Proceedings of Symposia in Pure Mathematics
Volume **73**, 2005

The ring of differential operators on forms in noncommutative calculus

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2000 *Mathematics Subject Classification.* 16E40.
Key words and phrases. Hochschild cohomology, Gerstenhaber algebras, operads.

1. Introduction

Many standard geometric objects associated to a manifold M can be defined in terms of the algebra A of functions on M. Such definitions can be often made in a manner that makes sense for any associative algebra A, commutative or not. The study and applications of these generalized geometric constructions is the subject of noncommutative geometry [**C**], [**M**].

For example, a vector field on a smooth manifold M can be viewed as a derivation of the algebra $A = C^\infty(M)$. If we require such derivations to be local, i.e. to preserve supports, then the Lie algebra of all such derivations is precisely the Lie algebra $\mathrm{Vect}(M)$ of vector fields. One can say that the noncommutative version of $\mathrm{Vect}(M)$ is $\mathrm{Der}(A)$, the Lie algebra of derivations of A. Depending on the nature of A, one can impose on derivations some conditions like locality, continuity, etc.

Now let us try to define in a similar way the algebra $\mathcal{V}^\bullet(M)$ of multivector fields on M. The space of multivector fields has a structure of a Gerstenhaber algebra. In other words, it is a graded commutative associative algebra, i.e. the multiplication satisfies

$$ba = (-1)^{|a||b|}ab;$$

$\mathcal{V}^\bullet[1]$ is a graded Lie algebra, i.e.

$$[b, a] = -(-1)^{(|a|-1)(|b|-1)}[a, b]$$

and

$$[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]];$$

and the two operations satisfy the Leibnitz identity

$$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|}b[a, c]$$

(cf. [**G**]).

Throughout the paper, for a complex $\mathcal{V}^\bullet$ with differential d, $\mathcal{V}[1]^k = \mathcal{V}^{k+1}$ is the complex with the differential $-d$; the ground ring k will be of characteristic zero. For any associative algebra A, one can construct a Gerstenhaber algebra [**G**]; the underlying space of this algebra is the Hochschild cohomology of A. We denote it by $H^\bullet(A, A)$ or simply by $H^\bullet(A)$. It was essentially proven in [**HKR**] that, when $A = C^\infty(M)$, this Gerstenhaber algebra becomes $\mathcal{V}^\bullet(M)$ if one understands the Hochschild cohomology properly. More precisely, $H^\bullet(A, A)$ for $A = C^\infty(M)$ is the cohomology of the complex of Hochschild cochains given by multi-differential expressions.

The problem with the above construction is that the corresponding algebra shrinks considerably as soon as A becomes noncommutative. Indeed, $H^0(A, A)$ is the center of A, and $H^1(A, A) = \mathrm{Der}^{\mathrm{out}}(A)$. So, for example, $H^0 = \mathbb{C}$ and $H^1 = 0$ for such an important algebra as $A = D(\mathbb{R}^n)$, the ring of differential operators on $\mathbb{R}^n$. (In fact for this algebra $H^i = 0$ for all $i > 0$).

The problem of constructing a noncommutative analog of the algebra of multivector fields cannot be very easy because the algebra A of "zero-fields" is noncommutative. Consider, for example, the standard Hochschild cochain complex $C^\bullet(A, A)$ (see §2.3). We denote it also by $C^\bullet(A)$. It is well known that $C^\bullet(A)[1]$ carries a bracket (called the Gerstenhaber bracket) which makes it a dg (differential graded) Lie algebra; $C^\bullet(A)$ carries the cup product which makes it a dg associative algebra; at the level of cohomology these two operations induce the standard

Gerstenhaber algebra structure on $H^\bullet(A)$. At the cochain level, however, the associative algebra $C^\bullet(A)$ is not commutative (it contains $A = C^0$ as a subalgebra).

A solution to this problem was proposed in [**T**]. It was shown there that the Gerstenhaber bracket and the cup product on $C^\bullet(A)$ are part of a much richer algebraic structure, namely that of a G_∞ algebra whose underlying L_∞ structure is given by the Gerstenhaber bracket (cf. Theorem 2.1.2 of the present paper). There are two equivalent ways to say that a complex $\mathcal{C}^\bullet$ is a G_∞ algebra. One can define a G_∞ structure on $\mathcal{C}^\bullet$ explicitly in terms of some multilinear operations on $\mathcal{C}^\bullet$, subject to some quadratic relations. Or, equivalently, one can say that there is a differential graded Gerstenhaber algebra $\mathcal{V}^\bullet$, quasi-isomorphic to $\mathcal{C}^\bullet$ as a complex. Applying this to $C^\bullet(A)$, one gets a a dg Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ together with a quasi-isomorphism $\mathcal{V}^\bullet(A)[1] \to C^\bullet(A)[1]$ of dg Lie algebras. The above quasi-isomorphism identifies the cohomology of the complex $\mathcal{V}^\bullet(A)$ with $H^\bullet(A)$, and this identification is a Gerstenhaber algebra isomorphism. If $A = C^\infty(M)$, then the dg Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ is quasi-isomorphic to the dg Gerstenhaber algebra $\mathcal{V}^\bullet(M)$, the algebra of multivector fields on M with zero differential. (In particular one gets a chain of quasi-isomorphisms of dg Lie algebras

$$C^\bullet(A)[1] \leftarrow \mathcal{V}^\bullet(A)[1] \to \mathcal{V}^\bullet(M)[1]$$

which implies the formality theorem of Kontsevich [**K**]).

The Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ is given by a standard tensor construction independent of anything but the vector space A. The difficult part is to construct the differential on $\mathcal{V}^\bullet(A)$. It is given by a universal formula involving the product on A and some universal coefficients. For these coefficients there seems to be no canonical choice; one can define them if one chooses a Drinfeld associator [**D**].

An alternative way to formulate the theorem from [**T**] is to say that $C^\bullet(A)$ is a G_∞ algebra. The notion of a G_∞ algebra was introduced in [**GJ**]. By definition, a complex $C^\bullet$ is a G_∞ algebra if it carries multi-linear operations

$$m_{k_1,\ldots,k_n}: \ (C^\bullet)^{\otimes(k_1+\ldots+k_n)} \to C^\bullet$$

for every $n > 0$, $k_1, \ldots, k_n > 0$; these operations are assumed to satisfy certain symmetry conditions under permutations and certain quadratic equations (which amount to the Maurer-Cartan equation in a certain dg Lie algebra).

For the Hochschild complex $C^\bullet(A)$, m_1 is the differential, $m_{1,1}$ is the Gerstenhaber bracket, and m_2 is the symmetrized cup product. The higher operations are defined by universal formulas involving the product in A and some coefficients. The choice of these coefficients depends on a choice of a Drinfeld associator.

One can see from this discussion how difficult, inexplicit, and non-canonical the construction of the dg Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ is. In view of applications to index theory and other topics, it is natural to ask whether this algebra has some features that are explicit and canonical. For example, as a dg Lie algebra, $\mathcal{V}^\bullet(A)[1]$ is quasi-isomorphic to $C^\bullet(A)[1]$.

In this paper we propose an answer which has a clear geometric meaning. For any Gerstenhaber algebra $\mathcal{A}^\bullet$, one can define an enveloping algebra $Y(\mathcal{A}^\bullet)$, which is a graded associative algebra equipped with a differential d. If $\mathcal{A}^\bullet = \mathcal{V}^\bullet(M)$, the algebra of multivector fields on M, then $Y(\mathcal{A}^\bullet) = D(\Omega^\bullet(M))$, the algebra of differential operators on differential forms on M. The differential d acts by commuting an operator with the De Rham differential.

For a dg Gerstenhaber algebra $(\mathcal{A}^\bullet, \delta)$, $Y(\mathcal{A}^\bullet)$ inherits a differential which we still denote by δ. By a dg algebra $Y(\mathcal{A}^\bullet)$ we always mean $(Y(\mathcal{A}^\bullet), \delta)$ (the differential d being ignored).

The first new result of this paper is Theorem 2.6.1. We construct an explicit canonical A_∞ algebra whose underlying complex is the Hochschild **chain** complex $C_\bullet(C^\bullet(A))$ of the dg associative algebra $C^\bullet(A)$. (Here, the dg algebra structure on $C^\bullet(A)$ is given by the differential and the cup product). In other words, one can construct canonically a dg associative algebra $\mathrm{D}(A)$, together with a quasi-isomorphism of complexes $\mathrm{D}(A) \to C_\bullet(C^\bullet(A))$. We show (Theorem 2.7.1) that there is an A_∞ quasi-isomorphism $Y(\mathcal{V}^\bullet(A)) \to C_\bullet(C^\bullet(A))$. One can interpret that as a dg algebra quasi-isomorphism of dg algebras $\tilde{Y} \to D(A)$ where $\tilde{Y}$ is a canonically constructed dg algebra quasi-isomorphic to $Y(\mathcal{V}^\bullet(A))$.

The above discussion does not take into account the differential d on $Y(\mathcal{V}^\bullet(A))$. To include d into the picture, note that the A_∞ structure on $C_\bullet(C^\bullet(A))$ can be extended to $C_\bullet(C^\bullet(A))[[u]]$, *the negative cyclic complex* of the dg algebra $C^\bullet(A)$. We show that there is an A_∞ quasi-isomorphism

$$(Y(\mathcal{V}^\bullet(A))[[u]], \delta + ud) \to C_\bullet(C^\bullet(A))[[u]]$$

Here δ is the differential on $Y(\mathcal{V}^\bullet(A))$ induced by the differential on $\mathcal{V}^\bullet(A)$.

In other words one can say that the cyclic differential B on the Hochschild complex extends to an A_∞ derivation of $C_\bullet(C^\bullet(A))$ (or, if one prefers, to a derivation of $\mathrm{D}(A)$). This derivation is intertwined with the differential d on $Y(\mathcal{V}^\bullet(A))$.

An extensive sketch of the proof of the main theorem 2.7.1 is given in section 3. A complete proof will be given in a more detailed exposition.

Let us give one example of computing the cohomology ring of the algebra $C_\bullet(C^\bullet(A))$ for a noncommutative algebra A. Let M be a symplectic manifold and $A = (C^\infty(M)[[h]], *)$ its deformation quantization ([**BFFLS**]). The following is contained in [**NT**].

THEOREM 1.0.1. *If M is simply connected, then the cohomology ring of the algebra $C_\bullet(C^\bullet(A))$ is isomorphic to $H^\bullet(M^{S^1})[[h]]$ where M^{S^1} is the free loop space of M.*

Let us say a few words about the A_∞ algebras $C_\bullet(C^\bullet(A))$ and $C_\bullet(C^\bullet(A))[[u]]$. The binary products in these algebras were first introduced in [**NT**]. They were applied to index theorems in [**BNT**] and in [**NT1**]. In fact their existence and properties were indications that the Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ might exist (cf., for example, Theorem 4.3 from [**NT**]). Some work on the higher operations in $C_\bullet(C^\bullet(A))$ was done in [**Ma**].

In fact one constructs both a G_∞ structure on $C^\bullet$ and a canonical A_∞ structure on $C_\bullet(C^\bullet)[[u]]$ where $C^\bullet$ is a dg algebra of a special type, namely *a brace algebra.* An interpretation of the algebra $C_\bullet(C^\bullet)[[u]]$ which is more invariant than ours is given in [**Kh**]. When $C^\bullet$ is commutative, one gets the standard shuffle product on Hochschild chains, extending to the A_∞ product of Getzler-Jones on negative cyclic chains.

It would be interesting to recover the Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ from a less subtle associative algebra $Y(\mathcal{V}^\bullet(A))$. Note that $Y(\mathcal{V}^\bullet(A))$ has an increasing filtration F_n, $n \geq 0$, such that:

- $F_0 = \mathcal{V}^\bullet(A)$; $F_m F_n \subset F_{m+n}$;
- $\mathrm{gr}_F\, Y$ is graded commutative;

- $dF_n \subset F_{n+1}$

Given any algebra with such filtration, one can recover the Gerstenhaber algebra structure on F_0: the product comes from the one on Y, and the bracket is the derived bracket $[a, db]$. It would be interesting to understand how to construct directly a family of filtrations on the canonical algebra Y, indexed by Drinfeld associators.

Let us finish by outlining a few possible areas of study in the future.

1. Connes-Moscovici type index theorems. Computations similar to those in $C_\bullet(C^\bullet(A))[[u]]$ are used in [**CM**], [**C1**]. It would be very interesting to find a unified framework for both approaches. In particular, the symmetry group acting on the space of possible choices of $\mathcal{V}^\bullet(A)$ is the Grothendieck-Teichmuller group which is closely related to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In [**CM**] and [**C1**], the symmetry with respect to the renormalization group was used. Part of Connes' program of noncommutative geometry is to unite the renormalization group and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (the second author thanks Alain Connes for helpful comments on this subject).

2. Quantum cohomology and the Fukaya category. Starting from a compact symplectic manifold M, one can construct the quantum cohomology ring $HQ^\bullet(M)$ (cf., for example, [**MDS**] or [**KM**]) and the A_∞ category $\mathcal{F}(M)$ (cf. [**F1**]). Conjecturally, the former is the Hochschild cohomology of the latter (cf. [**Sei**]).

One can generalize the construction of the algebra $Y(C^\bullet(A))$, or in fact the construction of both sides in Theorem 2.7.1, and replace an algebra A by an A_∞ category $\mathcal{F}$. Thus, one gets an associative dg algebra

$$Y(M) = Y(C^\bullet(\mathcal{F}(M)))$$

Its cohomology algebra, possibly noncommutative, we denote by $\mathcal{H}Q^\bullet(M)$. If the conjecture from [**Sei**] is true, then we get a morphism of algebras

$$HQ^\bullet(M) \to \mathcal{H}Q^\bullet(M)$$

The right hand side should be closely related to the cohomology of the free loop space of M, as suggested by Theorem 1.0.1.

3. Topological string theory of Chas-Sullivan. It is strongly believed that, for an oriented compact manifold X, the chain complex $C_\bullet(X^{S^1})$ is a G_∞ algebra, even a BV_∞ algebra, cf. [**CS**]. It would be interesting to study the enveloping algebra $Y(\mathcal{V}^\bullet(X))$ where $\mathcal{V}^\bullet(X)$ is the standard resolution of $C_\bullet(X^{S^1})$. Denote its cohomology algebra by $\mathcal{H}^\bullet_{\mathrm{loop}}(X)$. One gets a morphism of algebras

$$H_{n-\bullet}(X^{S^1}) \to \mathcal{H}^\bullet_{\mathrm{loop}}(X)$$

The algebra $\mathcal{H}^\bullet_{\mathrm{loop}}(X)$, possibly noncommutative, should be related to the double loop space of X.

A link between points **2** and **3** seems to be indicated in [**Sei**].

Acknowledgements. Both authors' work was partially supported by NSF grants. The authors would like to thank A. Beilinson, P. Bressler, A. Connes, V. Drinfeld, B. Feigin, K. Fukaya, E. Getzler, M. Khalkhali, R. Nest and A. Voronov for stimulating discussions. The second author thanks the Dennisfest organizers for their hospitality and for the wonderful scientific atmosphere.

2. Statement of the main theorem

2.1. Gerstenhaber algebras. Let k be the ground ring of characteristic zero. A *Gerstenhaber algebra* is a graded space $\mathcal{V}^\bullet$ together with

- A graded commutative associative algebra structure on $\mathcal{V}^\bullet$;
- a graded Lie algebra structure on $\mathcal{V}^{\bullet+1}$ such that

$$[a, bc] = [a, b]c + (-1)^{deg(a)deg(b)} b[a, c]$$

Example 2.1.1. Let $\mathfrak{g}$ be a Lie algebra. Then

$$C_\bullet(\mathfrak{g}) = \wedge^\bullet \mathfrak{g}$$

is a Gerstenhaber algebra.

The product is the exterior product, and the bracket is the unique bracket which turns $C_\bullet(\mathfrak{g})$ into a Gerstenhaber algebra and which is the Lie bracket on $\mathfrak{g} = \wedge^1(\mathfrak{g})$.

Example 2.1.2. Let M be a smooth manifold. Then

$$\mathcal{V}_M^\bullet = \wedge^\bullet T_M$$

is a sheaf of Gerstenhaber algebras.

The product is the exterior product, and the bracket is the Schouten bracket. We denote by $\mathcal{V}^\bullet(M)$ the Gerstenhaber algebra of global sections of this sheaf. The previous example is the algebra of left-invariant multivector fields on the Lie group of $\mathfrak{g}$.

2.2. Enveloping algebra of a Gerstenhaber algebra. The following construction is motivated by Example 2.1.2. For a Gerstenhaber algebra $\mathcal{V}^\bullet$, let $Y(\mathcal{V}^\bullet)$ be the associative algebra generated by two sets of generators i_a, L_a, $a \in \mathcal{V}^\bullet$, both i and L linear in a,

$$|i_a| = |a|;\ |L_a| = |a| - 1$$

subject to relations

$$i_a i_b = i_{ab};\ \ [L_a, L_b] = L_{[a,b]};$$

$$[L_a, i_b] = i_{[a,b]};\ L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b$$

The algebra $Y(\mathcal{V}^\bullet)$ is equipped with the differential d of degree one which is defined as a derivation sending i_a to L_a and L_a to zero.

For a smooth manifold M one has a homomorphism

$$Y(\mathcal{V}^\bullet(M)) \to \mathrm{D}(\Omega^\bullet(M))$$

The right hand side is the algebra of differential operators on differential forms on M. It is easy to see that this is in fact an isomorphism.

2.3. The Hochschild cochain complex. Let A be a graded associative algebra with unit 1 over a commutative unital ring k of characteristic zero. A Hochschild d-cochain is a linear map $A^{\otimes d} \to A$. Put, for $d \geq 0$,

$$C^d(A) = C^d(A, A) = \mathrm{Hom}_k(\overline{A}^{\otimes d}, A)$$

where $\overline{A} = A/(k \cdot 1)$ is the quotient linear k-space. Elements of $C^d(A)$ are called *normalized cochains*. We prefer to work with normalized cochains because the formulas for pairings between chains and cochains are simpler.

Put

$$|D| = (\text{degree of the linear map } D) + d$$

Put for cochains D and E from $C^\bullet(A,A)$

$$(D \smile E)(a_1,\dots,a_{d+e}) = (-1)^{|E|\sum_{i\le d}(|a_i|+1)} D(a_1,\dots,a_d)\times$$
$$\times E(a_{d+1},\dots,a_{d+e});$$
$$(D\circ E)(a_1,\dots,a_{d+e-1}) = \sum_{j\ge 0}(-1)^{(|E|+1)\sum_{i=1}^{j}(|a_i|+1)}$$
$$D(a_1,\dots,a_j,E(a_{j+1},\dots,a_{j+e}),\dots);$$
$$[D,\ E] = D\circ E - (-1)^{(|D|+1)(|E|+1)}E\circ D$$

These operations define the graded associative algebra $(C^\bullet(A,A)\ ,\smile)$ and the graded Lie algebra $(C^{\bullet+1}(A,A),\ [\ ,\ ])$ (cf. [**CE**]; [**G**]). Let

$$m(a_1,a_2) = (-1)^{\deg a_1}\ a_1a_2;$$

this is a 2-cochain of A (not in C^2). Put

$$\delta D = [m, D];$$
$$(\delta D)(a_1,\dots,a_{d+1}) = (-1)^{|a_1||D|+|D|+1}\times$$
$$\times a_1 D(a_2,\dots,a_{d+1})+$$
$$+\sum_{j=1}^{d}(-1)^{|D|+1+\sum_{i=1}^{j}(|a_i|+1)}D(a_1,\dots,a_ja_{j+1},\dots,a_{d+1})$$
$$+(-1)^{|D|\sum_{i=1}^{d}(|a_i|+1)}D(a_1,\dots,a_d)a_{d+1}$$

One has

$$\delta^2 = 0;\quad \delta(D\smile E) = \delta D\smile E + (-1)^{|degD|}D\smile\delta E$$
$$\delta[D,E] = [\delta D, E] + (-1)^{|D|+1}\ [D,\delta E]$$

($\delta^2=0$ follows from $[m,m]=0$).

Thus $C^\bullet(A,A)$ becomes a complex; we will denote it also by $C^\bullet(A)$. The cohomology of this complex is $H^\bullet(A,A)$ or the *Hochschild cohomology.*

We denote it also by $H^\bullet(A)$. The $\smile$ product induces the *Yoneda product* on $H^\bullet(A,A) = Ext^\bullet_{A\otimes A^0}(A,A)$. The operation $[\ ,\ ]$ is the Gerstenhaber bracket [**G**].

If $(A,\ \partial)$ is a differential graded algebra then one can define the differential ∂ acting on $C^\bullet(A)$ by:

$$\partial D\ =\ [\partial, D]$$

THEOREM 2.3.1. [**G**] *The cup product and the Gerstenhaber bracket induce a Gerstenhaber algebra structure on* $H^\bullet(A)$.

For cochains D and D_i define a new Hochschild cochain by the following formula of Gerstenhaber ([**G**]) and Getzler ([**G1**]):

$$D_0\{D_1,\dots,D_m\}(a_1,\dots,a_n) =$$
$$=\sum(-1)^{\sum_{k\le i_p}(|a_k|+1)(|D_p|+1)}D_0(a_1,\dots,a_{i_1},D_1(a_{i_1+1},\dots),\dots,$$
$$D_m(a_{i_m+1},\dots),\dots)$$

PROPOSITION 2.3.2. *One has*

$$(D\{E_1, \ldots, E_k\})\{F_1, \ldots, F_l\} = \sum (-1)^{\sum_{q \le i_p}(|E_p|+1)(|F_q|+1)} \times$$
$$\times D\{F_1, \ldots, E_1\{F_{i_1+1}, \ldots, \}, \ldots, E_k\{F_{i_k+1}, \ldots, \}, \ldots, \}$$

The above proposition can be restated as follows. For a cochain D let $D^{(k)}$ be the following k-cochain of $C^\bullet(A)$:

$$D^{(k)}(D_1, \ldots, D_k) = D\{D_1, \ldots, D_k\}$$

PROPOSITION 2.3.3. *The map*

$$D \mapsto \sum_{k \ge 0} D^{(k)}$$

is a morphism of differential graded algebras

$$C^\bullet(A) \to C^\bullet(C^\bullet(A))$$

2.4. The Gerstenhaber algebra $\mathcal{V}^\bullet(A)$. Below is the theorem from [**T**]. We sketch its proof in 3.1.

THEOREM 2.4.1. *For every associative algebra A there exists a dg Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ such that:*

- *There is a quasi-isomorphism of dg Lie algebras*
$$\mathcal{V}^\bullet(A)[1] \to C^\bullet(A)[1]$$
- *The above quasi-isomorphism induces an isomorphism of Gerstenhaber algebras*
$$H^\bullet(\mathcal{V}^\bullet(A)) \to H^\bullet(A)$$
where the Gerstenhaber structure on the right hand side is the standard one from 2.3.
- *For $A = C^\infty(M)$ there is a quasi-isomorphism of dg Gerstenhaber algebras*
$$\mathcal{V}^\bullet(A) \to \mathcal{V}^\bullet(M)$$

2.5. Hochschild chains. Let A be an associative dg algebra with unit 1 over a ground ring k. The differential on A is denoted by δ. Recall that by definition

$$\overline{A} = A/(k \cdot 1)$$

Set

$$C_p(A, A) = C_p(A) = A \otimes \overline{A}^{\otimes p}$$

Define the differentials $\delta : C_\bullet(A) \to C_\bullet(A)$, $b : C_\bullet(A) \to C_{\bullet-1}(A)$, $B : C_\bullet(A) \to C_{\bullet+1}(A)$ as follows.

$$\delta(a_0 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p} (-1)^{\sum_{k<i}(|a_k|+1)+1}(a_0 \otimes \cdots \otimes \delta a_i \otimes \cdots \otimes a_p)$$

$$b(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p-1} (-1)^{\sum_{i=0}^{k}(|a_i|+1)+1} a_0 \ldots \otimes a_k a_{k+1} \otimes \ldots a_p \tag{2.1}$$
$$+(-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)} a_p a_0 \otimes \ldots \otimes a_{p-1}$$

$$B(a_0\otimes\ldots\otimes a_p)=\sum_{k=0}^{p}(-1)^{\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)}1\otimes a_{k+1}\otimes\ldots a_p\otimes \otimes a_0\otimes\ldots\otimes a_k \tag{2.2}$$

The complex $C_\bullet(A)$ is the total complex of the double complex with the differential $b+\delta$.

Let u be a formal variable of degree -2. The complex $(C^\bullet(A)[[u]], b+\delta+uB)$ is called *the negative cyclic complex* of A.

One can define a product

$$\mathrm{sh}: C^\bullet(A)\otimes C^\bullet(A)\to C^\bullet(A) \tag{2.3}$$

and its extension

$$\mathrm{sh}+u\,\mathrm{sh}': C^\bullet(A)[[u]]\otimes C^\bullet(A)[[u]]\to C^\bullet(A)[[u]] \tag{2.4}$$

[**L**] by the following explicit formulas:

$$(a_0\otimes\ldots\otimes a_p)\otimes(c_0\otimes\ldots\otimes c_q)\stackrel{\mathrm{sh}}{\mapsto} a_0c_0\otimes \mathrm{sh}_{pq}(a_1,\ldots,a_p,\,c_1,\ldots,c_q) \tag{2.5}$$

where

$$\mathrm{sh}_{pq}(x_1,\ldots,x_{p+q})=\sum_{\sigma\in\mathrm{Sh}(p,q)}\mathrm{sgn}(\sigma)x_{\sigma^{-1}1}\otimes\ldots\otimes x_{\sigma^{-1}(p+q)} \tag{2.6}$$

and

$$\mathrm{Sh}(p,q)=\{\sigma\in\Sigma_{p+q}\,|\sigma1<\ldots<\sigma p;\ \sigma(p+1)<\ldots<\sigma(p+q)\}$$

In the graded case, $\mathrm{sgn}(\sigma)$ gets replaced by the sign computed by the following rule: in all transpositions, the parity of a_i is equal to $|a_i|+1$ if $i\neq 0$ and $|a_0|$ if $i=0$, and similarly for c_i. A transposition contributes a product of parities.

$$(a_0\otimes\ldots\otimes a_p)\otimes(c_0\otimes\ldots\otimes c_q)\stackrel{\mathrm{sh}'}{\mapsto} 1\otimes \mathrm{sh}'_{p+1,\,q+1}(a_0,\ldots,a_p,\,c_0,\ldots,c_q) \tag{2.7}$$

where

$$\mathrm{sh}'_{p+1,q+1}(x_0,\ldots,x_{p+q+1})=\sum_{\sigma\in\mathrm{Sh}'(p+1,q+1)}\mathrm{sgn}(\sigma)x_{\sigma^{-1}0}\otimes\ldots\otimes x_{\sigma^{-1}(p+q+1)} \tag{2.8}$$

and $\mathrm{Sh}'(p+1,q+1)$ is the set of all permutations $\sigma\in\Sigma_{p+q+2}$ such that $\sigma0<\ldots<\sigma p$, $\sigma(p+1)<\ldots<\sigma(p+q+1)$, and $\sigma0<\sigma(p+1)$

2.6. The A_∞ algebra $C_\bullet(C^\bullet(A))$. Recall [**LS**], [**St1**] that an A_∞ algebra is a graded vector space $\mathcal{C}$ together with a Hochschild cochain m of total degree 1,

$$m=\sum_{n=1}^{\infty}m_n$$

where $m_n\in C^n(\mathcal{C})$ and

$$[m,m]=0$$

Consider the Hochschild cochain complex of a graded algebra A as a differential graded associative algebra $(C^\bullet(A),\ \smile,\ \delta)$. Consider the Hochschild *chain* complex of this differential graded algebra. The total differential in this complex is $b+\delta$; the degree of a chain is given by

$$|D_0 \otimes \ldots \otimes D_n| = |D_0| + \sum_{i=1}^{n}(|D_i|+1)$$

where D_i are Hochschild cochains.

The complex $C_\bullet(C^\bullet(A))$ contains the Hochschild cochain complex $C^\bullet(A)$ as a subcomplex (of zero-chains) and has the Hochschild chain complex $C_\bullet(A)$ as a quotient complex:

$$C^\bullet(A) \xrightarrow{i} C_\bullet(C^\bullet(A)) \xrightarrow{\pi} C_\bullet(A)$$

(this sequence is not by any means exact). The projection on the right splits if A is commutative. If not, $C_\bullet(A)$ is naturally a graded subspace but not a subcomplex.

THEOREM 2.6.1. *There is an A_∞ structure on $C_\bullet(C^\bullet(A))[[u]]$ such that:*

- *All m_n are $k[[u]]$-linear, (u)-adically continuous*
- $m_1 = b + \delta + uB$
  For $x,\ y \in C_\bullet(A)$:
- $(-1)^{|x|}m_2(x,y) = (\mathrm{sh} + u\,\mathrm{sh}')(x,y)$
  For $D,\ E \in C^\bullet(A)$:
- $(-1)^{|D|}m_2(D,E) = D \smile E$
- $m_2(1\otimes D,\ 1\otimes E) + (-1)^{|D||E|}m_2(1\otimes E,\ 1\otimes D) = (-1)^{|D|}1\otimes[D,\ E]$
- $m_2(D,\ 1\otimes E) + (-1)^{(|D|+1)|E|}m_2(1\otimes E,\ D) = (-1)^{|D|+1}[D,\ E]$

Here is an explicit description of the above A_∞ structure. We define for $n \geq 2$

$$m_n = m_n^{(1)} + m_n^{(2)} + u m_n^{(3)}$$

where, for

$$a^{(k)} = D_0^{(k)} \otimes \ldots \otimes D_{N_k}^{(k)},$$
$$m_n^{(1)} = 0$$

for $n \geq 3$;

$$m_2^{(1)}(a^{(1)}, a^{(2)}) = (-1)^{|a^{(1)}|}\sum \pm D_0^{(1)} \smile D_0^{(2)}\underline{\{\ \}} \otimes \underline{\ldots}\otimes$$
$$\otimes D_1^{(2)}\underline{\{\ \}} \otimes \underline{\ldots}\ldots \otimes D_{N_2}^{(2)}\underline{\{\ \}} \otimes \underline{\ldots}$$

The space designated by $\_$ is filled with $D_1^{(1)}, \ldots, D_{N_1}^{(1)}$ whose order is preserved. The sign rule is as follows: the parity of $D_j^{(i)}$ is $|D_j^{(i)}|$ for $j=0$ and $|D_j^{(i)}|+1$ otherwise.

$$m_n^{(2)}(a^{(1)}, \ldots, a^{(n)}) =$$
$$= (-1)^{\sum_{i=1}^{n-1}|a_1|+n}\sum \pm D_{N_n}^{(n)}\underline{\{\ldots, D_0^{(1)}, \ldots, D_0^{(n-1)}\{\_\}, \ldots\}} \smile$$
$$\smile D_0^{(n)}\{\_\} \otimes \underline{\ldots} \otimes D_1^{(n)}\underline{\{\ \}} \otimes \underline{\ldots}\ldots \otimes D_{N_n-1}^{(n)}\underline{\{\ \}} \otimes \underline{\ldots}$$

The space designated by $\_$ is filled with $D_i^{(j)}$ for $j < n$ in such a way that:

- the cyclic order of each group $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ is preserved
- $D_0^{(1)}, \ldots, D_0^{(n-1)}$ are all inside the braces in $D_{N_n}^{(n)}\{\ \}$
- $D_0^{(i)}$ is to the left of $D_0^{(j)}$ for $i < j$

- any cochain $D_j^{(i)}$ may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form $D_q^{(p)}$ with $p < i$

The parity of $D_j^{(i)}$ is $|D_j^{(i)}|$ if $i = n$ and $j = 0$; it is $|D_j^{(i)}| + 1$ otherwise. Note that the formula for $m_n^{(2)}$ gives the Hochschild chain differential b for $n = 1$.

Finally, define

$$m_n^{(3)}(a^{(1)}, \ldots, a^{(n)}) = (-1)^{n+1} \sum \pm 1 \otimes \underline{\ldots \otimes D_0^{(0)} \otimes \ldots \otimes D_0^{(n)}\{\} \otimes \ldots}$$

The underlined space is filled with $D_i^{(j)}$ in such a way that:

- the cyclic order of each group $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ is preserved
- $D_0^{(i)}$ is to the left of $D_0^{(j)}$ for $i < j$
- any cochain $D_j^{(i)}$ may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form $D_q^{(p)}$ with $p < i$. The parity of $D_j^{(i)}$ is always $|D_j^{(i)}| + 1$.

One checks by a direct computation that the above formulas provide an A_∞ structure on $C_\bullet(C^\bullet(A))[[u]]$.

Remark 2.6.2. Let A be a commutative algebra. Then $C_\bullet(A)[[u]]$ is not only a subcomplex but an A_∞ subalgebra of $C_\bullet(C^\bullet(A))[[u]]$. This A_∞ structure on $C_\bullet(A)[[u]]$ was introduced in [**GJ1**].

2.7. Statement of the main theorem.

Theorem 2.7.1. *There is a A_∞ quasi-isomorphism*

$$Y(\mathcal{V}^\bullet(A)) \to C_\bullet(C^\bullet(A))$$

which extends to a $k[[u]]$-linear, (u)-adically continuous A_∞ quasi-isomorphism

$$(Y(\mathcal{V}^\bullet(A))[[u]], \delta + ud) \to C_\bullet(C^\bullet(A))[[u]]$$

3. Sketch of the proof of the main theorem

We will start by introducing several operads used in the proof of Theorem 2.1.2. We will then extend the notion of a Gerstenhaber algebra to that of a *calculus* (3.2). Next we will introduce some basic pairings between Hochschild chains and cochains, extending the definitions from 2.5. Then we will state Theorem 3.4.1 which asserts that the Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ can be extended to a calculus $\mathrm{Calc}(A)$.

Next, we will introduce a notion of a two-colored operad which is suitable for working with objects like calculi (3.5). After that we extend the contents of 3.1 by introducing corresponding two-colored operads and stating relations among them (3.6).

Next, we define a notion of the enveloping algebra of an algebra $\mathcal{A}^\bullet$ over an operad $\mathcal{O}$, provided that $\mathcal{O}$ is part of a two-colored operad $\mathcal{P}$. This is an associative dg algebra which we denote by $Y_{\mathcal{P}}(\mathcal{A}^\bullet)$. Because of Theorem 3.4.1, one has morphisms of dg algebras

$$Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(\mathcal{V}^\bullet(A)) \overset{\phi}{\leftarrow} Y_{\mathrm{Calc}^0_\infty}(C^\bullet(A)) \overset{\tau}{\leftarrow} Y_{\mathrm{Calc}^0_\infty}(\mathcal{V}^\bullet(A)) \overset{\psi}{\to} Y_{\mathrm{Calc}^0}(\mathcal{V}^\bullet(A)) \tag{3.1}$$

where two-colored operads $\mathrm{Calc}^0_{\mathrm{alg}}$, Calc^0_∞, Calc^0 are defined in 3.6.

Next, we note that $C_\bullet(A)$ is an A_∞ module over the A_∞ algebra $C_\bullet(C^\bullet(A))[[u]]$. We interpret this result as the existence of an A_∞ morphism

$$C_\bullet(C^\bullet(A)) \to Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(\mathcal{V}^\bullet(A))$$

Then we observe that this is a quasi-isomorphism. The maps ψ and τ in (3.1) are also quasi-isomorphisms; therefore we get an A_∞ morphism from $Y(\mathcal{V}^\bullet(A))$ to $C_\bullet(C^\bullet(A))$. To prove that this is a quasi-isomorphism, we are reduced to proving that ϕ is a quasi-isomorphism.

To that end, we study in some more detail the homology of the algebra $Y_{\mathrm{Calc}^0_\infty}(\mathcal{A}^\bullet)$ for any Gerstenhaber algebra $\mathcal{A}^\bullet$ (Appendix). We show that this is a twisted version of the Hochschild homology, i.e. there is a spectral sequence starting with the latter and converging to the former. We observe that the map ϕ is filtered with respect to some filtration, and the fact that it is a quasi-isomorphism follows from considering the corresponding spectral sequence.

In the last subsection before Appendix, we modify the above arguments to prove the cyclic case of the main theorem.

3.1. Operads G, G_∞, G_{alg}, G_{geom}. Here we recall the scheme of the proof of Theorem 2.4.1 which was used in [**T**], [**T1**].

Gerstenhaber algebras are algebras over an operad which we will denote by G.. In other words, $G(n)$ is the graded k-module of all n-ary operations composed of the product and the bracket in a Gerstenhaber algebra, subject to all relations following from Gerstenhaber algebra axioms. The operad G is often denoted also by e_2.

By G_∞ we denote the standard free resolution of G. This is an operad in the category of complexes. One description of it is as follows. Consider a graded space $\mathcal{A}^\bullet$. Let us pretend for a moment that $\mathcal{A}^\bullet$ is finite-dimensional. Consider the free graded Lie algebra $\mathrm{Lie}(\mathcal{A}^\bullet[1]^*)$ generated by the dual space to $\mathcal{A}^\bullet[1]$. Then the space $\mathcal{F}^\bullet(\mathcal{A}^\bullet) = \wedge^\bullet \mathrm{Lie}(\mathcal{A}^\bullet[1]^*)$ carries the structure of a Gerstenhaber algebra (Example 2.1.1). In fact $\mathcal{F}^\bullet(\mathcal{A}^\bullet)$ is the free Gerstenhaber algebra generated by $\mathcal{A}^\bullet[1]^*$. A G_∞ structure on $\mathcal{A}^\bullet$ is by definition a derivation δ of the Gerstenhaber algebra $\mathcal{F}^\bullet(\mathcal{A}^\bullet)$ such that $|\delta| = 1$ and $\delta^2 = 0$.

Remark 3.1.1. As stated, this definition has a problem if $\mathcal{A}^\bullet$ is infinite-dimensional: it involves various linear maps from $\mathcal{A}^\bullet[1]^*$ to tensor powers of $\mathcal{A}^\bullet[1]^*$, satisfying certain relations. What we actually mean are dual maps from tensor powers of $\mathcal{A}^\bullet[1]$ to $\mathcal{A}^\bullet[1]$, satisfying dual relations. A rigorous definition can be given in the dual language of coalgebras. The same remark applies to the definitions and constructions of 3.6, 3.10, 4 below.

Another, equivalent way of defining the operad G_∞ is to put

$$G_\infty = \mathrm{Cobar}(G^{\mathrm{dual}})$$

where G^{dual} is the Koszul dual operad as in [**GK**]. Note that the operad G is Koszul [**GJ**].

The operad G_{geom} is the little discs operad [**May**]. To pass from it to an operad in the category of complexes, one defines the operad $C_\bullet(G_{\mathrm{geom}})$. It is known that the homology operad $H_\bullet(G_{\mathrm{geom}})$ is isomorphic to G, see [**A**], [**Co**], [**GJ**].

The operad in the category of complexes G_{alg} is the operad of all universal operations on the Hochschild cochain complex of an associative algebra. It has various versions each of which suits our purposes ([**KS**], [**MS**], [**T1**]). Let us start with the version from [**MS**]. Consider all multi-linear operations on Hochschild cochains which are linear combinations of iterated compositions of the following elementary operations:

$$op(D)(a_1, \ldots, a_d) = a_1 D(a_2, \ldots, a_d)$$

$$op(D)(a_1, \ldots, a_d) = D(a_1, \ldots, a_i a_{i+1}, \ldots, a_d)$$

$$op(D)(a_1, \ldots, a_d) = D(a_1, \ldots, a_{d-1}) a_d$$

$$op(D, E)(a_1, \ldots, a_{d+e-1}) = D(a_1, \ldots, a_i, E(a_{i+1}, \ldots, a_{i+e}), \ldots)$$

$$op(D, E)(a_1, \ldots, a_{d+e}) = D(a_1, \ldots, a_d) E(a_{d+1}, \ldots, a_{d+e})$$

(a minor technical point: this construction makes sense if one works with non-normalized Hochschild cochains $C^\bullet(A) = \mathrm{Hom}(A^{\otimes n}, A)$). One can arrange these operations into an operad in the category of complexes, cf. [**MS**]. In [**KS**], a different version of this operad is proposed, namely the minimal operad $\mathcal{M}$. It consists of all universal operations on Hochschild cochains which are linear combinations of compositions of the brace operations from 2.5, the cup product, and the higher cup products which are defined if A is an A_∞ algebra. Such n-ary operations are naturally indexed by rooted trees whose vertices are labeled by symbols $1, \ldots, n, m_i$, $i \geq 2$, in such a way that each label from 1 to n enters exactly once, and m_i may only label vertices with i outgoing edges. Finally, by $\mathcal{G}_{\mathrm{alg}}$ we denote the standard free resolution of G_{alg}, the operad $\mathrm{Bar\,Cobar}(G_{\mathrm{alg}})$. For any operad $\mathcal{O}$ in the category of complexes, the operad $\mathrm{Bar\,Cobar}(\mathcal{O})$ admits an explicit description as in [**KS**] (cf. also 3.6).

The relation between the above operads is as follows.

$$G_\infty \xrightarrow{f_1} G_{\mathrm{alg}} \xleftarrow{g_1} \mathcal{G}_{\mathrm{alg}} \xrightarrow{g_2} C_\bullet(G_{\mathrm{geom}}) \xrightarrow{F} G \tag{3.2}$$

The quasi-isomorphism g_2 can be deduced from [**KS**], from [**MS**], or [**T1**]; g_1 is the standard quasi-isomorphism between a resolution of an operad and the operad itself. F is the formality quasi-isomorphism from [**K1**], [**T1**]; it depends on a choice of a Drinfeld associator. The existence of f_1 follow from the fact that , thanks to the existence of F, G_∞ and $\mathcal{G}_{\mathrm{alg}}$ are two free resolutions of G.

Therefore, since G_{alg} acts on $C^\bullet(A)$, we see that $C^\bullet(A)$ is a G_∞ algebra. This summarizes one of the versions of the proof of Theorem 2.4.1.

3.2. Calculi.

DEFINITION 3.2.1. *A precalculus is a pair of a Gerstenhaber algebra $\mathcal{V}^\bullet$ and a graded space $\Omega^\bullet$ together with*

- *a structure of a graded module over the graded commutative algebra $\mathcal{V}^\bullet$ on $\Omega^{-\bullet}$ (corresponding action is denoted by i_a, $a \in \mathcal{V}^\bullet$);*
- *a structure of a graded module over the graded Lie algebra $\mathcal{V}^{\bullet+1}$ on $\Omega^{-\bullet}$ (corresponding action is denoted by L_a, $a \in \mathcal{V}^\bullet$) such that*

$$[L_a, i_b] = i_{[a,b]}$$

and

$$L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b$$

Definition 3.2.2. *A calculus is a precalculus together with an operator d of degree 1 on $\Omega^\bullet$ such that $d^2 = 0$ and*

$$[d, i_a] = L_a.$$

Example 3.2.3. For any manifold one defines a calculus $\mathrm{Calc}(M)$ with $\mathcal{V}^\bullet$ being the algebra of multivector fields, $\Omega^\bullet$ the space of differential forms, and d the De Rham differential.

Example 3.2.4. For any associative algebra A one defines a calculus $\mathrm{Calc}_0(A)$ by putting $\mathcal{V}^\bullet = H^\bullet(A, A)$ and $\Omega^\bullet = H_\bullet(A, A)$. The five operations from Definition 3.2.2 are the cup product, the Gerstenhaber bracket, the pairings i_D and L_D, and the differential B, as in 3.3 below.

A differential graded (dg) calculus is a calculus with extra differentials δ of degree 1 on $\mathcal{V}^\bullet$ and b of degree -1 on $\Omega^\bullet$ which are derivations with respect to all the structures.

3.3. Pairings between chains and cochains. For a graded algebra A, for $D \in C^d(A, A)$, define

$$i_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D| \sum_{i \leq d}(|a_i|+1)} a_0 D(a_1, \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n \tag{3.3}$$

Proposition 3.3.1.

$$[b, i_D] = i_{\delta D}$$

$$i_D i_E = (-1)^{|D||E|} i_{E \smile D}$$

Now, put

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum_{k=1}^{n-d} \epsilon_k a_0 \otimes \ldots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \ldots \otimes a_n + \tag{3.4}$$

$$\sum_{k=n+1-d}^{n} \eta_k D(a_{k+1}, \ldots, a_n, a_0, \ldots) \otimes \ldots \otimes a_k$$

(The second sum in the above formula is taken over all cyclic permutations such that a_0 is inside D). The signs are given by

$$\epsilon_k = (|D| + 1) \sum_{i=0}^{k} (|a_i| + 1)$$

and

$$\eta_k = |D| + 1 + \sum_{i \leq k} (|a_i| + 1) \sum_{i \geq k} (|a_i| + 1)$$

Proposition 3.3.2.

$$[L_D, L_E] = L_{[D,E]}$$

$$[b, L_D] + L_{\delta D} = 0$$

$$[L_D, B] = 0$$

Now let us extend the above operations to the cyclic complex. Define

$$S_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j \geq 0;\ k \geq j+d} \epsilon_{jk} 1 \otimes a_{k+1} \otimes \ldots a_0 \otimes \ldots \otimes \otimes D(a_{j+1}, \ldots, a_{j+d}) \otimes \ldots \otimes a_k \tag{3.5}$$

(The sum is taken over all cyclic permutations for which a_0 appears to the left of D). The signs are as follows:

$$\epsilon_{jk} = |D|(|a_0| + \sum_{i=1}^{n}(|a_i|+1)) + (|D|+1)\sum_{j+1}^{k}(|a_i|+1) + \sum_{i \leq k}(|a_i|+1)\sum_{i \geq k}(|a_i|+1)$$

As we will see later, all the above operations are partial cases of a unified algebraic structure for chains and cochains, cf. 3.8; the sign rule for this unified construction was explained in 2.6.

Proposition 3.3.3. *(*[**R**]*)*

$$[b + uB, i_D + uS_D] - i_{\delta D} - uS_{\delta D} = L_D$$

Proposition 3.3.4. *(*[**DGT**]*) There exists a linear transformation $T(D,E)$ of the Hochschild chain complex, bilinear in $D,\ E \in C^\bullet(A,A)$, such that*

$$[b + uB, T(D,E)] - T(\delta D, E) - (-1)^{|D|}T(D, \delta E) =$$

$$= [L_D, i_E + uS_E] - (-1)^{|D|+1}(i_{[D,E]} + uS_{[D,E]})$$

3.4. The calculus $\mathrm{Calc}(A)$**.**

Theorem 3.4.1. [**TT**] *For any associative algebra A one can define a dg calculus* $\mathrm{Calc}(A)$ *such that:*

1). As dg Lie algebras, $\mathcal{V}^{\bullet+1}(A)$ is quasi-isomorphic to the Hochschild cochain complex $C^{\bullet+1}(A)$. As dg modules over the dg Lie algebra $\mathcal{V}^{\bullet+1}(A)$, $\Omega^\bullet(A)[[u]]$ with the differential $\delta + uB$ is quasi-isomorphic to the negative cyclic complex $C_\bullet(A)[[u]]$ with the differential $b + uB$.

2). If $A = C^\infty(M)$ then there is a quasi-isomorphism of calculi

$$\mathrm{Calc}(A) \to \mathrm{Calc}(M)$$

3). For any A, the calculus $H^\bullet(\mathrm{Calc}(A))$ is isomorphic to the calculus $\mathrm{Calc}_0(A)$ *from Example 3.2.4.*

3.5. Two-colored operads. The notion of a two-colored operad formalizes the situation when one has an algebra A over an operad $\mathcal{O}$, an object B, and a set $\mathcal{M}(n)$ of operations $A^{\otimes n} \otimes B \to B$. The union of $\mathcal{M}(n)$ is supposed to be closed under some natural operations.

More precisely, a two-colored operad $(\mathcal{O}, \mathcal{M})$ consists of an operad $\mathcal{O}$ and a collection of objects $\mathcal{M}(n)$, $n \geq 0$, together with an action of the symmetric group S_n on $\mathcal{M}(n)$ for all n, and with the operations

$$\mathcal{M}(k) \otimes \mathcal{O}(n_1) \otimes \ldots \otimes \mathcal{O}(n_k) \to \mathcal{M}(n_1 + \ldots + n_k) \tag{3.6}$$

$$\mathcal{M}(k) \otimes \mathcal{M}(l) \to \mathcal{M}(k+l) \tag{3.7}$$

subject to natural conditions of associativity and symmetry with respect to permutations.

An algebra over a two-colored operad $(\mathcal{O}, \mathcal{M})$ is a pair of objects (A, B) such that A is an $\mathcal{O}$-algebra, together with operations

$$\mathcal{M}(n) \otimes A^{\otimes n} \otimes B \to B \tag{3.8}$$

subject to natural relations.

One can easily adapt the basic notions of the theory of operads to this context. For example, the free two-colored operad generated by a collection of $k[S_n]$-modules $\mathcal{Q}(n)$, $\mathcal{R}(n)$ is the pair $(\mathcal{O}, \mathcal{M})$ where $\mathcal{O}$ is the free operad generated by $\{\mathcal{Q}(n)\}$ and $\mathcal{M}(n)$ is described as follows. Consider all rooted trees with a chosen path $\mathbf{p}_0$ from the root to an external vertex $\mathbf{v}_0$. Let the external vertices other than $\mathbf{v}_0$ be numbered by $1, \ldots, n$. We will call such objects *two-colored trees*. For such a tree, put

$$\mathcal{M}(T) = \bigotimes_{\text{internal vertices } v} \mathcal{M}(v),$$

where

$$\mathcal{M}(v) = \mathcal{Q}(\#(\text{edges outgoing from } v))$$

if v is not on $\mathbf{p}_0$;

$$\mathcal{M}(v) = \mathcal{R}(\#(\text{edges outgoing from } v))$$

if v is on $\mathbf{p}_0$. $\mathcal{M}(n)$ is the direct sum of $\mathcal{M}(T)$ over all isomorphism classes of such trees.

If $(\mathcal{O}, \mathcal{M})$ is a two-colored dg operad, then the cofree two-colored cooperad cogenerated by $(\mathcal{O}, \mathcal{M})$ acquires a differential. This differential is the sum of the one induced from $(\mathcal{O}, \mathcal{M})$ and the new one, which sends an element of the direct summand corresponding to a (two-colored) tree T to the sum of elements in direct summands corresponding to trees T' obtained from T by contracting an internal edge. These elements are obtained from the original element by applying an appropriate composition in the operad $(\mathcal{O}, \mathcal{M})$. Thus one gets a two-colored cooperad in the category of complexes $\mathrm{Cobar}(\mathcal{O}, \mathcal{M})$.

Dually, if $(\mathcal{O}, \mathcal{M})$ is a two-colored dg cooperad, one constructs a two-colored dg operad $\mathrm{Bar}(\mathcal{O}, \mathcal{M})$. Its underlying space is a direct sum over (two-colored) trees, and the new component of the differential consists of inserting an internal edge in all possible positions, combined with an appropriate cooperadic cocomposition.

Composing these two constructions, one produces for a dg two-colored operad $(\mathcal{O}, \mathcal{M})$ a new dg two-colored operad $\mathrm{Bar}\,\mathrm{Cobar}(\mathcal{O}, \mathcal{M})$. This is the standard free resolution of $(\mathcal{O}, \mathcal{M})$, which means that it is free as an operad in the category of graded vector spaces and that there is a canonical quasi-isomorphism of operads

$$\mathrm{Bar}\,\mathrm{Cobar}(\mathcal{O}, \mathcal{M}) \to (\mathcal{O}, \mathcal{M}).$$

Explicitly (compare [**KS**]), this resolution is the direct sum of components numbered by (two-colored) trees whose edges are labeled by one of the two labels, $finite$ or $infinite$. All external edges are infinite. The terms in the new differential are of two types: a) contracting a finite edge, combined with an operadic composition; b) making a finite edge infinite.

One can also define, following [**GK**], a Koszul dual cooperad of a two-colored operad, and extend the notion of a Koszul operad into the two-colored setting. For a Koszul two-colored operad $(\mathcal{O}, \mathcal{M})$, the dg operad $\mathrm{Bar}((\mathcal{O}, \mathcal{M})^{\mathrm{dual}})$ is a free resolution of $(\mathcal{O}, \mathcal{M})$.

3.6. Two-colored operads Calc**,** Calc_∞**,** $\mathrm{Calc}_{\mathrm{geom}}$**,** $\mathrm{Calc}_{\mathrm{alg}}$**.** In this section we will extend the method that was outlined in 3.1. This will enable us to prove both Theorem 3.4.1, 1)-3), and Theorem 2.7.1. To prove statement 4) of Theorem 3.4.1, some additional work is needed; we will give a proof in a subsequent paper.

By Calc, resp. Calc^0, we denote the two-colored operad in the category of graded spaces such that algebras over them are calculi (resp. precalculi). In other words, $\mathcal{O} = G$ and $\mathcal{M}(n)$ consists of all n-ary operations composed of i_a, L_a, and d (resp. i_a and L_a).

By Calc_∞ we denote the standard resolution of Calc. One can write it as

$$\mathrm{Calc}_\infty = \mathrm{Bar}\,\mathrm{Calc}^{\mathrm{dual}}$$

Similarly,

$$\mathrm{Calc}^0_\infty = \mathrm{Bar}\,{\mathrm{Calc}^0}^{\mathrm{dual}}$$

(one can show that Calc and Calc^0 are Koszul).

Alternatively, one can give the following explicit definition.

A Calc_∞ algebra is a pair of graded vector spaces $(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ where $\mathcal{A}^\bullet$ is a G_∞ algebra, together with the following extra data. As in 3.1, let us pretend that $\mathcal{A}^\bullet, \mathcal{B}^\bullet$ are finite-dimensional (cf. Remark 3.1.1). Recall from 3.1 that one can define the dg Gerstenhaber algebra $\mathcal{F}^\bullet(\mathcal{A}^\bullet)$. Let $\Omega^{-\bullet}_{\mathcal{A}^\bullet,\mathcal{B}^\bullet}$ be the free graded $Y(\mathcal{F}^\bullet(\mathcal{A}^\bullet))$-module generated by $(\mathcal{B}^{-\bullet}[1])^*$. In other words, let $(\mathcal{F}^\bullet(\mathcal{A}^\bullet), \Omega^\bullet_{\mathcal{A}^\bullet,\mathcal{B}^\bullet})$ be the free precalculus generated by $(\mathcal{A}^{-\bullet}[1])^*$ and $(\mathcal{B}^{-\bullet}[1])^*$.

A Calc^0_∞ algebra structure on $(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ is a linear operator d on $\Omega^\bullet_{\mathcal{A}^\bullet,\mathcal{B}^\bullet}$ of square zero and degree one, such that $(\Omega^{-\bullet}_{\mathcal{A}^\bullet,\mathcal{B}^\bullet}, d)$ is a dg module over the dg algebra $(Y(\mathcal{F}^\bullet(\mathcal{A}^\bullet)), \delta)$.

Let u be a formal parameter of degree two. A Calc_∞ algebra structure on $(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ is a $k[[u]]$-linear, (u)-adically continuous operator d on $\Omega^\bullet_{\mathcal{A}^\bullet,\mathcal{B}^\bullet}[[u]]$ of square zero and degree one, such that $(\Omega^{-\bullet}_{\mathcal{A}^\bullet,\mathcal{B}^\bullet}[[u]], d)$ is a dg module over the dg algebra $(Y(\mathcal{F}^\bullet(\mathcal{A}^\bullet))[[u]], \delta + ud)$.

We define $\mathrm{Calc}^0_{\mathrm{geom}} = (\mathcal{O}, \mathcal{M})$ where $\mathcal{O}$ is the little discs operad and $\mathcal{M}$ is the configuration space, up to dilations, of cylinders $[0, r] \times S^1$ with n disjoint discs in the interior. The operations of type (3.6) consist of inserting little discs into the discs on the cylinder. The operations of type (3.7) consist of putting one cylinder on top of the other.

As for $\mathrm{Calc}_{\mathrm{geom}}$, $\mathcal{O}$ is the little discs operad and $\mathcal{M}$ is the configuration space, up to dilations and horizontal rotations, of cylinders $[0, r] \times S^1$ with a marked point on each component of the boundary and with n disjoint discs in the interior. The operations of type (3.6) consist of inserting little discs into the discs on the cylinder. The operations of type (3.7) consist of aligning marked points and then putting one cylinder on top of the other.

Now let us indicate how one defines $\mathrm{Calc}^0_{\mathrm{alg}}$ and $\mathrm{Calc}_{\mathrm{alg}}$. For them, $\mathcal{O} = G_{\mathrm{alg}}$. To describe $\mathcal{M}$, consider all universal operations $C^\bullet(A)^{\otimes n} \otimes C_\bullet(A) \to C_\bullet(A)$ which are linear combinations of iterated compositions of the operations from G_{alg} with the following elementary operations:

$$a_0 \otimes \ldots \otimes a_p \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_p$$

$$a_0 \otimes \ldots \otimes a_p \mapsto a_p \otimes a_0 \otimes \ldots \otimes a_{p-1}$$

$$a_0 \otimes \ldots \otimes a_p \mapsto a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_p$$

$$(D,\ a_0 \otimes \ldots \otimes a_p) \mapsto a_0 \otimes D(a_1, \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_p$$

(As in 3.1, to make it correct, one has to work with non-normalized chains and cochains).

For the two-colored operad $\mathrm{Calc}^0_{\mathrm{alg}}$, the $\mathcal{M}$ part consists only of the operations for which the term containing a_0 remains on the position number zero.

For example, the operation

$$(D_1, D_2, D_3, a_0 \otimes \ldots \otimes a_7) \mapsto D_1(a_7, a_0)D_3(a_1, D_2(a_2, a_3))a_4 \otimes a_5 a_6$$

is in $\mathcal{M}$ for both $\mathrm{Calc}_{\mathrm{alg}}$ and $\mathrm{Calc}^0_{\mathrm{alg}}$; the operation

$$(D_1, D_2, a_0 \otimes \ldots \otimes a_5) \mapsto a_1 a_2 \otimes a_3 \otimes D_2(a_4, D_1(a_5), a_0))$$

is in $\mathcal{M}$ for $\mathrm{Calc}_{\mathrm{alg}}$ but not for $\mathrm{Calc}^0_{\mathrm{alg}}$ (the term containing a_0 is on the position number two); the operation

$$(D_1, D_2, D_3, a_0 \otimes \ldots \otimes a_5) \mapsto D_1(a_5 a_0)a_1 \otimes a_2 D_3(a_4, D_2(a_3))$$

is not in $\mathcal{M}$ for either (because the cyclic order of a_i 's is broken).

As in 3.1, a minimal version in the manner of [**KS**] can be defined (and is necessary for the current version of the proof of the main theorem). We will discuss it in full in a more detailed exposition.

Finally, put

$$\mathcal{C}_{\mathrm{alg}} = \mathrm{Bar}\,\mathrm{Cobar}(\mathrm{Calc}_{\mathrm{alg}})$$

Adapting the arguments from [**MS**], [**T1**] or from [**KS**] to our purposes, one proves that the following chain of quasi-isomorphisms extends that of 3.1:

$$\mathrm{Calc}_\infty \xrightarrow{f_1} \mathrm{Calc}_{\mathrm{alg}} \xleftarrow{g_1} \mathcal{C}_{\mathrm{alg}} \xrightarrow{g_2} C_\bullet(\mathrm{Calc}_{\mathrm{geom}}) \xrightarrow{F} \mathrm{Calc} \tag{3.9}$$

and similarly for Calc^0. As a corollary, one gets Theorem 3.4.1.

3.7. Enveloping algebra of an algebra over a two-colored operad. Let $\mathcal{P} = (\mathcal{O}, \mathcal{M})$ be a two-colored dg operad. For an $\mathcal{O}$-algebra $\mathcal{A}^\bullet$ put

$$Y_{\mathcal{P}}(\mathcal{A}^\bullet) = \oplus_{n>0} {\mathcal{A}^\bullet}^{\otimes n} \otimes_{k[S_n]} \mathcal{M}(n)/\sim$$

where the equivalence relation $\sim$ is generated by the following:

for $a_1, \ldots, a_n \in \mathcal{A}^\bullet$, $m \in \mathcal{M}(k)$, $o \in \mathcal{O}(l)$, $k + l - 1 = n$, and for all i

$$(a_1 \otimes \ldots \otimes a_n) \otimes (o \circ_i m) \sim (a_1 \otimes \ldots \otimes a_{i-1} \otimes o(a_i, \ldots, a_{i+l-1}) \otimes \ldots \otimes a_n) \otimes m$$

taken with the appropriate sign, where $\circ_i : \mathcal{O}(l) \otimes \mathcal{M}(k) \to \mathcal{M}(n)$ is the ith elementary composition.

The operations $\mathcal{M}(k) \otimes \mathcal{M}(l) \to \mathcal{M}(k + l)$ turn $Y_{\mathcal{P}}(\mathcal{A}^\bullet)$ into an associative algebra.

Example 3.7.1. . Let $\mathcal{P} = (\mathrm{As}, \mathcal{M})$ be the two-colored operad algebras over which are pairs (A, B), where A is an associative algebra and B is a left A-module (resp. a right module, resp. a bimodule). Then $Y_{\mathcal{P}}(A) = A$ (resp. A^{op}, resp. $A \otimes A^{\mathrm{op}}$).

Example 3.7.2. . Let $\mathcal{P} = (\mathrm{Lie}, \mathcal{M})$ be the two-colored operad algebras over which are pairs $(\mathfrak{g}, B)$ where $\mathfrak{g}$ is a Lie algebra and B is a $\mathfrak{g}$-module. Then $Y_{\mathcal{P}}(\mathfrak{g}) = U(\mathfrak{g})$.

Example 3.7.3. Let $\mathcal{P} = \mathrm{Calc}^0$. Then, for a Gerstenhaber algebra $\mathcal{A}^\bullet$, $Y_{\mathcal{P}}(\mathcal{A}^\bullet) = Y(\mathcal{A}^\bullet)$.

EXAMPLE 3.7.4. Let $\mathcal{P} = \mathrm{Calc}$. Denote by $\mathfrak{a}$ a one-dimensional Abelian graded Lie algebra concentrated in degree one. This algebra acts on $Y(\mathcal{A}^\bullet)$ by derivations, the generator acting by d. One can form a cross product $U(\mathfrak{a}) \ltimes Y(\mathcal{A}^\bullet)$. For a Gerstenhaber algebra $\mathcal{A}^\bullet$,

$$Y_{\mathcal{P}}(\mathcal{A}^\bullet) \simeq U(\mathfrak{a}) \ltimes Y(\mathcal{A}^\bullet).$$

3.8. The A_∞ module structure on Hochschild chains. Recall the definition of A_∞ modules over A_∞ algebras. First, note that for a graded space $\mathcal{M}$, the Gerstenhaber bracket $[\ ,\ ]$ can be extended to the space

$$\mathrm{Hom}(\overline{\mathcal{C}}^{\otimes\bullet}, \mathcal{C}) \oplus \mathrm{Hom}(\mathcal{M} \otimes \overline{\mathcal{C}}^{\otimes\bullet}, \mathcal{M})$$

For a graded k-module $\mathcal{M}$, a structure of an A_∞ module over an A_∞ algebra $\mathcal{C}$ on $\mathcal{M}$ is a cochain

$$\mu = \sum_{n=1}^{\infty} \mu_n$$

$$\mu_n \in Hom(\mathcal{M} \otimes \overline{\mathcal{C}}^{\otimes n-1}, \mathcal{M})$$

such that

$$[m + \mu, m + \mu] = 0$$

THEOREM 3.8.1. *On $C_\bullet(A)[[u]]$, there exists a structure of an A_∞ module over the A_∞ algebra $C_\bullet(C^\bullet(A))[[u]]$ such that:*

- *All μ_n are $k[[u]]$-linear, (u)-adically continuous*
- *$\mu_1 = b + uB$ on $C_\bullet(A)[[u]]$*
  For $a \in C_\bullet(A)[[u]]$:
- *$\mu_2(a, D) = (-1)^{|a||D|+|a|}(i_D + uS_D)a$*
- *$\mu_2(a, 1 \otimes D) = (-1)^{|a||D|} L_D a$*
  For $a,\ x \in C_\bullet(A)[[u]]$:

$$(-1)^{|a|}\mu_2(a, x) = (\mathrm{sh} + u\,\mathrm{sh}')(a, x)$$

To obtain formulas for the structure of an A_∞ module from Theorem 3.8.1, one has to assume that, in the formulas for the A_∞ structure from Theorem 2.6.1, all $D_j^{(1)}$ are elements of A; then one has to replace braces { } by the usual parentheses ( ) symbolizing evaluation of a multi-linear map at elements of A.

3.9. The algebra $Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(C^\bullet(A))$ and the Hochschild complex. Note that the construction from 3.8 can be interpreted as existence of an A_∞ morphism

$$C_\bullet(C^\bullet(A)) \to Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(C^\bullet(A)) \tag{3.10}$$

From (3.9) one gets an algebra homomorphism

$$Y_{\mathrm{Calc}^0_\infty}(C^\bullet(A)) \to Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(C^\bullet(A)) \tag{3.11}$$

It is easy to show that both maps

$$Y_{\mathrm{Calc}^0_\infty}(C^\bullet(A)) \leftarrow Y_{\mathrm{Calc}^0_\infty}(\mathcal{V}^\bullet(A)) \to Y_{\mathrm{Calc}^0}(\mathcal{V}^\bullet(A)) \tag{3.12}$$

are quasi-isomorphisms. To prove the first part of Theorem 2.7.1, it remains to show that (3.11) and (3.10) are quasi-isomorphisms under our assumptions.

3.10. The algebra $Y_{\mathrm{Calc}^0_\infty}(C^\bullet(A))$ and the Hochschild complex. To show that (3.11) is a quasi-isomorphism, we introduce the following filtrations: on $C_\bullet(C^\bullet(A))$, let

$$\mathrm{filt}_n = C_{\leq n}(C^\bullet(A)) = C^\bullet(A) \otimes \overline{C^\bullet(A)}^{\otimes \leq n} \tag{3.13}$$

On $Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(C^\bullet(A))$ and $Y_{\mathrm{Calc}^0_\infty}(C^\bullet(A))$, let

$$\mathrm{filt}_n = \sum_{m \leq n} \mathcal{Q}(m) \otimes_{k[S_m]} C^\bullet(A)^{\otimes m} / \sim \tag{3.14}$$

It is easy to see that (3.10) preserves the filtration and therefore induces an isomorphism of associated graded quotients. The morphism (3.11) preserves the filtration by definition. At the level of associated graded quotients, (3.11) induces a morphism of complexes

$$Y_{\mathrm{Calc}^0_\infty}(H^\bullet(A)) \to C_\bullet(H^\bullet(A)) \tag{3.15}$$

where, in the left hand side, $H^\bullet(A)$ is the Hochschild cohomology, viewed as a Gerstenhaber algebra on which all the operations are zero. The fact that (3.15) is an isomorphism follows from Proposition 4.2.4 in the Appendix.

3.11. Another proof of quasi-isomorphicity of the map (3.11). We are going to prove a slightly more general statement.

Proposition 3.11.1. *Let U be an arbitrary G_{alg}-algebra. Then the map $Y_{\mathrm{Calc}^0_\infty}(U) \to Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(U)$ constructed in the same way as the map (3.11) is a quasi-isomorphism.*

This proof is based on two facts. The first one is that the map of colored operads f_1 from (3.9) is a quasi-isomorphism. The second fact says that M^0_{alg} is free over G_{alg}. This means that one can choose S_n-equivariant subspaces $E(n) \subset M^0_{\mathrm{alg}}(n)$ (for any n) in such a way that for all N the insertion map

$$\bigoplus_{n, M_1+M_2+\cdots+M_n=N} M^0_{\mathrm{alg}}(n) \otimes_{S_n} G_{\mathrm{alg}}(M_1) \otimes \cdots \otimes G_{\mathrm{alg}}(M_n) \otimes_{S_{M_1}\times\cdots\times S_{M_n}} k[S_N] \to M^0_{\mathrm{alg}}(N)$$

is an isomorphism. This fact follows from the explicit construction of $\mathrm{Calc}_{\mathrm{alg}}$.

Having these two facts, we prove the Proposition as follows. First, let us translate the definition of a universal enveloping algebra into the language of PROPs. Let $(\mathcal{O}, M)$ be a two-colored operad and U be an $\mathcal{O}$-algebra. Let $P_{\mathcal{O}}$ be the PROP generated by $\mathcal{O}$. Then the structure maps

$$\mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_m) \otimes M(m) \to M(n_1 + \cdots + n_m)$$

endow M with a structure of a functor $M' : P^{\mathrm{op}}_{\mathcal{O}} \to$ Complexes, where for $[n] \in \mathrm{Ob}P_{\mathcal{O}}$, $M'([n]) = M(n)$. Further on, we will denote M' by M. Put $F_U([n]) = U^{\otimes n}$. Then, since U is a $P_{\mathcal{O}}$-algebra, F_U is a functor $P_{\mathcal{O}} \to$ Complexes.

We see that

$$Y_{\mathcal{O},M}(U) \cong F_U \otimes_{P_{\mathcal{O}}} M,$$

where on the right hand side we use the MacLane tensor product.

We have a quasi-isomorpism $f_1 : \mathrm{Calc}^0_\infty \to \mathrm{Calc}^0_{\mathrm{alg}}$. It produces a symmetric monoidal functor $F : P_{G_\infty} \to P_{G_{\mathrm{alg}}}$ which induces a quasi-isomorphism on the spaces of homomorphisms; also, f_1 gives rise to a natural transformation $G : M^0_\infty \to M^0_{\mathrm{alg}} \circ F$ which is a quasi-isomorphism.

Let us come back to our G_{alg}-algebra U. Let U' be U considered as a G_∞-algebra, where the corresponding structure is induced by f_1. We have:

$$F_{U'} \cong F_U \circ F. \tag{3.16}$$

In this light, the map $Y_{\mathrm{Calc}^0_\infty}(U') \to Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(U)$, which coincides with the map (3.11), is described as the composition:

$$M^0_\infty \otimes_{P_{G_\infty}} F_{U'} \to M^0_{\mathrm{alg}} \circ F \otimes_{P_{G_\infty}} F_U \circ F \to M^0_{\mathrm{alg}} \otimes_{P_{G_{\mathrm{alg}}}} F_U, \tag{3.17}$$

where the first map is induced by G and (3.16).

We need to show that this composition produces a quasi-isomorphism. To this end, we introduce a functor $F_!$ from the category of functors $P_{G_\infty} \to$ Complexes to the category of functors $P_{G_{\mathrm{alg}}} \to$ Complexes. Denote $h_{[m]}([n]) = \hom_{P_{G_{\mathrm{alg}}}}([n],[m])$. Each $h_{[m]}$ is a functor $P^{\mathrm{op}}_{G_{\mathrm{alg}}} \to$ Complexes. For a functor $N : P_{G_\infty} \to$ Complexes set $F_!N([m]) := N \otimes_{P_{G_\infty}} h_{[m]}$. We have canonical isomorphisms

$$\hom_{P_{G_{\mathrm{alg}}}}(F_!N, M) \cong \hom_{P_{G_\infty}}(N, M \circ F)$$

for any $M : P_{G_{\mathrm{alg}}} \to$ Complexes; and

$$F_!M \otimes_{P_{G_{\mathrm{alg}}}} L \cong M \otimes_{P_{G_\infty}} (L \circ F).$$

In particular, the map G induces a map $G' : F_!M^0_\infty \to M^0_{\mathrm{alg}}$. Since F is a quasi-isomorphism and M^0_∞ is semi-free and has a set of generators centered in non-positive degrees, it follows that G' is a quasi-isomorphism. One sees that the map in (3.17) can be rewritten as the following composition:

$$M^0_\infty \otimes_{P_{G_\infty}} F_{U'} \cong M^0_\infty \otimes_{P_{G_\infty}} F_U \circ F \cong F_!M^0_\infty \otimes_{P_{G_{\mathrm{alg}}}} F_U \to M^0_{\mathrm{alg}} \otimes_{P_{G_{\mathrm{alg}}}} F_U.$$

where the second map is induced by G'. Since G' is a quasi-isomorphism and M^0_{alg} is semi-free and has a set of generators centered in non-positive degrees,the second map is a quasi-isomorphism, therefore, the whole composition is a quasi-isomorphism.

3.12. (3.10) is a quasi-isomorphism. Again, we we will replace $C^\bullet(A,A)$ with an arbitrary G_{alg}-algebra U. U has an associative cup-product, and we can form the Hochschild chain complex $C_\bullet(U)$ with respect to this product. The map (3.10) is then generalized to a map $\phi := \phi_U : C_\bullet(U) \to Y_{\mathrm{Calc}^0}(U)$, and we need to show that this map is a quasi-isomorphism. We are going to make a couple of reductions. Firstly, we can use the filtration as in (3.13) by the number of tensor factors of U. Then it suffices to check that ϕ induces a quasi-isomorphism of the associated graded spaces. This implies that it suffices to consider the case in which all operations on U vanish. To prove this statement we need one more reduction. Note that any commutative algebra B can be considered as a G_{alg}-algebra in which the cup product is commutative and all braces vanish. Denote thus obtained G_{alg}-algebra by B'. We have $U = B'$, where B has zero product. We now want to reduce this case to the case in which $U = (SV)'$, where SV is a free commutative algebra. To this end we notice that for B having zero product, there exists a semi-free resolution $p : SW \to B$ having the property that SW can be endowed with an increasing exhausting filtration F such that the associated graded algebra $\mathrm{Gr}\, SW$ is free. Since the functors $C_\bullet$ and Y_{Calc^0} preserve quasi-isomorphisms, it suffices to show that ϕ_{SW} is a quasi-isomorphism; using the filtration induced by F, we reduce the statement to quasi-isomorphicity of $\phi_{\mathrm{Gr}\, SW}$, where $\mathrm{Gr}\, SW$ is free.

Thus, from now on we set $U = (SV)'$. In this case we have the Kostant-Hochschild-Rosenberg quasi-isomorphism $\omega : SV \otimes S(V[1]) \to C_\bullet(SV)$. Compute the composition $\omega\phi : SV \otimes S(V[1]) \to Y_{\mathrm{Calc}^0}(SV)$. To this end, we are going to use the elements $i, L \in M^0_{\mathrm{alg}}(1)$ defined in (3.2.1). Then it is easy to check that $\omega\phi$ is homotopy equivalent to the map ψ, which on an element $u \otimes (a_1 \wedge \cdots \wedge a_n)$, where $u \in SV$ and $a_i \in V$, takes the value

$$i_u \mathrm{Alt}(L_{a_{i_1}} L_{a_{i_2}} ... L_{a_{i_n}}),$$

where Alt means alternation.

Let us now use the quasi-isomorphisms $f_1 : \mathrm{Calc}^0_\infty \to \mathrm{Calc}^0_{\mathrm{alg}}$ from (3.10) and $f : Y_{\mathrm{Calc}^0_\infty}(SV) \to Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(SV)$ as in (3.11). Let $I' \in M^0_\infty(1)$, $L' \in M^0_\infty(1)$ be such that $f_1(I')$ (resp. $f_1(L')$) is homologous in $M^0_{\mathrm{alg}}(1)$ to i (resp. L).

Define a map $\chi : SV \otimes S(V[1]) \to Y_{\mathrm{Calc}^0_\infty}(SV)$ by setting

$$\chi(u \otimes (a_1 \wedge \cdots a_n)) = I'_u \mathrm{Alt}(L'_{a_{i_1}} L'_{a_{i_2}} ... L'_{a_{i_n}}).$$

We see that ψ is homotopy equivalent to $f\chi$. Hence, our task is to show that χ is a quasi-isomorphism. To this end we will use the natural map $r : Y_{\mathrm{Calc}^0_\infty}(SV) \to Y_{\mathrm{Calc}^0}(SV)$ induced by the map of calculi $\mathrm{Calc}^0_\infty \to \mathrm{Calc}^0$. It is easy to see that this map is a quasi-isomorphism and that $r\chi$ is an isomorphism, therefore χ is a quasi-isomorphism, hence the statement.

3.13. The algebra $Y_{\mathrm{Calc}_\infty}(C^\bullet(A))$ and the cyclic complex. To adapt the above arguments to the cyclic case, notice first that one can interpret the cyclic part of Theorem 2.6.1 as follows: there is an L_∞ action of the graded Lie algebra $\mathfrak{a}$ (cf. 3.7.4) on the A_∞ algebra $C_\bullet(C^\bullet(A))$, and one can form a corresponding A_∞ algebra cross product $U(\mathfrak{a}) \ltimes C_\bullet(C^\bullet(A))$. One of the ways to explain this is the following. The A_∞ algebra $C_\bullet(C^\bullet(A))$ is quasi-isomorphic to a dg associative algebra $\mathcal{R}$; the free dg Lie algebra resolution $\mathcal{L}$ of $\mathfrak{a}$ acts on $\mathcal{R}$ by derivations; form a cross product $U(\mathcal{L}) \ltimes \mathcal{R}$. As a complex, it is quasi-isomorphic to $U(\mathfrak{a}) \otimes C_\bullet(C^\bullet(A))$; from this, one can recover the A_∞ structure on the latter. For example, the binary product of the generator of $\mathfrak{a}$ with an element $c \in C_\bullet(C^\bullet(A))$ is Bc where B is the cyclic differential.

By Example 3.7.4,

$$Y_{\mathrm{Calc}}(\mathcal{V}^\bullet(A)) \simeq U(\mathfrak{a}) \ltimes Y(\mathcal{V}^\bullet(A))$$

One extends (3.10) to a quasi-isomorphism

$$U(\mathfrak{a}) \ltimes C_\bullet(C^\bullet(A)) \to Y_{\mathrm{Calc}_{\mathrm{alg}}}(C^\bullet(A)) \tag{3.18}$$

One can also construct an A_∞ quasi-isomorphism

$$U(\mathfrak{a}) \ltimes Y_{\mathrm{Calc}^0_{\mathrm{alg}}}(C^\bullet(A)) \to Y_{\mathrm{Calc}_{\mathrm{alg}}}(C^\bullet(A))$$

Both maps

$$Y_{\mathrm{Calc}_\infty}(C^\bullet(A)) \leftarrow Y_{\mathrm{Calc}_\infty}(\mathcal{V}^\bullet(A)) \to Y_{\mathrm{Calc}}(\mathcal{V}^\bullet(A)) \tag{3.19}$$

are quasi-isomorphisms. Thus, we have an A_∞ quasi-isomorphism from $U(\mathfrak{a}) \ltimes Y(\mathcal{V}^\bullet(A))$ to $U(\mathfrak{a}) \ltimes C_\bullet(C^\bullet(A))$. From this it is easy to deduce the statement of Theorem 2.7.1 regarding the cyclic complexes.

4. Appendix. Hochschild-Gerstenhaber homology

4.1. Introductory remarks. Let $\mathcal{A}^\bullet$ be a Gerstenhaber algebra (or, more generally, a G_∞ algebra). In this section we will construct $HG_\bullet(\mathcal{A}^\bullet)$, a new homology functor of $\mathcal{A}^\bullet$. It is defined by means of an explicit complex and is a limit of a spectral sequence whose E_1 term is the Hochschild homology of the graded associative algebra $\mathcal{A}^\bullet$. This spectral sequence degenerates at E_1 in important partial cases. The homology $HG_\bullet(\mathcal{A}^\bullet)$ is an associative algebra. When the spectral sequence does degenerate, one can view the associative algebra $HG_\bullet(\mathcal{A}^\bullet)$ as a deformation of the graded commutative algebra $H_\bullet(\mathcal{A}^\bullet)$ with the shuffle product sh, cf. (2.3).

One can extend the definition of $HG_\bullet$ and define the negative cyclic Gerstenhaber homology $HGC_\bullet^-(\mathcal{A}^\bullet)$. It relates to $HG_\bullet$ exactly as the Hochschild homology to the negative cyclic homology.

4.2. Hochschild-Gerstenhaber homology vs Hochschild homology. To define $HG_\bullet(\mathcal{A}^\bullet)$, recall the two-colored operad Calc^0 and its canonical free resolution Calc^0_∞ from 3.6.

Definition 4.2.1. *For a G_∞ algebra $\mathcal{A}^\bullet$, let $HG_\bullet(\mathcal{A}^\bullet)$ be the homology of the complex $Y_{\mathrm{Calc}^0_\infty}(\mathcal{A}^\bullet)$.*

Let us start by realizing $Y_{\mathrm{Calc}^0_\infty}(\mathcal{A}^\bullet)$ as an explicit complex. Recall from 3.1 that a G_∞ algebra structure on $\mathcal{A}^\bullet$ determines a derivation δ of the Gerstenhaber algebra $\mathcal{F}^\bullet(\mathcal{A}^\bullet) = \wedge^\bullet \mathrm{Lie}(\mathcal{A}^\bullet[1]^*)$. (Here and in all the computations below, as usual, one has to understand the duals properly; cf. Remark 3.1.1). For an associative augmented dg algebra Y, we denote by $\mathrm{Bar}(Y)$ its standard bar (bi)complex which computes $\mathrm{Ext}^\bullet(k,k)$.

Proposition 4.2.2. *There is a natural isomorphism of complexes*

$$Y_{\mathrm{Calc}^0_\infty}(\mathcal{A}^\bullet) \to \mathrm{Bar}(Y(\wedge^\bullet \mathrm{Lie}(\mathcal{A}^\bullet[1]^*)))$$

This can be seen directly from the definitions. The right hand side in (4.2.2) stands for the enveloping algebra of the dg Gerstenhaber algebra. It is an associative dg algebra, with the differential induced by δ.

In what follows, A_∞ is a Gerstenhaber algebra.

Theorem 4.2.3. *There is a natural spectral sequence converging to $HG_\bullet(\mathcal{A}^\bullet)$ for which*

$$E_1 = E_2 = H_\bullet(\mathcal{A}^\bullet),$$

the Hochschild homology of the graded algebra A_∞.

An important partial case is the following:

Proposition 4.2.4. *Let A_∞ is a Gerstenhaber algebra on which both operations are zero. Then*

$$HG_\bullet(\mathcal{A}^\bullet) \xrightarrow{\sim} H_\bullet(\mathcal{A}^\bullet) = C_\bullet(\mathcal{A}^\bullet)$$

Proof of Theorem 4.2.2. Introduce a filtration on the standard complex $Y_{\mathrm{Calc}^0_\infty}(\mathcal{A}^\bullet)$: note that $Y(\wedge^\bullet \mathrm{Lie}(\mathcal{A}^\bullet[1]^*))$ is graded by the number of generators i_a occurring in a monomial, and let F_n consist of all those cochains that annihilate all elements of degree greater than n.

Proposition 4.2.5. *The spectral sequence associated to the filtration F has the properties as in Theorem 4.2.3.*

Denote

$$\mathcal{L}(\mathcal{A}^\bullet) = \mathrm{Lie}(\mathcal{A}^\bullet[1]^*)$$

To prove the above Proposition, let us start by reinterpreting $Y(\wedge^\bullet \mathcal{L}(\mathcal{A}^\bullet))$.

Let ϵ be a formal parameter of degree -1 and square zero. Consider the dg Lie algebra $\mathcal{L}(\mathcal{A}^\bullet)[\epsilon]$. The universal enveloping algebra $U(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon])$ admits a derivation which is characterized by the following. On $U(\mathcal{L}(\mathcal{A}^\bullet)\epsilon) = \wedge^\bullet(\mathcal{L}(\mathcal{A}^\bullet))$, this is just the derivation δ determined by the G_∞ structure; and it is the only such derivation which commutes with the derivation induced by $\frac{\partial}{\partial\epsilon}$. Thus, $U(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon])$ becomes a dg associative algebra.

Lemma 4.2.6. *There is a natural isomorphism of dg algebras*

$$Y(\wedge^\bullet(\mathcal{L}(\mathcal{A}^\bullet))) \to U(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon])$$

The proof is straightforward.

Now we have an identification

$$Y_{\mathrm{Calc}^0_\infty}(\mathcal{A}^\bullet) \simeq \mathrm{Bar}(U(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon]))$$

Under this identification, the filtration F becomes as follows. Note that $U(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon])$ is graded by the number of factors ϵ in a monomial, and F_n consists of cochains annihilating all elements of degree greater than n.

In the situation of Proposition 4.2.4, the differential on $U(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon])$ is zero. So, to finish the proof of Proposition 4.2.5, and therefore of Theorem 2.7.1, on can go directly to Lemma 4.2.10.

Note that the graded Lie algebra $\mathcal{L}(\mathcal{A}^\bullet)$ possesses a Lie algebra derivation determined by the commutative product on $\mathcal{A}^\bullet$ (or, in the general G_∞ case, from the C_∞ structure on $\mathcal{A}^\bullet$). Thus $\mathcal{L}(\mathcal{A}^\bullet)$, as well as $\mathcal{L}(\mathcal{A}^\bullet)[\epsilon]$, becomes a dg Lie algebra. One sees easily that the following is true.

Lemma 4.2.7. *The first term of the spectral sequence associated to the filtration F is equal to $H^\bullet(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon])$, the cohomology of the differential graded Lie algebra with trivial coefficients.*

Lemma 4.2.8. *In the above spectral sequence $E_2 = E_1$.*

To prove this, we need some notation. Let $\mathfrak{g}$ be a dg Lie algebra. By PBW, there is an $\mathrm{ad}(\mathfrak{g})$- invariant isomorphism $U(\mathfrak{g}) \to S(\mathfrak{g})$. For any i, let $U^{>i}(\mathfrak{g})$ be the pre-image of $S^{>i}(\mathfrak{g})$ under this isomorphism. In particular, $U^{>1}(\mathfrak{g})$ is a $\mathrm{ad}(\mathfrak{g})$-complement of $\mathfrak{g}$ in the augmentation ideal.

The differential on $\mathfrak{g}$ induces a differential on $U(\mathfrak{g})$. Denote this differential by δ. Let δ_1 be another derivation of degree one of $U(\mathfrak{g})$. Assume that, for any $X \in \mathfrak{g}$,

$$\delta X - \delta_1 X \in U^{>1}(\mathfrak{g}) \tag{4.1}$$

Since the cohomology $H^\bullet(\mathfrak{g})$ is computed by the bar complex $\mathrm{Bar}(U(\mathfrak{g}))$, the new differential δ_1 acts on $H^\bullet(\mathfrak{g})$.

Lemma 4.2.9. *Under the assumption (4.1), the action of δ_1 on $H^\bullet(\mathfrak{g})$ is trivial.*

Proof. Recall how the isomorphism between $H^\bullet(\mathfrak{g})$ and the cohomology of the bar complex is constructed. One has two standard resolutions of the trivial $U(\mathfrak{g})$-module k: The bar resolution $\mathrm{Bar}_n = U(\mathfrak{g}) \otimes \overline{U(\mathfrak{g})}^{\otimes n}$ and the Koszul resolution

$K_n = U(\mathfrak{g}) \otimes \wedge^n(\mathfrak{g})$. One has the standard embedding of complexes $i : K_\bullet \to \mathrm{Bar}_\bullet$ defined by

$$f \otimes (x_1 \wedge \ldots \wedge x_n) \mapsto f \otimes \mathrm{Alt}(x_1 \otimes \ldots \otimes x_n)$$

for $f \in U(\mathfrak{g})$ and $x_i \in \mathfrak{g}$. By the standard techniques of homological algebra, one can split this embedding and construct a projection $j : \mathrm{Bar}_\bullet \to K_\bullet$, so that $ji = 1$ and ij is homotopic to the identity as a morphism of complexes of $U(\mathfrak{g})$-modules. It is not difficult to see that one can pick j that annihilates modulo $U^{>0}(\mathfrak{g}) \cdot K_\bullet$ all elements $f \otimes \mathrm{Alt}(x_1 \otimes \ldots \otimes x_n)$ where $f \in U(\mathfrak{g})$, $x_1 \in U^{>1}(\mathfrak{g})$, and $x_i \in \mathfrak{g}$ for $i > 1$. But the differential induced by δ_1 sends the image of i precisely to such elements. This shows that δ_1 acts by zero on the cohomology of the complex $\mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{Bar}_\bullet, k)$ which computes $H^\bullet(\mathfrak{g})$.

It remains to show that

LEMMA 4.2.10. *For a unital Gerstenhaber algebra* $\mathcal{A}^\bullet$

$$H^\bullet(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon]) \simeq H_\bullet(\mathcal{A}^\bullet)$$

Proof. First, $C^\bullet(\mathcal{L}(\mathcal{A}^\bullet)[\epsilon])$ is isomorphic to $C^\bullet(\mathcal{L}(\mathcal{A}^\bullet), S^\bullet(\mathcal{L}(\mathcal{A}^\bullet))^*)$, the standard cochain complex with coefficients in $S^\bullet(\mathcal{L}(\mathcal{A}^\bullet))^*$. Since $\mathcal{L}(\mathcal{A}^\bullet)$ is free as a graded Lie algebra, the embedding into
$C^\bullet(\mathcal{L}(\mathcal{A}^\bullet), S^\bullet(\mathcal{L}(\mathcal{A}^\bullet))^*)$ of the subcomplex

$$C^o \to \mathrm{Ker}(C^1 \to C^2) \tag{4.2}$$

is a quasi-isomorphism where

$$C^i = \mathrm{Hom}(\wedge^i(\mathcal{L}(\mathcal{A}^\bullet)), S^\bullet(\mathcal{L}(\mathcal{A}^\bullet))^*)$$

Finally, the complex (4.2) can be written explicitly: let, as before, $C_\bullet(\mathcal{A}^\bullet)$ be the usual Hochschild complex, and let $C'_\bullet(\mathcal{A}^\bullet)$ be the same complex equipped with the bar differential b', cf. [**L**]. Then one can identify C_0 with $C'_\bullet(\mathcal{A}^\bullet)$, $\mathrm{Ker}(C^1 \to C^2)$ with $C_\bullet(\mathcal{A}^\bullet)$, and the differential between the two with the map $1-t$ from [**L**]. The identification is done as follows:

$$S(\mathcal{L}(\mathcal{A}^\bullet)) \xrightarrow{PBW} U(\mathcal{L}(\mathcal{A}^\bullet)) \simeq T(\mathcal{A}^\bullet[1]^*);$$

therefore

$$C^0 \simeq T(\mathcal{A}^\bullet[1]) = C'_\bullet(\mathcal{A}^\bullet)$$

and

$$\mathrm{Ker}(C^1 \to C^2) \simeq \mathcal{A}^\bullet[1] \otimes T(\mathcal{A}^\bullet[1]) \simeq C_\bullet(\mathcal{A}^\bullet)[1]$$

Since $\mathcal{A}^\bullet$ is unital, $C'_\bullet(\mathcal{A}^\bullet)$ is contractible, and the theorem is proven.

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