CL}(X[1]) \otimes \mathbf{CL}(X[1]) \to \mathbf{CL}(X[1]) \to X[1]$$

does not vanish only on $X[1] \otimes X[1] \subset \mathbf{CL}(X[1]) \otimes \mathbf{CL}(X[1])$ and is defined by the bracket on X.

Thus, $\mathbf{CL}(X[1])$ is a Lie bialgebra. One sees that it is conilpotent. This construction defines a functor $B : \mathscr{LBA} \to Pe_2^{\frown}$ so that $G^\# : Pe_2\vee \to L$ and the composition $G^\# : Pe_2\vee \to L \to \mathscr{LBA}\vee$ is a strong generalized map of PROPs.

6.13.7. Let X be a conilpotent Lie bialgebra. We have a map $Z : X \to \mathbf{CL}(C^\bullet(X)[1])$ such that $Z : X \xrightarrow{Z'} \mathbf{CL}(X) \to \mathbf{CL}(C^\bullet(X)[1])$ and

$$Z' \in \hom(X, \mathbf{CL}(X)) = \hom(X, \oplus \sum \mathbf{lie}(n)^* \otimes X^{\otimes n})$$

is defined by the Lie coalgebra structure on X.

This construction defines a map Id $\to B^\# G^\#$.

6.13.8. Functors $\Sigma G^\# : L' \to Pe_2^{\frown}$ and $\Sigma B^\# : Pe_2^{\frown} \to L'$. are t-functors. Each of them is a homotopic symmetric monoidal equivalence.

The proof of there statements goes along the same line as the corresponding proof for $H' : PM^{\frown} \to S'$ and $B' : S' \to PM^{\frown}$ (6.10.9-6.11)

6.13.9. Let $G' = T_{\Sigma G^\#}$ and $B' = T_{\Sigma B^\#}$. They establish a symmetric monoidal equivalence between L_0 and $T_{Pe_2^{\frown}}$.

6.13.10. We thus have symmetric monoidal equivalences $\mathbf{Q} : L_0 \to T_{Pe_2^{\frown}} \to S_0$ and $\mathbf{DQ} : S_0 \to T_{Pe_2^{\frown}} \to L_0$ so that $\mathbf{Q}$ and $\mathbf{DQ}$ are mutually inverse strong generalized maps of PROPs (this follows from the fact that both $\mathbf{Q}$ and $\mathbf{DQ}$ are induced by t-functors which are compositions of strong generalized maps of PROPs.

6.13.11. Let C be an additive symmetric monoidal category and let $C' \subset C^{op}\vee$ be the full subcategory of functors $C^{op} \to \mathbf{vect}$. We have the Ioneda embedding $C \to C'$ Let $X : \mathscr{LBA} \to C'$ be a conilpotent algebra. We can rewrite $X : \mathscr{LBA} \otimes C \to \mathbf{vect}$ or $X : C^{op} \to \mathscr{LBA}$. Since X is conilpotent, $X : C^{op} \to L_0$. Set $Q(X) = \mathbf{Q} \circ X : C^{op} \to S_0$. $Q(X)$ is a conilpotent bialgebra in C'. (it is a strong $\mathscr{BA}$-algebra because Q is the restriction onto the core of an essentially strong generalized map of PROPs). Thus Q is a map from the category of conilpotent Lie bialgebras in C' to the category of conilpotent Hopf algebras in C'

6.13.12. Similarly, the composition with $\mathbf{DQ}$ defines the quasi-inverse functor DQ from the category of conilpotent Hopf algebras in C' to the category of conilpotent Lie bialgebras in C'. (Again $\mathrm{DQ}(X)$ is always a strong $\mathscr{LBA}$-algebra, because DQ is the restriction onto the core of an essentially strong generalized map of PROPs).

6.13.13. The two previous subsections imply that the pair of functors Q and DQ establishes an equivalence between the categories of conilpotent Lie bialgebras in C' and conilpotent Hopf algebras in C'.

6.14. Action of GT

6.14.1. Recall that we have fixed a quasi-isomorphism $u : \mathbf{hoe}_2 \to \mathscr{M}$. Therefore, we have a T-equivalence $Q_u : P\widehat{\mathbf{hoe}_2} \to S'$. In the same way, we have a quasi-isomorphism $v : \mathbf{ho}\mathscr{M} \to e_2$ which is homotopy inverse to u . We can construct a map $Q_v : S' \to PM \to P\mathbf{ho}\mathscr{M} \to P\mathbf{hoe}_2$ so that Q_uQ_v and Q_vQ_u are homotopy equivalent to identity.

6.14.2. Let $\mathbf{hoe}_2$ be a resolution of e_2 and let $m : \mathbf{hoe}_2 \to \mathbf{hoe}_2[[t]]$ be an action of a formal one-parametric group on $\mathbf{hoe}_2$ such that its derivative at $t = 0$ produces a non-trivial element in the cohomology of the deformation complex of e_2. Then we have the map $Q_{um} : P\widehat{\mathbf{hoe}_2} \to S'[[t]]$.

Theorem 6.28. *There is no equivalence between $T_{Q_{um}}$ and iT_{Q_u} which reduces to identity modulo t.*

Proof. Assume there is. Then $T_{Q_uQ_v}$ and $T_{Q_{um}Q_v}$ are also equivalent in such a way that the reduction of the equivalence modulo t is identity. Hence, in virtue of Proposition 6.19, Q_uQ_v and $Q_{um}Q_v$ are deformationally homotopy equivalent and so are Q_u and Q_{um}. Let $m_* : \widehat{\mathbf{hoe}_2} \to \widehat{\mathbf{hoe}_2}[[t]]$ be the induced map. Then $Q_{um} = Q_um_*$, therefore m_* is deformationally homotopy equivalent to i. Let $e \in \widehat{\mathbf{hoe}_2}$ be such that $e(X) = \hom([1],x)$. Note that $m_*e = e$. The full operad $\mathscr{O}_e$ is equivalent to the operad of $\mathbf{hoe}_2$ algebras and the induced map $\mathscr{O}_e \to \mathscr{O}_e[[t]]$ is just m. Since m_* is deformationally homotopy equivalent to i and i is deformationally trivial, so does m according to 4. This contradicts to the assumption. $\square$

6.14.3. In [9] it was proven that the Lie algebra $\mathfrak{g}rt$ of the graded Grotendieck-Teichmüller group acts on $\mathbf{hoe}_2$ by derivations so that the induced map $\mathfrak{g}rt \to H^0$ **def** $\mathbf{hoe}_2$ is injective. Therefore, the conclusion of Theorem 6.28 is true when m is induced by a one parametric subgroup of the graded Grotendieck-Teichmüller group.

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