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Generalized Topological Index

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Abstract. This paper deals with an index which is different in general from the topological index defined by Atiyah and Singer because we loosen the normalization property. The intrinsic relation of this new index with operations in K-theory is explained. It is also shown that if we change the normalization axiom, the corresponding index is well-defined and may be expressed in terms of the topological index.

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1. Introduction

The purpose of this paper is to define an index which is a generalization of the topological equivariant index introduced by Atiyah and Singer. The new index is allowed to have a different normalization condition. The main results of the paper are presented in Theorems 4.2 and 4.3.

It turns out that our modification of the normalization axiom cannot be completely arbitrary. There are strong relations imposed on the set of parameters involved. In the third section of the paper we find all those relations. We prove that under those restrictions the generalized topological index exists and, moreover, it is unique. The final formula for the new index is written in terms of an operation in K-theory.

The normalization proposed by Atiyah and Singer is the most natural one for the index theorem purposes and our results imply the following statements. First, the normalization axiom for the index is not redundant and it does not follow from the other axioms. Second, all possible changes of the normalization property do not have substantially new features because of the close relations to the classical theory which we exhibit.

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2. Equivariant K-Theory and Topological Index

In this section, we shall briefly recall some facts about K-theory in the case when a compact Lie group G acts on a manifold as well as on complex vector bundles E over it. This subject was developed in [3] and [5]. We assume that diffeomorphic actions of G commute with the projection $\pi: E \to X$. If we allow subtraction, the set of equivalence classes of G-bundles under the natural equivalence relation generates the Grothendieck group $K_G(X)$. As it was shown in [5], this K-theory has such features as Bott periodicity and Thom isomorphism. Unless stated otherwise, we always consider K-theory with compact support which is formed by G-triples (E, F, σ), where E and F are G-bundles on our manifold and σ is a G-morphism between them that is an isomorphism outside a compact set. Also when we are talking about a G-bundle E over a G-manifold X we always assume that the group G acts on E and its action covers the action of G on X. The expression *G-bundle* does *not* refer to a bundle with the structure group G.

We denote by $R(G)$ the ring of characters for a compact Lie group G. If our manifold is just a point O, then one has $K_G(\mathrm{O}) = R(G)$. The group $K_G(X)$ is canonically an $R(G)$-module.

For each G-manifold X, its tangent bundle has the obvious G-action, and its complexification $T^cX = TX \otimes_{\mathbb{R}} \mathbb{C}$ is also a G-bundle over X.

Now we describe (cf. [1]) several important morphisms in the equivariant K-theory. We will consider a G-bundle E over X whose structure group is H such that E corresponds to a principal P with the structure group H. Thus P is acted upon by $G \times H$ and if F is a fiber of E then of course $E = F \times_H P$. Now we suppose that the action of H on F can be extended to be an action of $G \times H$. Then one can form a multiplication

$$K_G(TX) \otimes K_{G\times H}(TF) \to K_G(TE) \tag{2.1}$$

as follows. Given an element $a \in K_{G\times H}(TF)$ we obtain an element $p^*a \in K_{G\times H}(P \times TF)$ (p is a projection on the second factor). This element can be treated as an element of the K_G-group of $P \times_H TF$ which is the vertical tangent bundle corresponding to the bundle E over X: $E \xrightarrow{\pi} X$. Finally, one takes its cup-product with an element of $K_G(\pi^*TX)$ and one ends up with an element of $K_G((P \times_H TF) \oplus \pi^*TX) \simeq K_G(TE)$.

It is easy to see that $K_G(TX)$ is a module over the ring $K_G(X)$ in a usual fashion. This allows us to define a multiplication

$$K_G(X) \otimes K_G(TX) \to K_G(X \times TX) \xrightarrow{j^*} K_G(TX) \tag{2.2}$$

as the composition of the cup product with the pullback of the indicative map $j: TX \to X \times TX$, $j(t) = (\pi t, t)$. In terms of triples [KB] it looks like

$$(\pi^*E_1 \otimes E_2, \pi^*F_1 \otimes F_2, \pi^*\sigma_1 \otimes \sigma_2),$$

where $(E_1, F_1, \sigma_1) \in K_G(X)$ and $(E_2, F_2, \sigma_2) \in K_G(TX)$.

Besides we shall need a map

$$\mu_P: R(G \times H) \to K_G(X), \tag{2.3}$$

which is simply the change of fiber in our principal H-bundle P (on which G acts as well) by the virtual H-module which represents an element of $R(G \times H)$.

The index theorem proved by Atiyah and Singer [1] asserts that the analytical and topological indices coincide as $R(G)$-module homomorphisms $K_G(TX) \to R(G)$. Essentially their proof was based on a set of three axioms which uniquely characterize the topological index. They showed that the analytical index also satisfies those axioms. We will see how the index changes if we modify one of the axioms. For our purposes we shall reproduce the definition of an index function:

DEFINITION 2.1. An index function is an $R(G)$-module homomorphism $\mathrm{ind}_G^X\colon K_G(TX) \to R(G)$ functorial with respect to G-diffeomorphisms of X and if $\phi\colon G_1 \to G_2$ is a group homomorphism, the diagram

$$\begin{array}{ccc} K_{G_2}(TX) & \xrightarrow{\phi^*} & K_{G_1}(TX) \\ \mathrm{ind}_{G_2}^X \big\downarrow & & \mathrm{ind}_{G_1}^X \big\downarrow \\ R(G_2) & \xrightarrow{\phi^*} & R(G_1) \end{array}$$

is commutative.

An index function is called simply *an index* (denoted $\mathcal{J}_G^X$) if it satisfies the following two axioms. If $i\colon U \hookrightarrow X$ is an open G-embedding, there is the naturally induced homomorphism of K_G-groups $K_G(TU) \xrightarrow{i_!} K_G(TX)$.

AXIOM A (*Excision axiom*). Under the above assumptions $\mathcal{J}_G^U = \mathcal{J}_G^X \circ i_!$.

AXIOM B (*Multiplicative axiom*). For $a \in K_G(TX)$ and $b \in K_{G\times H}TF$

$$\mathcal{J}_G^E(ab) = \mathcal{J}_G^X(a \cdot \mu_P(\mathcal{J}_{G\times H}^F(b))),$$

where μ_P comes from Equation (2.3), the product ab is taken as in Equation (2.1) and the structure of $K_G(TX)$ as a $K_G(X)$-module is determined by Equation (2.2).

For each closed G-embedding of manifolds $i\colon X \hookrightarrow Y$, the Gysin homomorphism $i_!\colon K_G(TX) \to K_G(TY)$ can be naturally defined. It is well-known that for each G-space X there exists a G-embedding of X into an underlying space of some real G-module Y. The topological index is now the composite map

$$\text{t-ind} \stackrel{\text{def}}{=} (j_!)^{-1} \circ i_!\colon K_G(TX) \to R(G),$$

where $j\colon \mathrm{O} \to Y$ is the inclusion of the origin and $K_G(T\mathrm{O})$ is identified with $R(G)$. It is important to notice that the topological index does not depend on a choice of a G-module Y. Finally, the topological index has the following normalization feature:

AXIOM N. Let the inclusion of the origin $j\colon \mathrm{O} \hookrightarrow \mathbb{R}^n$ induce the Gysin homomorphism $j_!\colon R(\mathrm{O}(n)) \to K_{\mathrm{O}(n)}(T\mathbb{R}^n)$. Then $\mathrm{ind}_{\mathrm{O}(n)}^{\mathbb{R}^n}(j_!(1)) = 1 \in R(\mathrm{O}(n))$.

It was shown by Atiyah and Singer in [1] that an index function satisfying axioms A, B, and N coincides with the topological index.

3. Modified Normalization

For our index $\mathcal{J}$ we modify the normalization axiom introduced above in the following fashion. Suppose that we have a family $\{b_n\}_{n\in\mathbb{N}}$, where $b_n \in R(\mathrm{O}(n))$.

AXIOM N*. With the notation and assumptions in Axiom N $\mathcal{J}_{\mathrm{O}(n)}^{\mathbb{R}^n}(j_!(1)) = b_n \in R(\mathrm{O}(n))$.

In the present section we will determine all possible $\{b_n\}$ in accordance to Axioms A, B, and N*.

An embedding $\mathrm{O}(n) \times \mathrm{O}(m) \hookrightarrow \mathrm{O}(n+m)$ gives us the following consequence of the multiplicative axiom.

LEMMA 3.1. *Set* $b_n \otimes b_m(x, y) =_{\mathrm{def}} b_n(x)b_m(y)$, *where* $x \in \mathrm{O}(n)$, $y \in \mathrm{O}(m)$, *and* $(x, y) \in \mathrm{O}(n+m)$. *Then* $b_{n+m}|_{\mathrm{O}(n)\times\mathrm{O}(m)} = b_n \otimes b_m$.

Proof. If $a_i \in K_{G_i}(TX_i)$, where $i = 1, 2$, then from the multiplicative axiom we have $\mathcal{J}_{G_1\times G_2}^{X_1\times X_2}(a_1a_2) = \mathcal{J}_{G_1}^{X_1}(a_1)\mathcal{J}_{G_2}^{X_2}(a_2)$.

Now take

$$X_1 = \mathbb{R}^n, \quad G_1 = \mathrm{O}(n), \quad X_2 = \mathbb{R}^m, \quad G_2 = \mathrm{O}(m),$$
$$a_1 = j_!^{(m)}(1) \text{ and } a_2 = j_!^{(n)}(1).$$

Here $j^{(m)}\colon \mathrm{O} \hookrightarrow \mathbb{R}^m$ and $j^{(n)}\colon \mathrm{O} \hookrightarrow \mathbb{R}^n$ are inclusions of the origins. □

If we consider a simple case of a bundle such that both the base and the fiber are just a point, then the multiplicative axiom yields $b_0 = 1$ or $b_0 = 0$. In the latter case, $\mathcal{J} = 0$ identically. So we only need to investigate the former case.

Let us find all the sets of b_n obeying Lemma 3.1.

LEMMA 3.2. *There exists a polynomial* $q(x)$ *such that for* $A \in \mathrm{O}(n)$

$$b_n(A) = \det q(A). \tag{3.1}$$

Furthemore, $q(x)q(1/x) \in \mathbb{Z}[x, 1/x]$ *and* $q(1), q(-1) \in \mathbb{Z}$ *and any polynomial satisfying these conditions gives rise* (*by Equation* (3.1)) *to a family* $b_n \in R(\mathrm{O}(n))$ *satisfying Lemma 3.1.*

Proof. Every matrix $M \in \mathrm{O}(n)$ is conjugate to a matrix A consisting of several blocks placed along the main diagonal. Every block is either

$$B_\psi = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \quad \text{or} \quad C_\pm = (\pm 1).$$

Due to Lemma 3.1 we only need to find a polynomial q such that $b_2(B_\psi) = \det q(B_\psi)$ and

$$b_1(\pm 1) = q(\pm 1). \tag{3.2}$$

Note that $b_2 \in R(\mathrm{O}(2))$, hence $b_2(B_\psi) = p(\mathrm{e}^{i\psi}) + p(\mathrm{e}^{-i\psi})$ for some polynomial p with integer coefficients. It is a fact that there exists a polynomial $q(x)$ such that

$$p(x) + p(1/x) = q(x)q(1/x). \tag{3.3}$$

We see that $b_2(B_\psi) = \det q(B_\psi)$. Now Lemma 3.1 implies that

$$b_2\left(\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = b_1(C_\pm)^2,$$

hence $q(1) = \pm b_1(C_+)$ and $q(-1) = \pm b_1(C_-)$. Since $b_1(C_\pm)$ are integer numbers, $q(\pm 1)$ are also integers. Note that the identity (3.3) still holds if we replace $q(x)$ by $\pm q(x)$ or $\pm x q(x)$. By means of these substitutions we can get rid of the signs and get Equation (3.2).

Now let us prove the inverse statement. First, we shall show that the corresponding b_n determined by $b_n(A) = \det q(A)$ for $A \in \mathrm{O}(n)$ are in $R(\mathrm{O}(n))$. Recall that on a maximal torus all $\chi \in R(\mathrm{O}(n))$ look as follows:

(1) For $n = 2k$, one has

$$\chi\begin{pmatrix} B_{\psi_1} & 0 & \cdots & 0 \\ 0 & B_{\psi_2} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & B_{\psi_k} \end{pmatrix}$$
$$= P(\mathrm{e}^{i\psi_1} + \mathrm{e}^{-i\psi_1}, \mathrm{e}^{i\psi_2} + \mathrm{e}^{-i\psi_2}, ..., \mathrm{e}^{i\psi_k} + \mathrm{e}^{-i\psi_k}),$$

where P is a symmetric polynomial with integer coefficients; and if I is the identity of $\mathrm{O}(n)$ then

$$\chi\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix} \equiv \chi(I) (\mathrm{mod}\ 2).$$

(2) For $n = 2k + 1$ one has

$$\chi\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & B_{\psi_1} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & B_{\psi_k} \end{pmatrix}$$
$$= P(\mathrm{e}^{i\psi_1} + \mathrm{e}^{-i\psi_1}, \mathrm{e}^{i\psi_2} + \mathrm{e}^{-i\psi_2}, ..., \mathrm{e}^{i\psi_k} + \mathrm{e}^{-i\psi_k}),$$

$$\chi\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 1 \end{pmatrix} \equiv \chi(I)(\mathrm{mod}\ 2).$$

Now it is easy to check that $\chi(A) = \det q(A)$ satisfies the conditions of the lemma. For example, if $n = 2k+1$ we have

$$b_n(Y) = q(1)\prod_{j=1}^{k} q(\mathrm{e}^{i\psi_j})q(\mathrm{e}^{-i\psi_j}), \quad \text{where } Y = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & B_{\psi_1} & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & B_{\psi_k} \end{pmatrix}.$$

It is a fact that for each polynomial $q(x)$ such that $q(x)q(1/x) \in \mathbb{Z}[x, 1/x]$ there exists another polynomial $a(x) \in \mathbb{Z}[x]$ such that $q(x)q(1/x) = a(x+1/x)$. So we get $b_n(Y) = q(1)\prod_{j=1}^{k} a(\mathrm{e}^{i\psi_j} + \mathrm{e}^{-i\psi_j})$, which is a symmetric polynomial in $\mathrm{e}^{i\psi_j} + \mathrm{e}^{-i\psi_j}$ with integer coefficients. Finally,

$$\chi\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= q(-1)q^{2k}(1) \equiv q^2(-1)q^{2k}(1) \equiv q^2(1)q^{2k}(1) \equiv q(I)(\mathrm{mod}\ 2).$$

Here we used the fact that since $q(x)q(1/x) \in \mathbb{Z}[x, 1/x]$, one has

$$q^2(-1) = q(-1)q(1/-1) \equiv q(1)q(1/1)(\mathrm{mod}\ 2).$$

The case $n = 2k$ can be treated similarly and the conditions of Lemma 3.1 can be checked directly. □

4. Operation and Index

Our next goal is to define an operation in K-theory [4] which is determined by the set $\{b_n\}$ and to prove the existence and uniqueness of the new index. First, for a space X and a real vector bundle V over it coming from a principal bundle P we can construct an element of complex K-theory by changing a fiber to B_n, where n is the dimension of V. Then this operation may be extended to the whole K-group of X. That is how we get the operation $\nu\colon K^{\mathbb{R}}(X) \to K(X)$. This operation can be constructed in equivariant case too when a group G acts on $V = P \times_{\mathrm{O}(n)} \mathbb{C}^n$ and the action of G on $\mathbb{C}^n$ is trivial.

LEMMA 4.1. *If* $a, b \in K_G^{\mathbb{R}}(X)$, *then* $\nu(a+b) = \nu(a)\nu(b)$.

Proof. The fact that for vector bundles over X the equality $\nu(V \oplus W) = \nu(V)\nu(W)$ holds is a straightforward consequence of Lemma 3.1. Therefore it is true for the whole group $K_G^{\mathbb{R}}(X)$. $\square$

Let us suppose that our space X is G-equivariantly embedded in a G-module $\mathbb{R}^k$: $i: X \hookrightarrow \mathbb{R}^k$ in such a way that the group G may be viewed as a subgroup of $\mathrm{O}(k)$. If the dimension of X is m, put $n = k - m$ for the dimension of the normal bundle NX to X which is isomorphic to a tubular neighbourhood of X in $\mathbb{R}^k$. If $k : NX \hookrightarrow \mathbb{R}^k$ is the corresponding embedding of this tubular neighbourhood, then one has for $z \in K_G(TX)$:

$$\mathcal{J}_G^{NX}(z\tilde{b}_n) = \mathcal{J}_G^{NX}(i_!z) = \mathcal{J}_G^{\mathbb{R}^k}(k_!i_!z) = b_k|_G\text{t-ind}_G^{\mathbb{R}^k}(k_!i_!z) = b_k|_G\text{t-ind}_G^X(z),$$

by excision axiom and definition of topological index which respects Gysin homomorphism. The multiplicative axiom thus produces the following equality:

$$\mathcal{J}_G^{NX}(z\tilde{b}_n) = \mathcal{J}_G^X(z\nu(NX)) = b_k|_G\text{t-ind}_G^X(z).$$

Now put $a\nu(TX)$ instead of z and notice that $\nu(TX)\nu(NX) = b_k$, because $TX \oplus NX = X \times \mathbb{R}^k$ with the component-wise G-action. So we arrive to the fact that for $a \in K_G(TX)$

$$b_k|_G\mathcal{J}_G^X(a) = b_k|_G\text{t-ind}_G^X(a\nu(TX)). \tag{4.1}$$

THEOREM 4.2. *The index $\mathcal{J}$ defined by $\mathcal{J}_G^X(a) = \text{t-ind}_G^X(a\nu(TX))$ satisfies axioms A, B and N*.*

Proof. The modified normalization axiom obviously holds for $\mathcal{J}$. The excision axiom is a consequence of the observation that for an open embedding $U \hookrightarrow Y$ the tangent bundle of U is the restriction of the tangent bundle of Y. Let us check the multiplicative axiom. If E is a fiber bundle over X with a fiber F: $F \hookrightarrow E \xrightarrow{\pi} X$, then the tangent space TE as a bundle over E allows the G-equivariant splitting: $TE = \pi^*TX \oplus V$, where V is the corresponding vertical bundle. Moreover, $\nu(TE) = \nu(\pi^*(TX))\nu(V)$. From the definition of the multiplication (2.1) we get for $a \in K_G(TX)$ and $b \in K_{G\times H}(TF)$ that $ab\nu(TE) = (a\nu(TX))(b\nu(TF))$. $\square$

Now the question is whether we can always omit $b_k|_G$ in the both sides of the equality (4.1), i.e. to prove the uniqueness of the index.

THEOREM 4.3. *An index which satisfies axioms A, B and N* is unique.*

Proof. First, consider the case when $b_1(1) \neq 0$ which in accordance to Lemma 3.1 implies that $b_n(1) \neq 0$. Note that from Equation (4.1) it immediately follows that our index $\mathcal{J}$ is uniquely defined at the identity of the group. For $g \in G$, let X^g be a fixed point set and M_g be the localization of an $R(G)$-module M with respect to an ideal generated by all characters vanishing at the conjugacy class of g. Proposition 2.8 from [2] says that

$$i_!: K_G(TX^g)_g \to K_G(TX)_g, \tag{4.2}$$

is an isomorphism, where $i\colon X^g \to X$ is the inclusion. Let $j\colon NX^g \to X$ be an open embedding of a tubular neighbourhood and $k\colon X^g \to NX^g$ be the embedding onto zero section. Excision axiom and isomorphism (4.2) yield that to define $\mathcal{J}$ on X at the class of g, it suffices to do it for elements $k_! a$, $a \in K_G(TX^g)_g$. Multiplicative axiom reduces that definition just to the computation of the index on $K_G(TX^g)_g \cong K(TX^g) \otimes R(G)_g$. To do it let us notice that $\mathcal{J}$ is an $R(G)$-module homomorphism and that $\mathcal{J}$ is uniquely defined at the identity of G.

To treat the case $b_1(1) = 0$ we need some preparation first. In this case the polynomial $q(x)$ from Lemma 3.1 is such that $q(1) = 0$, hence $q(x) = (x-1)r(x)$, and $r(x)r(1/x) \in \mathbb{Z}[x, 1/x]$. Thus, according to Lemma 3.2, the formula in (3.3) where q is substituted by r defines a set of $\mathrm{O}(n)$-modules C_n and an operation $\mu\colon K^{\mathbb{R}}(X) \to K(X)$ similarly to the operation ν: for a bundle $E = P \times_{\mathrm{O}(n)} V$ over X we set $\mu(E) = P \times_{\mathrm{O}(n)} C_n$.

LEMMA 4.4. *Let $g \in G$ and $g \neq e$, the identity element. Then*

$$\mathcal{J}_G^X(a)|_g = \mathcal{J}_G^{X^g}(i^* a\mu(NX^g))|_g,$$

where NX^g is the normal bundle for the embedding $i\colon X^g \to X$.

Proof. Proposition 2.8 from [2] says that the isomorphism in (4.2) looks like

$$i_! i^* \frac{a}{\sum(-1)^i \Lambda^i (NX^g)^{\mathbb{C}}} = a, \quad a \in K_G(TX).$$

Moreover, from the multiplicative axiom it follows that

$$\mathcal{J}_G^X(i_! b) = \mathcal{J}_G^{X^g}(b\nu(NX^g))$$

for any $b \in K_G(TX^g)$. Therefore,

$$\begin{aligned}\mathcal{J}_G^X(a) &= \mathcal{J}_G^X(i_! i^* \frac{a}{\sum(-1)^i \Lambda^i (NX^g)^{\mathbb{R}}})|_g \\ &= \mathcal{J}_G^X(i^* \frac{a}{\sum(-1)^i \Lambda^i (NX^g)^{\mathbb{C}}} \nu(NX^g))|_g = \mathcal{J}_G^{X^g}(i^* a\mu(NX^g))|_g. \quad \square\end{aligned}$$

Let us pass to the proof of the theorem. The proof for the case $b_1(1) \neq 0$ implies that to prove the uniqueness of the index it suffices to do it only for the nonequivariant index. Thus now, in the case $b_1(1) = 0$, we are done if we prove that $\mathcal{J}_e^X \equiv 0$. To show this vanishing, consider the product

$$\underbrace{X \times \cdots \times X}_{p}$$

for some prime number p. The cyclic group $\mathbb{Z}_p$ acts on this product by cyclic permutations. Obviously for any $g \in \mathbb{Z}_p$, $g \neq e$,

$$(\underbrace{X \times \cdots \times X}_{p})^g = \Delta,$$

where Δ is the diagonal. Let $a \in K(TX)$. Consider the $\mathbb{Z}_p$-equivariant element $a \times \cdots \times a \in K(T(X \times \cdots \times X))$ and compute its index by Lemma 4.4. One has

$$\mathcal{J}_{\mathbb{Z}_p}^{X \times \cdots \times X}(a \times \cdots \times a)|_g = \mathcal{J}_{\mathbb{Z}_p}^{\Delta}(\xi \mu(N\Delta))|_g,$$

where $\xi = a \times \cdots \times a|_{T\Delta}$ with the obvious action of $\mathbb{Z}_p$. Hence, we have the character defined on the whole group:

$$\chi = \mathcal{J}_{\mathbb{Z}_p}^{X \times \cdots \times X}(a \times \cdots \times a) - \mathcal{J}_{\mathbb{Z}_p}^{\Delta}(\xi \mu(N\Delta)) \in R(\mathbb{Z}_p)$$

and $\chi(g)$ vanishes for all $g \neq e$. It is well-known that in this case p divides $\chi(e)$. Let us compute $\chi(e)$. Note that ξ as a nonequivariant bundle is isomorphic to $a^p \in K(TX)$ and $K(TX)$ is a nilpotent ring. Hence, ξ vanishes for p big enough. The multiplicative axiom yields $\mathcal{J}_e^{X \times \cdots \times X}(a \times \cdots \times a) = (\mathcal{J}_e^X(a))^p$. Thus, for p big enough we have $\chi(e) = (\mathcal{J}_e^X(a))^p \equiv 0(\mathrm{mod}\ p)$ which is only possible when $\mathcal{J}_e^X(a) = 0$, whence the vanishing of the nonequivariant index follows. $\square$

Remark. The proof of the Theorem 4.3 implies that in nonequivariant situation $\mathcal{J} \equiv 0$ is provided by $b_1(1) = 0$. On the account of uniqueness it follows that $\text{t-ind}^X(a\nu(TX)) \equiv 0$, though it is just due to $\mathrm{ch}(\nu(TX)) = 0$.

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