What do DG-categories form?

Dmitry Tamarkin

Abstract

We introduce a homotopy 2-category structure on the collection of dg-categories, dg-functors, and their derived transformations. This construction provides for a conceptual proof of Deligne's conjecture on Hochschild cochains.

1. Introduction

It is well known that categories form a 2-category: 1-arrows are functors and 2-arrows are their natural transformations.

In a similar way, dg-categories also form a 2-category: 1-arrows $A \to B$ are dg-functors; given a pair of dg-functors $F, G : A \to B$ one can define a complex of their natural transformations $\mathrm{hom}(F, G)$ which naturally generalizes the notion of a natural transformation in the usual setting. Thus, we can use $\mathrm{hom}(F, G)$ as the space of 2-arrows[1].

However, this construction has a serious drawback: the spaces $\mathrm{hom}(F, G)$ are not homotopically invariant in any way. For example: let $W : B \to C$ be a weak equivalence of dg-categories, we then have a natural map

$$\mathrm{hom}(F, G) \to \mathrm{hom}(WF, WG)$$

which, in general, is not a quasi-isomorphism of complexes.

Drinfeld [Dri04] proposes another construction, in which the role of dg-functors $A \to B$ is played by $A^{\mathrm{op}} \times B$-bi-modules. By choosing an appropriate class of such bi-modules, one can achieve a good homotopy behavior. Unfortunately, this class does not contain identity functors $A \to A$ but only their resolutions which satisfy the properties of identity only up to homotopies.

The goal of this paper is to provide for a homotopy invariant structure on the category of dg-category which, on one hand, would be as close to the 2-category structure as possible; on the other hand, this structure should be free of the above mentioned drawbacks.

In order to achieve a homotopy invariant behavior, one has to pass to a derived version of the notion of a natural transformation between two functors. This can be done in a very well known way (see Sec. 3.0.2 and 3.1)

As it is common in such situations, these derived transformations of functors cannot be composed as nicely as the usual transformations of functors do ..., so they don't form a 2-category.

Our result is that, informally speaking, derived natural transformations form a certain homotopy version of 2-category. Let us now sketch the idea of the notion of a homotopy 2-category, precise definitions will be given in the main body of the paper.

A good starting point is to reformulate the definition of a dg-2-category as follows:

A small 2-category is

2000 Mathematics Subject Classification 18D05, 18D50, 55U10, 55U40

Keywords: differential graded category, 2-operad, multisimplicial set

[1] As usual, a *dg-category* is by definition a category enriched over the symmetric monoidal category of complexes over a fixed ground field k.

1) a set of objects C, a set of 1-arrows $\hom(X, Y)$ for every pair $X, Y \in \mathcal{C}$. These data should form a usual category;

2) a complex of 2-arrows $\hom(F, G)$ for all 1-arrows $F, G \in \hom(A, B)$.

These data should have the following structure:

3) given objects $A_0, A_1, \ldots, A_n$ and 1-arrows $F_{ij} : A_i \to A_{i+1}$, $i = 0, 1, \ldots, n-1$; $j = 0, 1, \ldots, m_i$, one should have a composition map

$$c : \bigotimes_{ij} \hom(F_{ij}; F_{i,j+1}) \to \hom(F, G), \tag{1}$$

where $F, G : A_0 \to A_n$,

$$F = F_{n-1,0}F_{n-2,0} \cdots F_{10}F_{00}, \text{ and}$$
$$G = G_{n-1,m_{n-1}}G_{n-1,m_{n-2}} \cdots G_{1m_1}G_{0m_0}.$$

There should be a certain coherence axiom saying that these compositions are closed under iterations. Instead of giving a precise formulation, let us consider an example, as on the picture (2).

$$\begin{tikzcd}
A_0 \arrow[r, "F_{01}"] \arrow[r, bend left=30, "F_{02}"] \arrow[r, bend left=70, "F_{03}"] \arrow[r, bend right=50, "F_{00}"'] & A_1 \arrow[r, "F_{11}"] \arrow[r, bend left=50, "F_{12}"] \arrow[r, bend right=50, "F_{10}"'] & A_2 \arrow[r, "F_{20}"] & A_3 \arrow[r, "F_{32}"] \arrow[r, bend left=30, "F_{33}"] \arrow[r, bend left=70, "F_{34}"] \arrow[r, bend right=30, "F_{31}"'] \arrow[r, bend right=70, "F_{30}"'] & A_4
\end{tikzcd} \tag{2}$$

Let us split this picture into four sub-pictures as follows:

$$\begin{tikzcd}
A_0 \arrow[r, bend left=30, "F_{02}"] \arrow[r, bend left=70, "F_{03}"] & A_1 \arrow[r, "F_{11}"] \arrow[r, bend left=50, "F_{12}"] & A_2 & & A_2 \arrow[r, "F_{20}"] & A_3 \arrow[r, bend left=30, "F_{33}"] \arrow[r, bend left=70, "F_{34}"] & A_4 \\
A_0 \arrow[r, "F_{01}"] \arrow[r, bend left=50, "F_{02}"] \arrow[r, bend right=50, "F_{00}"'] & A_1 \arrow[r, "F_{11}"] \arrow[r, bend right=50, "F_{10}"'] & A_2 & & A_2 \arrow[r, "F_{20}"] & A_3 \arrow[r, "F_{32}"] \arrow[r, bend left=50, "F_{33}"] \arrow[r, bend right=30, "F_{31}"'] \arrow[r, bend right=70, "F_{30}"'] & A_4
\end{tikzcd} \tag{3}$$

These sub-pictures yield the following composition maps:

$$\hom(F_{02}, F_{03}) \otimes \hom(F_{11}, F_{12}) \to \hom(F_{11}F_{02}; F_{12}F_{03});$$
$$\hom(F_{33}, F_{34}) \to \hom(F_{33}F_{20}; F_{34}F_{20});$$
$$\hom(F_{00}, F_{01}) \otimes \hom(F_{01}, F_{02}) \otimes \hom(F_{10}, F_{11}) \to \hom(F_{10}F_{00}; F_{11}F_{02});$$
$$\hom(F_{30}, F_{31}) \otimes \hom(F_{31}, F_{32}) \otimes \hom(F_{32}, F_{33}) \to \hom(F_{30}F_{20}; F_{33}F_{20});$$

These maps can be composed with the composition map determined by the following "quotient-picture":

$$
\begin{tikzcd}
 & {} \arrow[dr, bend left, "F_{12}"] & & {} \arrow[dr, bend left, "F_{34}"] & \\
A_0 \arrow[ur, bend left, "F_{03}"] \arrow[r, "F_{02}"] \arrow[dr, bend right, "F_{00}"'] & {} \arrow[r, "F_{11}"] & A_2 \arrow[ur, "F_{20}"] \arrow[r, "F_{20}"] \arrow[dr, "F_{20}"'] & {} \arrow[r, "F_{33}"] & A_4 \\
 & {} \arrow[ur, bend right, "F_{10}"'] & & {} \arrow[ur, bend right, "F_{30}"'] &
\end{tikzcd}
\tag{4}
$$

$$
\begin{aligned}
&\hom(F_{11}F_{02}; F_{12}F_{03}) \otimes \hom(F_{10}F_{00}; F_{11}F_{02}) \otimes \hom(F_{30}F_{20}; F_{33}F_{20}) \hom(F_{33}F_{20}; F_{34}F_{20}) \\
&\qquad\qquad\qquad\qquad \to \hom(F_{30}F_{20}F_{10}F_{00}; F_{34}F_{20}F_{12}F_{03})
\end{aligned}
$$

so as to get a map

$$
\begin{aligned}
&\hom(F_{00}, F_{01}) \otimes \hom(F_{00}, F_{01}) \otimes \hom(F_{01}, F_{02}) \otimes \hom(F_{02}, F_{03}) \otimes \hom(F_{10}, F_{11}) \\
&\quad \otimes \hom(F_{11}, F_{12}) \otimes \hom(F_{30}, F_{31}) \otimes \hom(F_{31}, F_{32}) \otimes \hom(F_{32}, F_{33}) \otimes \hom(F_{33}, F_{34}) \\
&\qquad\qquad\qquad\qquad \to \hom(F_{30}F_{20}F_{10}F_{00}; F_{34}F_{20}F_{12}F_{03})
\end{aligned}
$$

The coherence axiom then requires that *the map that we have just constructed should coincide with the map* (1) *determined by the picture* (2).

This definition can be homotopized in the following way (this is just a particular case of the notion of an algebra over a 2-operad from [Bat07]):

A homotopy 2-category is a collection of data 1), 2) (as in the above definition) with 3) modified as follows:

h3) for each collection of non-negative integers $m_0, m_1, m_2, \dots, m_n$ there should be given a contractible complex $\mathcal{O}(m_0, m_1, \dots, m_n)$ and a map

$$
c : \mathcal{O}(m_0, m_1, \dots, m_n) \otimes \bigotimes_{ij} \hom(F_{ij}; F_{i,j+1}) \to \hom(F, G),
$$

where the notations are the same as in 3).

In order to formulate the coherence axiom we need some operad-like structure on the collection of complexes $\mathcal{O}(m_0, m_1, \dots, m_n)$. Let us briefly discuss this structure.

First, with every picture P as in (2), one naturally associates a complex $\mathcal{O}(P)$. (Example: for the picture from (2), $\mathcal{O}(P) := \mathcal{O}(3, 2, 0, 4)$, where the numbers represent the numbers of arrows in each column of the picture); next, suppose we have a subdivision of a picture P into a number of sub-pictures $P_1, P_2, \dots, P_k$ with the corresponding quotient-picture Q. (We do not define the precise meaning of these words hoping that the spirit can be felt from the above example of a subdivision of the picture (2) into four sub-pictures (3) with the corresponding quotient-picture (4).)

We then should have a composition map

$$
\mathcal{O}(P) \otimes \mathcal{O}(P_1) \otimes \mathcal{O}(P_2) \cdots \otimes \mathcal{O}(P_k) \to \mathcal{O}(Q) \tag{5}
$$

Having these maps, one can formulate the coherence axiom in this new setting in a natural way.

The maps (5) along with certain natural associativity properties constitute a so-called structure of 2-operad [Bat07]; we will reproduce a precise definition.

Our main result is that *dg-categories form a homotopy 2-category* in which objects are dg-categories, 1-arrows are functors and the complexes of 2-arrows are defined using the derived version of the complex of natural transformations.

We conclude the paper with an observation that this result immediately implies that for every category $\mathcal{C}$, the complex $\mathbf{Rhom}(\mathrm{Id}_C, \mathrm{Id}_C)$ (the homotopy center of $\mathcal{C}$) is an algebra over the above mentioned 2-operad $\mathcal{O}$. A result from [Bat07] implies that *an algebra structure over any contractible 2-operad ($\mathcal{O}$ is such) implies a structure of an algebra over some resolution of the chain operad of little disks*, thus yielding another proof of Deligne's conjecture on Hochschild cochains [KS00], [Vor00], [MS02], [BF04].

The plan of the paper is as follows. We begin with defining a cosimplicial complex of natural transformations $\mathbf{hom}^\bullet(F, G)$ for every pair of dg-functors $F, G : A \to B$. By taking the realization, one gets a complex $\mathbf{Rhom}(F, G) := |\,\mathbf{hom}^\bullet(F, G)|$ which we use as a replacement for the naive complex $\mathrm{hom}(F, G)$.

Next, we introduce some combinatorics in order to describe pictures like the one in (2).

Next, we make definitions of a 2-operad (which is equivalent to that in [Bat07]) and a homotopy 2-category.

After that, we proceed to constructing a homotopy 2-category of dg-categories. It turns out to be more convenient to start with constructing a certain structure on the cosimplicial complexes $\mathbf{hom}^\bullet(F, G)$, without passing to the realization. This structure will be given in terms of a collection of certain multisimplicial sets so that one can study them using some topology.

Finally, using the realization functors, this structure will be converted to a homotopy 2-category structure in which the complexes of 2-arrows are $\mathbf{Rhom}(F, G)$.

We conclude by showing that this result coupled with Batanin's theorem on contractible 2-operads readily implies Deligne's conjecture.

Acknowledgements

I would like to thank A. Beilinson for statement of the problem and M. Batanin for explaining me his results. Special thanks to the referee for his very helpful and very thoroughly written report.

2. Conventions, notations

Ordered sets Any finite non-empty totally ordered set will be called *an ordinal*.

Given a non-negative integer n, we denote by $[n]$ the ordinal $\{0 < 1 < \cdots < n\}$.

Given an ordinal I we denote by m_I its minimum and by M_I its maximum.

Denote by $\overrightarrow{I}$ the set of all pairs $\overrightarrow{ij}$, where $i, j \in I$ and j is the immediate successor of i. We have an induced total order on $\overrightarrow{I}$, but $\overrightarrow{I}$ may be empty. We have natural projections $s, t : \overrightarrow{I} \to I$; $s(\overrightarrow{ij}) = i$; $t(\overrightarrow{ij}) = j$.

Given $a, b \in I$, $a \leqslant b$, we define the interval $[a, b] \subset I$ in the usual way.

A monotone (:=non-decreasing) map of ordinals $f : I \to J$ will be called *dominant* if $f(m_I) = m_J$ and $f(M_I) = M_J$.

Ordinals and their monotone dominant maps form a category. In [Joy], this category is called *the category of 1-disks* and its opposite is identified with the category Δ of ordinals and their monotone maps.

3. Functors between dg categories, and their natural transformations

3.0.1 Let A be a dg-category. Let $X : I \to A$ be a family of objects in A indexed by an ordinal I.

Set $A(X) := \bigotimes_{\overrightarrow{\imath}\in\overrightarrow{I}} A(s(\overrightarrow{\imath});t(\overrightarrow{\imath}))$. If I is a one-element ordinal, we set $A(X)=k$.

Given a dominant monotone map $k : J \to I$, we have a natural map

$$k^* : A(X) \to A(X\circ k)$$

3.0.2 *Definition of the cosimplicial complex of natural transformations* Let A, B be small dg categories and $F, G : A \to B$ functors.

Let I be a finite non-empty totally ordered set. Set

$$\mathbf{hom}^I(F;G) := \prod_{X:I\to A} \hom_k\Big(A(X);\hom_B\big(F(X(m_I));G(X(M_I))\big)\Big).$$

Let I' be obtained from I by adding two elements m', M' such that $m' < I < M'$. Let $X' : I' \to A$, let X be the restriction of X' onto I We then have

$$A(X') \cong A(X(M_I);X'(M')) \otimes A(X) \otimes A(X'(m');X(m_I)). \tag{6}$$

We have a natural map

$$\mathbf{hom}^I(F,G) \to \mathbf{hom}^{I'}(F,G)$$

such that the chain $\Phi \in \mathbf{hom}^I(F,G)$ is mapped into a chain Φ' according to the rule

$$\Phi'(\omega\otimes U\otimes\alpha) = G(\omega)\circ\Phi(U)\circ F(\alpha),$$

where $\omega \in A(X(M), X'(M'))$; $U \in A(X)$; $\alpha \in A(X'(m'), X(m))$, and we use the identification (6).

3.0.3 *Cosimplicial structure* Let Δ be the category of ordinals and their monotone (=non-decreasing) maps. We are going to endow the collection of spaces $\mathbf{hom}^I(F,G)$ with a structure of functor

$$\Delta \to \mathbf{complexes}.$$

Let $\sigma : I \to J$ be a monotone map. Define a map

$$\sigma_* : \mathbf{hom}^I(F,G) \to \mathbf{hom}^J(F,G)$$

as follows.

Let $\sigma' : I' \to J$ be the extension of σ which sends m' and M' to the minimum and the maximum of J respectively.

Set $\sigma_*\Phi(U) := \Phi'((\sigma')^*U)$, where $U \in A(X)$ for some $X : J \to A$. In this way we get the desired cosimplicial structure.

3.1 Total complex of a cosimplicial comlplex

It is well known that given a cosimplicial complex one can produce its total complex by applying alternated sums of co-face maps.

We will use a slightly different definition of this total complex.

Let I be an ordinal, let Δ^I be the simplex whose vertices are labelled by I.

Let $C_*(\Delta^I)$ be its reduced chain complex. Let $\mathcal{S}^*(I) := C_{-*}(\Delta^I)$. It is clear that $\mathcal{S}^*(\bullet)$ is a cosimplicial complex (here $\bullet$ stands for the "cosimplicial" argument).

We will denote this cosimplicial complex simply by $\mathcal{S}$.

Given an arbitrary cosimplicial complex K we can form a complex $\hom_\Delta(\mathcal{S}, K)$ which will also be denoted by $|K|$. In this formula, $\hom_\Delta$ means "enriched hom-functor in the category of cosimplicial complexes."

Thus, we can construct a complex $|\mathbf{hom}^\bullet(F,G)|$ which will be denoted by $\mathbf{Rhom}(F,G)$.

3.1.1 We also have a "naive" notion of the complex of natural transformations of two functors. Indeed, given a pair of functors $F,G : A \to B$, define the complex

$$\hom(F,G)$$

as the equalizer of the diagram

$$\hom(F,G) \longrightarrow \mathbf{hom}^{[0]}(F,G) \overset{d_1}{\underset{d_0}{\rightrightarrows}} \mathbf{hom}^{[1]}(F,G)$$

where d_0, d_1 are the co-face maps.

We can define a constant cosimplicial complex

$$\hom^\bullet(F,G),$$

where

$$\hom^I(F,G) := \hom(F,G).$$

We then have a cosimplicial map

$$\hom^\bullet(F,G) \to \mathbf{hom}^\bullet(F,G).$$

4. Some combinatorics

We want to find an algebraic structure naturally possessed by complexes $\mathbf{Rhom}(F,G)$.

This structure will be given in terms of a family of multilinear maps between these complexes and some relations between them.

In order to formulate this structure we need some combinatorics.

4.1 Combinatorial data

4.1.1 *2-ordinals, 2-trees* By definition, a 2-ordinal $\mathcal{U}$ is a collection of the following data[2]:

- an ordinal $\mathcal{C}_\mathcal{U}$;
- for each $\overrightarrow{c} \in \overrightarrow{\mathcal{C}}_\mathcal{U}$, an ordinal $\mathcal{F}_{\mathcal{U},\overrightarrow{c}}$.

2-ordinals are meant to represent pictures of the type shown below:

$$\begin{tikzcd}
c_0 \arrow[r, "f_1^{01}"] \arrow[r, bend left=40, "f_2^{01}"] \arrow[r, bend left=80, "f_3^{01}"] \arrow[r, bend right=40, "f_0^{01}"'] & c_1 \arrow[r, "f_1^{12}"] \arrow[r, bend left=40, "f_2^{12}"] \arrow[r, bend right=40, "f_0^{12}"'] & c_2 \arrow[r, "f_0^{23}"] & c_3 \arrow[r, "f_2^{34}"] \arrow[r, bend left=40, "f_3^{34}"] \arrow[r, bend left=80, "f_4^{34}"] \arrow[r, bend right=40, "f_1^{34}"'] \arrow[r, bend right=80, "f_0^{34}"'] & c_4
\end{tikzcd} \tag{7}$$

This picture corresponds to the following 2-ordinal:

- $\mathcal{C} = \{c_0 < c_1 < c_2 < c_3 < c_4\}$;

[2]An equivalent notion is introduced in [Str00], where it is called "a globular 2-cardinal."

- $\mathcal{F}_{\overrightarrow{c_0c_1}} = \{f_0^{01} < f_1^{01} < f_2^{01} < f_3^{01}\}$;
- $\mathcal{F}_{\overrightarrow{c_1c_2}} = \{f_0^{12} < f_1^{12} < f_2^{12}\}$;
- $\mathcal{F}_{\overrightarrow{c_2c_3}} = \{f_0^{23}\}$;
- $\mathcal{F}_{\overrightarrow{c_3c_4}} = \{f_0^{34} < f_1^{34} < \cdots < f_4^{34}\}$

A picture of such a type can be drawn for any 2-ordinal in the obvious way.

We denote

$$\overrightarrow{\mathcal{F}}_{\mathcal{U}} := \sqcup_{\overrightarrow{c}\in\overrightarrow{\mathcal{C}}} \overrightarrow{\mathcal{F}}_{\mathcal{U},\overrightarrow{c}}.$$

Elements of this set can be visualized as 2-cells in the picture corresponding to $\mathcal{U}$. We then have an obvious monotone map of finite totally ordered sets.

$$\pi : \overrightarrow{\mathcal{F}}_{\mathcal{U}} \to \overrightarrow{\mathcal{C}}_{\mathcal{U}}.$$

According to [Bat07], let us call any map of finite totally ordered sets *a 2 stage tree* or, shortly, *a 2-tree*.

We will denote 2-trees by one letter, say t. We will refer to the elements of t as:

$$\pi_t : \overrightarrow{\mathcal{F}}_t \to \overrightarrow{\mathcal{C}}_t.$$

We have shown how, given a 2-ordinal $\mathcal{U}$, one constructs a 2-tree $\pi_U : \overrightarrow{\mathcal{F}}_{\mathcal{U}} \to \overrightarrow{\mathcal{C}}_{\mathcal{U}}$. Denote this 2-tree by $t_{\mathcal{U}}$, so that:

$$\pi_{t_{\mathcal{U}}} := \pi_{\mathcal{U}}; \quad \overrightarrow{\mathcal{F}}_{t_{\mathcal{U}}} := \overrightarrow{\mathcal{F}}_{\mathcal{U}};$$
$$\overrightarrow{\mathcal{C}}_{t_{\mathcal{U}}} := \overrightarrow{\mathcal{C}}_{\mathcal{U}}.$$

It is clear that a 2-ordinal is defined up-to a canonical isomorphism by its 2-tree[3].

4.1.2 Given a 2-ordinal $\mathcal{U}$ we can construct a strict 2-category $[\mathcal{U}]$, the universal one among the 2-categories V possessing the following properties:

- $\mathbf{Ob}V = \mathcal{C}$;
- there are fixed maps $\mathcal{F}_{\overrightarrow{c_1c_2}} \to \mathbf{Ob}\,\hom_V(c_1, c_2)$ for all $\overrightarrow{c_1c_2} \in \overrightarrow{\mathcal{C}}$. For an $f \in \mathcal{F}_{\overrightarrow{c_1c_2}}$ we denote by the same symbol the corresponding object in $\hom_V(c_1, c_2)$.
- For each $\overrightarrow{f_1f_2} \in \overrightarrow{\mathcal{F}}_{\overrightarrow{c_1c_2}}$ we have a fixed element in $\hom_{\hom_V(c_1,c_2)}(f_1, f_2)$.

This 2-category has a clear meaning in terms of the picture (7).

Objects are $c_0, c_1, \cdots$; the space of maps $c_i \to c_j$ is non-empty iff $c_i \leqslant c_j$, in which case an arrow $c_i \to c_j$ is just a directed path from c_i to c_j; let us define a partial order on the space of such paths by declaring that one path is less or equal to another iff the former lyes below the latter. We then have a category structure on $\hom(c_i, c_j)$ produced by the just defined poset of paths (each arrow goes from a smaller object to a greater one).

Here is a more formal description. Given $c, c' \in \mathcal{C}$, we have

1) $\hom(c, c') = \emptyset$ if $c > c'$;
2) $\hom(c, c) = \{\mathrm{Id}_c\}$;
3) if $c < c'$, then

$$\mathbf{Ob}\,\hom(c, c') := \prod_{\overrightarrow{c}\in\overrightarrow{[c,c']}} \mathcal{F}_{\overrightarrow{c}}.$$

[3]This identification of 2-ordinals and 2-trees is introduced in [Bat98], where it is called the $(-)*$-*construction.*

Given $f^1, f^2 \in \hom(c, c')$;

$$f^k = \{f^k_{\overrightarrow{c}}\}_{\overrightarrow{c} \in \overrightarrow{[c_1 c_2]}},\ k = 1, 2;$$

we have a unique arrow $f_1 \to f_2$ iff

$$f^1_{\overrightarrow{c}} \leqslant f^2_{\overrightarrow{c}}$$

for all $\overrightarrow{c} \in \overrightarrow{[c_1 c_2]}$. Thus, the set $\hom(c_1, c_2)$ is partially ordered and has the least and the greatest element.

4.1.3 We will often need a special 2-ordinal called *globe* and denoted by **globe**. We define **globe** by

$$\mathcal{C}_{\mathbf{globe}} = \{c_0 < c_1\};$$
$$\mathcal{F}_{\overrightarrow{c_0 c_1}} = \{f_0 < f_1\}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

Any 2-ordinal isomorphic to **globe** will also be called a globe.

4.1.4 *Balls in a 2-ordinal* Given a 2-ordinal $\mathcal{U}$ define *a ball* in $\mathcal{U}$ as any 2-ordinal $\mathcal{U}'$ the form:

- $\mathcal{C}_{\mathcal{U}'}$ is an interval in $\mathcal{C}_{\mathcal{U}}$;
- for each $\overrightarrow{c} \in \overrightarrow{\mathcal{C}}_{\mathcal{U}'}$, $\mathcal{F}_{\mathcal{U}', \overrightarrow{c}}$ is an interval in $\mathcal{F}_{\mathcal{U}, \overrightarrow{c}}$.

We then see that

$$[\mathcal{U}'] \subset [\mathcal{U}]$$

is a full subcategory.

The set of all balls in $\mathcal{U}$ is partially ordered; each minimal ball is a globe; the set of all these minimal balls is naturally identified with $\overrightarrow{\mathcal{F}}_{\mathcal{U}}$.

4.1.5 We write $[\mathcal{U}]_1$ for the underlying usual category of $\mathcal{U}$.

4.1.6 *Maps of 2-ordinals* Let $\mathcal{U}, \mathcal{V}$ be 2-ordinals. *A map*[4] $P : \mathcal{U} \to \mathcal{V}$ is a 2-functor $[P] : [\mathcal{V}] \to [\mathcal{U}]$ satisfying:

1) the induced map $[P] : \mathcal{C}_{\mathcal{V}} \to \mathcal{C}_{\mathcal{U}}$ is dominant (= monotone and preserves the minima and the maxima);
2) for all $c_1 < c_2$, $c_1, c_2 \in \mathcal{C}_V$ the induced map

$$\hom_{[\mathcal{V}]}(c_1, c_2) \to \hom_{[\mathcal{U}]}(P(c_1), P(c_2))$$

preserves the least and the greatest elements.

With this definition of a map, 2-ordinals form a category. Any globe is a terminal object in this category.

4.1.7 *Inverse images of balls* Let $P : \mathcal{U} \to \mathcal{V}$ be a map of 2-ordinals and $\mathcal{V}' \subset \mathcal{V}$ a ball in $\mathcal{V}$. Define $P^{-1}\mathcal{V}' =: \mathcal{U}'$ as a unique ball in $\mathcal{U}$ satisfying:

[4]See [Ber02] for the definition of an equivalent concept of *covering map of n-trees.*

There exists a map of 2-ordinals $P' : \mathcal{U}' \to \mathcal{V}'$ fitting into a commutative diagram:

$$\begin{tikzcd}
{[\mathcal{U}]} & {[\mathcal{V}]} \arrow[l, "{[P]}"'] \\
{[\mathcal{U}']} \arrow[u] & {[\mathcal{V}']} \arrow[l, "{[P']}"] \arrow[u]
\end{tikzcd}$$

with the vertical arrows being the natural inclusions.

Consider a pictorial example:

Let $\mathcal{U}$ be the same ordinal as in (7):

$$\begin{tikzcd}
c_0 \arrow[r, "f_1^{01}"] \arrow[r, bend right=50, "f_0^{01}"'] \arrow[r, bend left=50, "f_2^{01}"] \arrow[r, bend left=90, "f_3^{01}"] & c_1 \arrow[r, "f_1^{12}"] \arrow[r, bend right=50, "f_0^{12}"'] \arrow[r, bend left=50, "f_2^{12}"] & c_2 \arrow[r, "f_0^{23}"] & c_3 \arrow[r, "f_2^{34}"] \arrow[r, bend right=50, "f_1^{34}"'] \arrow[r, bend right=90, "f_0^{34}"'] \arrow[r, bend left=50, "f_3^{34}"] \arrow[r, bend left=90, "f_4^{34}"] & c_4
\end{tikzcd}$$

and let $\mathcal{V}$ be defined by the following picture

$$\begin{tikzcd}
d_0 \arrow[r, "g_1^{01}"] \arrow[r, bend right=50, "g_0^{01}"'] \arrow[r, bend left=50, "g_2^{01}"] & d_1 \arrow[r, bend right=50, "g_0^{12}"'] \arrow[r, bend left=50, "g_1^{12}"] & d_2
\end{tikzcd}$$

Consider a map $P : \mathcal{U} \to \mathcal{V}$ such that the corresponding map $[P] : [\mathcal{V}] \to [\mathcal{U}]$ is as follows:

$$[P]d_0 = c_0;\ [P]d_1 = c_2;\ [P]d_2 = c_4;$$
$$[P]g_0^{01} = f_0^{12} f_0^{01};\ [P]g_1^{01} = f_1^{12} f_0^{01}\ [P]g_2^{01} = f_2^{12} f_3^{01};$$
$$[P]g_0^{12} = f_0^{34} f_0^{23};$$
$$[P]g_1^{12} = f_4^{34} f_0^{23}.$$

Let us label globes in $\mathcal{V}$ by I, II, III as shown on the picture:

$$\begin{tikzcd}
d_0 \arrow[r, "II"] \arrow[r, bend right=50, "I"'] \arrow[r, bend left=50] & d_1 \arrow[r, bend right=50] \arrow[r, bend left=50] \arrow[r, phantom, "III"] & d_2
\end{tikzcd}$$

The preimages of these globes are then as follows

$$\begin{tikzcd}
II & c_0 \arrow[r, "f_1^{01}"] \arrow[r, bend right=50, "f_0^{01}"'] \arrow[r, bend left=50, "f_2^{01}"] \arrow[r, bend left=90, "f_3^{01}"] & c_1 \arrow[r, "f_1^{12}"] \arrow[r, bend left=50, "f_2^{12}"] & c_2 & III & c_2 \arrow[r, "f_0^{23}"] & c_3 \arrow[r, "f_2^{34}"] \arrow[r, bend right=50, "f_1^{34}"'] \arrow[r, bend right=90, "f_0^{34}"'] \arrow[r, bend left=50, "f_3^{34}"] \arrow[r, bend left=90, "f_4^{34}"] & c_4 \\
I & c_0 \arrow[r, bend right=50, "f_0^{01}"'] & c_1 \arrow[r, "f_1^{12}"] \arrow[r, bend right=50, "f_0^{12}"'] & c_2
\end{tikzcd}$$

4.1.8 *Maps of the corresponding 2-trees* According to Batanin, we define a map of 2-trees $P : t_1 \to t_2$ as a commutative diagram

$$\begin{tikzcd} \overrightarrow{\mathcal{F}}_{t_1} \arrow[r, "P_{\overrightarrow{\mathcal{F}}}"] \arrow[d, "\pi_{t_1}"] & \overrightarrow{\mathcal{F}}_{t_2} \arrow[d, "\pi_{t_2}"] \\ \overrightarrow{\mathcal{C}}_{t_1} \arrow[r, "P_{\overrightarrow{\mathcal{C}}}"] & \overrightarrow{\mathcal{C}}_{t_2} \end{tikzcd}$$

with the arrow $P_{\overrightarrow{\mathcal{C}}}$ being monotone and the arrow $P_{\overrightarrow{\mathcal{F}}}$ being monotone on each fiber of π_{t_1} (i.e. on each subset $\pi_{t_1}(c) \subset \overrightarrow{\mathcal{F}}_{t_1}$ for each $c \in \overrightarrow{\mathcal{C}}_{t_1}$).

4.1.9 A map of ordinals $\mathcal{U} \to \mathcal{V}$ naturally induces a map of the sets of two-cells:

$$\overrightarrow{\mathcal{F}}_{\mathcal{U}} \to \overrightarrow{\mathcal{F}}_{V};$$

it is not hard to see that this map lifts to a map of the corresponding 2-trees. We are going to give a formal definition of this map.

Given a map of 2-ordinals $P : \mathcal{U} \to \mathcal{V}$, we define an induced map of the associated 2-trees:

$$P^t : t_{\mathcal{U}} \to t_V$$

as follows:

1) define the map

$$P^t_{\overrightarrow{\mathcal{C}}} : \overrightarrow{\mathcal{C}}_{\mathcal{U}} \to \overrightarrow{\mathcal{C}}_{V}.$$

For $\overrightarrow{c_1c_2} \in \overrightarrow{\mathcal{C}}_U$ we set $P_{\overrightarrow{\mathcal{C}}}(\overrightarrow{c_1c_2}) = \overrightarrow{d_1d_2}$ iff

$$[c_1c_2] \subset [[P](d_1), [P](d_2)].$$

2) Given a globe $\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{U}}$ define its image $P^t_{\overrightarrow{\mathcal{F}}}(\overrightarrow{f}) =: \overrightarrow{g}$ in $\overrightarrow{\mathcal{F}}_{\mathcal{V}}$ as a unique globe such that the ball $P^{-1}\overrightarrow{g}$ contains $\overrightarrow{f}$.

4.1.10 We see that this way we have constructed a category of 2-ordinals and a category of 2-trees and a functor between them; as shown in [Ber02], this functor is an equivalence.

4.1.11 *Diagrams* Given a 2-ordinal $\mathcal{U}$ and a category $\mathcal{C}$, *a $\mathcal{U}$-diagram* in $\mathcal{C}$ is a functor

$$\mathcal{D} : [U]_1 \to \mathcal{C}$$

Given a map of 2-ordinals $P : \mathcal{U} \to \mathcal{V}$ and a $\mathcal{U}$-diagram $\mathcal{D}$, it naturally restricts to produce $P^{-1}\overrightarrow{f}$- diagrams $\mathcal{D}|_{P^{-1}\overrightarrow{f}}$, where $\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{V}}$, and a $\mathcal{V}$-diagram $P_*\mathcal{D}$. These induced diagrams come from the obvious functors

$$[P^{-1}\overrightarrow{f}] \hookrightarrow [\mathcal{U}];$$
$$[P] : [\mathcal{V}] \to [U].$$

5. 2-operads and their algebras

We are going to adjust notions of an operad and an algebra over an operad so that they work in our setting. In the usual setting, given an operad, we define its action on a complex; in our situation, instead of one complex, we have a family of complexes: a complex $\mathbf{Rhom}(F, G)$ for each globe formed by the pair of dg-categories A, B and a pair of dg-functors $F, G : A \to B$. We abstract this situation by introducing a notion of a $\mathcal{C}$-complex. Next, following [Bat07], we define the notion of a 2-operad, and, lastly, the notion of a structure of an $\mathcal{O}$- algebra on a $\mathcal{C}$-complex, where $\mathcal{O}$ is a 2-operad.

Using these notions, we will be able to formulate the definition of a homotopy 2-category as an algebra structure over a contractible 2-operad.

5.1 $\mathcal{C}$-complexes and their tensor products

We fix a small category in sets $\mathcal{C}$. Let $\mathcal{C}_0$ be the set of objects in $\mathcal{C}$ and $\mathcal{C}_1$ be the set of its arrows. Let $s, t : \mathcal{C}_1 \to \mathcal{C}_0$ be the source and target maps

We define *a globe in* $\mathcal{C}$ as any **globe**-diagram in $\mathcal{C}$ (= a pair of objects in $\mathcal{C}$ and a pair of arrows between these objects).

Let $\mathcal{C}_2$ be the set of all globes in $\mathcal{C}$. We have obvious maps

$$s', t' : \mathcal{C}_2 \to \mathcal{C}_1;$$

given a globe g

$$\begin{tikzcd} A_0 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_0"'] & A_1 \end{tikzcd}$$

where $A_0, A_1 \in \mathcal{C}$; $f_0, f_1 : A_0 \to A_1$, we set $s'(g) = f_0$; $t'(g) = f_1$. It is clear that

$$ss' = ts'; \quad st' = tt', \tag{8}$$

and $\mathcal{C}_2$ is the terminal object in the category of all sets Z endowed with maps $s', t' : Z \to \mathcal{C}_1$ satisfying (8)

5.1.1 We define *a $\mathcal{C}$-complex* as a family of complexes parameterized by $\mathcal{C}_2$.

5.1.2 *Tensor product of $\mathcal{C}$-complexes* Given a 2-ordinal $\mathcal{U}$ and a $\overrightarrow{\mathcal{F}}_{\mathcal{U}}$-family of $\mathcal{C}$- complexes $\mathcal{K} = \{K_{\overrightarrow{f}}\}_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{C}}}$, we define a new $\mathcal{C}$-complex

$$\bigotimes^{\mathcal{U}} \mathcal{K} := \bigotimes^{\mathcal{U}}_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{U}}} K_{\overrightarrow{f}}$$

as follows.

Pick a globe $g \in C_2$ and define

$$(\bigotimes^{\mathcal{U}} \mathcal{K})_g := \bigoplus_{\mathcal{D} | p_* \mathcal{D} = g} \bigotimes_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{U}}} K_{(\mathcal{D}|_{\overrightarrow{f}})},$$

where $p : \mathcal{U} \to$ **globe** is the terminal map, $\mathcal{D}$ is any $\mathcal{U}$-diagram in $\mathcal{C}$ with $p_* \mathcal{D} = g$, and $\overrightarrow{\mathcal{F}}_{\mathcal{U}}$ is identified with the set of globes in $\mathcal{U}$ so that $\mathcal{D}|_{\overrightarrow{f}}$ is a globe in $\mathcal{C}$.

5.1.3 Given a map of 2-ordinals $\mathcal{U} \to \mathcal{V}$, we have a canonical isomorphism

$$\bigotimes^{\mathcal{V}}_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{V}}} \bigotimes^{P^{-1}\overrightarrow{f}}_{\overrightarrow{g} \in \overrightarrow{\mathcal{F}}_{P^{-1}\overrightarrow{f}}} K_{\overrightarrow{g}} \to \bigotimes^{\mathcal{U}}_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{U}}} K_{\overrightarrow{f}};$$

call this isomorphism *a constraint*.

Given a chain of maps of 2-ordinals $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$, we get the associativity property of this constraint.

This can be reformulated so that the category of $\mathcal{C}$-complexes becomes a category with two monoidal structures such that one of them distributes over the other but we won't need it in this paper.

5.1.4 Fix a set **Chrom** to be called the set of colors. Let us also fix family of $\mathcal{C}$-complexes $\mathcal{K} := \{K_\chi\}_{\chi\in\mathbf{Chrom}}$.

Let $\mathcal{U}$ be a 2-ordinal. Define *an* **Chrom**-*coloring* χ of $\mathcal{U}$ as a prescription of a color $c^\chi_{\overrightarrow{f}} \in \mathbf{Chrom}$ for each $\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_\mathcal{U}$ and an additional color $c^\chi \in \mathbf{Chrom}$.

A 2-ordinal endowed with a coloring will be called a colored 2-ordinal. Given a colored 2-ordinal $\mathcal{U}' := (\mathcal{U}, \chi)$, we have a complex

$$\mathbf{full}_\mathcal{K}(\mathcal{U}') := \hom([\bigotimes_{\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_\mathcal{U}}^{\mathcal{U}} K_{c^\chi_{\overrightarrow{f}}}]; K_{c^\chi})$$

These complexes form an algebraic structure called a colored 2-operad. We are going to define this notion.

5.2 Colored 2-operads

We need a notion of a map of colored 2-ordinals:

Let $\mathcal{U}' = (\mathcal{U}, \chi_\mathcal{U})$; $\mathcal{V}' = (\mathcal{V}, \chi_\mathcal{V})$ be colored ordinals. We say that we have a map $P' : \mathcal{U}' \to \mathcal{V}'$ if:

- we are given a map $P : \mathcal{U} \to \mathcal{V}$ of 2-ordinals;
- $c^{\chi_\mathcal{U}} = c^{\chi_\mathcal{V}}$.

Given a globe $\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_\mathcal{V}$, we then have a natural coloring $\chi^{\overrightarrow{f}}$ on $P^{-1}\overrightarrow{f}$: we set $c^{\chi^{\overrightarrow{f}}} = c^{\chi_\mathcal{V}}_{\overrightarrow{f}}$; $(c^{\chi_{\overrightarrow{f}}})_{\overrightarrow{g}} := (c^{\chi_\mathcal{U}})_{\overrightarrow{g}}$.

5.2.1 *Definition of a colored 2-operad* A **Chrom**-colored operad $\mathcal{O}$ is a collection of the following data:

- a functor $\mathcal{O}$ from the isomorphism groupoid of the category of colored 2-ordinals to the category of complexes;
- given a map $P : \mathcal{U}' \to \mathcal{V}'$ of colored ordinals, set

  $$\mathcal{O}(P) := \bigotimes_{\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_\mathcal{V}} \mathcal{O}(P^{-1}\overrightarrow{f})$$

  we then should have a map

  $$\circ_P : \mathcal{O}(P) \otimes \mathcal{O}(\mathcal{V}') \to \mathcal{O}(\mathcal{U}').$$

Axioms:

The first axiom asks for covariance of the map $\circ_P$ under isomorphisms of 2-ordinals.

In order to formulate the next axiom, note that given a chain of maps of colored 2-ordinals

$$\mathcal{U} \xrightarrow{P} \mathcal{V} \xrightarrow{Q} \mathcal{W},$$

we have a natural map

$$\circ(Q,P):\mathcal{O}(P)\otimes\mathcal{O}(Q)\to\mathcal{O}(QP);$$

indeed, for each $\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_{\mathcal{W}}$ we have induced maps

$$P_{\overrightarrow{f}}:(QP)^{-1}\overrightarrow{f}\to Q^{-1}\overrightarrow{f},$$

and we can define our map as follows

$$\mathcal{O}(P)\otimes\mathcal{O}(Q)=\bigotimes_{\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_{\mathcal{W}}}[\mathcal{O}(P_{\overrightarrow{f}})\otimes\mathcal{O}(Q^{-1}\overrightarrow{f})]\to\bigotimes_{\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_{\mathcal{W}}}\mathcal{O}((QP)^{-1}\overrightarrow{f})=\mathcal{O}(QP).$$

The property then reads that the maps $\circ(Q,P)$ should be associative in the obvious way.

5.2.2 It is immediate that the complexes $\mathbf{full}_{\mathcal{K}}(U')$ form a colored 2-operad $\mathbf{full}_{\mathcal{K}}$

5.2.3 Given an **Chrom**-colored 2-operad $\mathcal{O}$, we define an $\mathcal{O}$-algebra as an **Chrom**-family $\mathcal{K}$ of $\mathcal{C}$-complexes along with a map of colored 2-operads

$$\mathcal{O}\to\mathbf{full}_{\mathcal{K}}.$$

5.2.4 Define a (non-colored) 2-operad as a colored operad with the set of colors to have only one element[5].

5.3 Main Theorem

DG-categories and their functors form a category. Fix a small sub-category $\mathcal{C}$ in this category. Given a globe g in $\mathcal{C}$, we have defined a complex $\mathbf{Rhom}(g)$. These complexes form a $\mathcal{C}$-complex, to be denoted **Rhom**. Likewise we have the functor of usual homomorphisms $\hom(g)$, these also form a $\mathcal{C}$-complex hom. We have a natural map of $\mathcal{C}$-complexes

$$\hom\to\mathbf{Rhom}.$$

We know that the pair $(C,\hom)$ is naturally a 2-category. This can be formulated in our language as follows. Define a trivial 2-operad **triv** as follows:

set $\mathbf{triv}(\mathcal{U})=k$ for each 2-ordinal $\mathcal{U}$ and demand that all structure maps preserve $1\in k$.

Then the 2-categorical structure on $(\mathcal{C},\hom)$ amounts to the fact that we have a **triv**-algebra structure on hom.

5.3.1 *Formulation of a theorem* Define a notion of *a contractible 2-operad* as a collection of the following data:

- a 2-operad $\mathcal{O}$;
- a quasi-isomorphism of 2-operads $\mathcal{O}\to\mathbf{triv}$.

THEOREM 5.1. *There exists a contractible 2-operad $\mathcal{O}$ and $\mathcal{O}$-algebra structures on* hom *and* **Rhom** *such that:*

1) *the map* hom $\to$ **Rhom** *is a map of $\mathcal{O}$-algebras;*
2) *the $\mathcal{O}$-algebra structure on* hom *is the pull-back of the* **triv**-*structure via the structure map* $\mathcal{O}\to\mathbf{triv}$.

The rest of the paper is devoted to proving this theorem.

[5]This definition is narrower than Batanin's one: in his papers such objects are called "1-terminal 2-operads"; however, all 2-operads in our paper are 1-terminal.

6. Constructing a colored 2-operad which acts on $\mathbf{hom}^\bullet$

Let $\mathbb{N}$ be the set of isomorphism classes of ordinals; $\mathbb{N} = \{[0], [1], [2], \ldots\}$. We then have a $\mathbb{N}$-family of $\mathcal{C}$-complexes $I \mapsto \mathbf{hom}^I$, $I \in \mathbb{N}$.

We will start with a construction of a colored 2-operad $\mathbf{seq}$ in the category of sets acting naturally[6] on $\mathbf{hom}^\bullet$. (Note that in the definition of colored operad of Sec. 5.2.1, the category of complexes can be replaced with any symmetric monoidal category.)

By default, all colorings are $\mathbb{N}$*-colorings.*

6.1 Construction of seq

Let $(\mathcal{U}, \chi)$ be a colored 2-ordinal. Let

$$\pi : \overrightarrow{\mathcal{F}}_{\mathcal{U}} \to \overrightarrow{\mathcal{C}}_{\mathcal{U}}$$

be the induced 2-tree.

Let us use the following notation for the ordinals which determine the coloring:

$$I_{\overrightarrow{f}} := c^{\chi}_{\overrightarrow{f}};$$
$$J := c^{\chi}.$$

Define a set $\mathbf{seq}(\mathcal{U})$ each of whose elements is a collection of the following data:

A) a total order on $\mathcal{I} := \mathcal{I}_{\mathcal{U}} := \coprod_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}} I_{\overrightarrow{f}}$;

B) a monotone map $W : \mathcal{I} \to J$.

The conditions are:

1) the total order on $\mathcal{I}$ agrees with the orders on each $I_{\overrightarrow{f}}$;

2) if $i_1, i_2 \in I_{\overrightarrow{f}}$ and $i_1 < i < i_2$, $i \in I_{\overrightarrow{f}_1}$ then $\pi(\overrightarrow{f}_1) < \pi(\overrightarrow{f})$;

3) if $\pi(\overrightarrow{f}_1) = \pi(\overrightarrow{f}_2) = \overrightarrow{c}$ and $\overrightarrow{f}_1 < \overrightarrow{f}_2$ in the sense of the order on $\overrightarrow{\mathcal{F}}_{\overrightarrow{c}}$, then $I_{\overrightarrow{f}_1} < I_{\overrightarrow{f}_2}$ with respect to the order on $\mathcal{I}$.

6.1.1 *Compositions* Let $P : \mathcal{U}' \to \mathcal{V}'$ be a map of $\mathbb{N}$-colored ordinals. Define the structure map

$$\circ_p : \prod_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{V}}} \mathbf{seq}(P^{-1}\overrightarrow{f}) \times \mathbf{seq}(\mathcal{V}) \to \mathbf{seq}(\mathcal{U}).$$

Let us pick elements $\lambda_{\overrightarrow{f}} \in \mathbf{seq}(P^{-1}\overrightarrow{f})$; $\lambda \in \mathbf{seq}(V)$ and define their composition. We have

$$\mathcal{I}_{\mathcal{U}} = \coprod_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_{\mathcal{V}}} \mathcal{I}_{P^{-1}\overrightarrow{f}}.$$

The elements λ_ϕ define total order on $\mathcal{I}_{P^{-1}\overrightarrow{f}}$ and monotone maps

$$\mathcal{I}_{P^{-1}\overrightarrow{f}} \to I_{\overrightarrow{f}}.$$

[6] We have a functor of taking the span, from the category of sets to the category of vector spaces. Treating vector spaces as complexes sitting in degree 0, we get a symmetric monoidal functor from the category of sets to the category of complexes. The latter functor induces a functor from the category of colored 2-operads in sets to the category of 2-operads in the category of complexes. Let $k[\mathbf{seq}]$ be the colored operad in the category of complexes obtained by applying this functor to $\mathbf{seq}$. Whenever we have a $\mathcal{C}$-complex $\mathcal{K}$ and a $k[\mathbf{seq}]$-algebra structure on $\mathcal{K}$, we will say that $\mathbf{seq}$ *acts on* $\mathcal{K}$.

The element λ defines a total order on

$$\mathcal{I}_{\mathcal{V}} := \coprod_{\overrightarrow{f}} I_{\overrightarrow{f}}.$$

We have a natural map

$$M : \mathcal{I}_{\mathcal{U}} = \coprod_{\overrightarrow{f}} \mathcal{I}_{P^{-1}\overrightarrow{f}} \to \coprod_{\overrightarrow{f}} I_{\overrightarrow{f}} = I_{\mathcal{V}}.$$

Lemma 6.1. *There is a unique total order on $\mathcal{I}_{\mathcal{U}}$ such that:*

- *the map M is monotone;*
- *this order agrees with those on each $\mathcal{I}_{P^{-1}\overrightarrow{f}}$;*

Proof. If such an order exists, it must be defined as follows:

- if $x, y \in \mathcal{I}_{\mathcal{U}}$, and $M(x) < M(y)$, then $x < y$;
- if $M(x) = M(y) \in I_\phi$, then $x, y \in \mathcal{I}_{P^{-1}\overrightarrow{f}}$.

It is clear that this way we indeed get a total order on $\mathcal{I}_{\mathcal{U}}$. The map M is automatically monotone. We only need to check the matching of this order with that on each $\mathcal{I}_{P^{-1}\overrightarrow{f}}$. This follows immediately from the monotonicity of the corresponding maps

$$\mathcal{I}_{P^{-1}\overrightarrow{f}} \to I_{\overrightarrow{f}}.$$

□

Next we define a map $W' : \mathcal{I}_{\mathcal{U}} \to J$ as a composition

$$\mathcal{I}_{\mathcal{U}} \to \mathcal{I}_{\mathcal{V}} \xrightarrow{W} J.$$

Lemma 6.2. *The constructed order on $\mathcal{I}_{\mathcal{U}}$ and the constructed map W' give rise to an element in* $\mathbf{seq}(\mathcal{U})$

Proof. Straightforward. □

We define the composition of the elements $\lambda_{\overrightarrow{f}}$ and λ to be the constructed element in $\mathbf{seq}(\mathcal{U})$. One can check that this composition satisfies the associativity property.

Remark. There is an equivalent description of the 2-operad **seq** in terms of planar trees. A detailed account will appear in [VDT].

6.2 Action of seq on Rhom

We need a couple of auxiliary constructions.

6.2.1 Given a dg-category A, an ordinal J, and a map $X : J \to A$ we call any element in $A(X)$ (see the very beginning of the paper) *a chain in* A or, more, specific, an X-chain in A. Fix a chain $h \in A(X)$.

Suppose we are given an $X : J \to A$ as above. Suppose that in addition, we are given an ordinal R and an R-family of functors $F_r : A \to B$.

Next, for each $\overrightarrow{r} = \overrightarrow{r_1 r_2} \in \overrightarrow{R}$, choose ordinals $I_{\overrightarrow{r}}$ and elements

$$h_{\overrightarrow{r}} \in \mathbf{hom}^{I_{\overrightarrow{r}}}(F_{r_1}; F_{r_2}). \tag{9}$$

Finally let us fix a monotone map

$$W : I = \coprod_{\overrightarrow{r} \in \overrightarrow{R}} I_{\overrightarrow{r}} \to J,$$

where the order on I is defined by those on $\overrightarrow{R}$ and on each of $I_{\overrightarrow{r}}$.

Given all these data, we will construct a chain c in B.

6.2.2 Let us now make a formal definition.

First of all we need to construct an ordinal K and a map $Y : K \to B$ so that $c \in B(Y)$.

Constructing K For $r \in R$ let m_r be the supremum in J of the set

$$\coprod_{\overrightarrow{r_1 r_2} | r_2 \leqslant r} W(I_{\overrightarrow{r_1 r_2}})$$

Let M_r be the infimum in J of the set

$$\coprod_{\overrightarrow{r_1 r_2} | r \leqslant r_1} W(I_{\overrightarrow{r_1 r_2}})$$

We define:

$$K := K(J, W) := \coprod_{r \in R} [m_r, M_r].$$

We then have natural maps

$$\pi : K \to R;$$
$$\kappa : K \to J.$$

Constructing a map $Y := Y(X, W) : K \to B$ Set

$$Y(j_r) = F_r(\kappa(j_r)),$$

where $j_r \in [m_r; M_r] \subset K$.

Constructing the resulting chain $c \in B(Y)$ We will define a map

$$\mu_W : A(X) \otimes \bigotimes_{\overrightarrow{r_1 r_2} \in \overrightarrow{R}} \mathbf{hom}^{I_{\overrightarrow{r_1 r_2}}}(F_{r_1}; F_{r_2}) \to B(Y).$$

so that

$$c = \mu_W(h; \{h_{\overrightarrow{r}}\}_{\overrightarrow{r} \in \overrightarrow{R}}).$$

For an interval $[a, b] \subset J$, let $X_{ab} := X|_{[a,b]}$.

Let $R = \{0 < 1 < 2 < \cdots < N\}$. We then have

$$A(X) = A(X_{m_0 M_0}) \otimes A(X_{M_0 m_1}) \otimes A(X_{m_1 M_1}) \otimes \cdots .$$

Let $W_{r,r+1} := W|_{I_{\overrightarrow{r,r+1}}}$. We then have a dominant map

$$W_{r,r+1} : I_{\overrightarrow{r,r+1}} \to [M_r; m_{r+1}].$$

Set

$$X'_{r,r+1} : I_{\overrightarrow{r,r+1}} \stackrel{W_{r,r+1}}{\to} [M_r; m_{r+1}] \xrightarrow{X} A.$$

Hence, we have an induced map

$$W^*_{r,r+1} : A(X_{M_r; m_{r+1}}) \to A(X'_{r,r+1});$$

via substitution, we get a map:

$$A(X_{M_r m_{r+1}}) \otimes \mathbf{hom}^{I_{\overrightarrow{r,r+1}}}(F_r; F_{r+1}) \to B(F_r(X(M_r)); F_{r+1}(X(m_{r+1}))).$$

The functor F_r induces a map

$$A(X_{m_r M_r}) \to B(Y|_{[m_r;M_r]})$$

Combining these maps we get the desired map μ_W. Let us consider an example in which:

$$J = \{0_j < 1_j < \cdots < 10_j\};$$
$$R = \{a < b < c\};$$
$$I_{\overrightarrow{ab}} = \{0_a < 1_a < 2_a < 3_a\};\ I_{\overrightarrow{bc}} = \{0_b < 1_b < 2_b\},$$

and the map W is given by the following table

$$0_a \mapsto 1_j;\ 1_a, 2_a \mapsto 3_j;\ 3_a \mapsto 4_j;$$
$$0_b \mapsto 5_j;\ 1_b \mapsto 7_j;\ 2_b \mapsto 8_j.$$

Then our map is constructed according to the following picture:

$$\begin{tikzcd}
\bullet^0 \arrow[r] \arrow[d, dash] \arrow[dr, phantom, "F_a"] & \bullet^1 \arrow[rrr] \arrow[d, dash] \arrow[drrr, phantom, "h_{\overrightarrow{ab}}"] & & & \bullet^2 \arrow[r] \arrow[d, dash] \arrow[dr, phantom, "F_b"] & \bullet^3 \arrow[rrr] \arrow[d, dash] \arrow[drrr, phantom, "h_{\overrightarrow{bc}}"] & & & \bullet^4 \arrow[r] \arrow[d, dash] \arrow[dr, phantom, "F_c"] & \bullet^5 \arrow[r] \arrow[d, dash] \arrow[dr, phantom, "F_c"] & \bullet^6 \arrow[d, dash] \\
0_j \arrow[r] & 1_j \arrow[r] \arrow[rr, bend left, "*"'] & 2_j \arrow[r] & 3_j \arrow[r] \arrow[r, bend left, "*"'] \arrow[loop above, "*"] & 4_j \arrow[r] & 5_j \arrow[r] \arrow[rr, bend left, "*"'] & 6_j \arrow[r] & 7_j \arrow[r] \arrow[r, bend left, "*"'] & 8_j \arrow[r] & 9_j \arrow[r] & 10_j
\end{tikzcd}$$

Explanation of the picture:

- in each cell marked by $*$ we compose the arrows on the bottom of the cell;
- in each cell marked by $h_{\overrightarrow{ab}}$, $h_{\overrightarrow{bc}}$ we apply the corresponding element from (9);
- in each cell marked by F_a, F_b or F_c we apply the corresponding functor.

The resulting chain c corresponds to the top line of the arrows on the picture.

Let us write an explicit formula for c Fix a map $X : J \to A$ and a chain $f_{01} \otimes f_{12} \otimes \cdots \otimes f_{910} \in A(X)$ so that $f_{k;k+1} : X(k) \to X(k+1)$. The ordinal K then corresponds to the fat points on the bottom line of the diagram, so $K = \{0 < \cdots < 6\}$. The map $Y : K \to B$ is sends $\{0, \ldots, 6\}$ to, respectively:

$$F_a(X(0)); F_a(X(1)); F_b(X(4)); F_b(X(5)); F_c(X(8)); F_c(X(9)); F_c(X(10)).$$

The resulting chain in $B(Y)$ is given by the following collection of arrows:

$$F_a(f_{01}); h_{\overrightarrow{ab}}(f_{23}f_{12}; \mathrm{Id}_{X(3)}; f_{34}); F_b(f_{45}); h_{\overrightarrow{bc}}(f_{67}f_{56}; f_{78}); F_c(f_{89}); F_c(f_{9,10}).$$

6.2.3 *Definition of a* **seq**-*algebra structure* Let us now construct the structure maps

$$A : \mathbf{seq}(\mathcal{U}) \to \mathbf{full}_{\mathbf{hom}^\bullet}(\mathcal{U}).$$

Equivalently, for each $\mathcal{U}$-diagram in $\mathcal{C}$, one has to construct a map

$$k[\mathbf{seq}(\mathcal{U})] \otimes \bigotimes_{\overrightarrow{f} \in \overrightarrow{\mathcal{F}}} \mathbf{hom}^{I_{\overrightarrow{f}}}(\mathcal{D}|_{\overrightarrow{f}}) \to \mathbf{hom}^J(p_* \mathcal{D}),$$

where $\mathcal{D}|_{\overrightarrow{f}}$ is the $\mathcal{C}$-globe obtained by the restriction of $\mathcal{D}$ onto $\overrightarrow{f}$ and $p_*\mathcal{D}$ is a $\mathcal{C}$-globe obtained by pre-composing $\mathcal{D}$ with the terminal map $p : \mathcal{U} \to \mathbf{globe}$.

Let μ be the minimal element in $\overrightarrow{\mathcal{C}}$. Consider the ordinals $I_{\overrightarrow{f}}$, $\overrightarrow{f} \in \overrightarrow{\mathcal{F}}_\mu$. It follows that they form intervals in $\mathcal{I}$ and $\overrightarrow{f}_1 < \overrightarrow{f}_2$ implies $I_{\overrightarrow{f}_1} < I_{\overrightarrow{f}_2}$ in $\mathcal{I}$.

We have a restriction

$$W:\coprod_{\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_\mu} I_{\overrightarrow{f}}\to J,$$

satisfying all the conditions of the previous section. Hence we have a map μ_W as explained above. Let us show that after the application of μ_W, the remaining ingredients form a similar structure to that with which we started.

The map μ_W only involves the complexes $\mathbf{hom}^{I_{\overrightarrow{f}}}$ with $\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_\mu$. The remaining complexes are labelled by the elements of the set

$$\mathcal{I}':=\mathcal{I}\backslash\coprod_{\overrightarrow{f}\in\overrightarrow{\mathcal{F}}_\mu} I_{\overrightarrow{f}}.$$

The map W naturally descends to a map

$$W':\mathcal{I}'\to K(J,W).$$

Let $\mathcal{C}':=\mathcal{C}\backslash m_{\mathcal{C}}$; We then get a diagram $\mathcal{U}'$ with $\mathcal{C}_{\mathcal{U}'}=\mathcal{C}'$ and $\mathcal{F}_{\mathcal{U}',\overrightarrow{c}}=\mathcal{F}_{\mathcal{U},\overrightarrow{c}}$ It then follows that $(\mathcal{I}',W')\in\mathbf{seq}(\mathcal{U}')$. Thus we have constructed a map

$$\nu_{\mathcal{U}}:k[\mathbf{seq}(\mathcal{U})]\otimes\bigotimes_{\overrightarrow{f}\in\overrightarrow{\mathcal{F}}}\mathbf{hom}^{I_{\overrightarrow{f}}}(\mathcal{D}|_{\overrightarrow{f}})\to k[\mathbf{seq}(\mathcal{U}')]\otimes\bigotimes_{\overrightarrow{f}'\in\overrightarrow{\mathcal{F}}_{\mathcal{U}'}}\mathbf{hom}^{I_{\overrightarrow{f}}'}(F_{P_*\mathcal{D}|_{\overrightarrow{f}'}})$$

One can now iterate this procedure thus exhausting all the arguments; in the end we will obtain a chain of morphisms in $C_{M_{\mathcal{C}}}$, and, finally, we can take the composition of all morphisms in this chain, which will produce the result.

We omit the proof that this is indeed a **seq**-algebra structure — this is pretty clear.

6.2.4 *Examples* Consider a pair of typical examples.

Example 1: The 2-ordinal U (see (10) is as follows $\mathcal{C}_U:=\{0<1\}$; $\mathcal{F}_{\overrightarrow{01}}=\{f_0<f_1<f_2\}$. We have $\mathcal{F}_{\mathcal{U}}:=\{\overrightarrow{f_0f_1},\overrightarrow{f_1f_2}\}$:

$$\begin{tikzcd} 0 \arrow[r, bend left=50, "f_2"] \arrow[r, "f_1"] \arrow[r, bend right=50, "f_0"'] & 1 \end{tikzcd} \tag{10}$$

Fix a $\mathcal{U}$-diagram $\mathcal{D}$ so that we have dg-categories C_0,C_1 and functors $F_0,F_1,F_2:C_0\to C_1$. Let $I_{\overrightarrow{f_0f_1}}:=\{0_f<\cdots<n_f\}$; $I_{\overrightarrow{f_1f_2}}:=\{0_g<\cdots<m_g\}$; $J:=\{0_j<\cdots<(n+m)_j\}$. Thus, we have a coloring of $\mathcal{U}$.

Fix the order on $\mathcal{I}:=I_f\sqcup I_g$ by prescribing $I_f<I_g$. Set

$$W:I_f\sqcup I_g\to J$$

as follows:

$$W(p_i):=p_j;\ W(q_j):=(n+q)_j.$$

The order on $\mathcal{I}$ and the map W determine an element $c\in\mathbf{seq}(\mathcal{U})$. This element determines an operation

$$m_c:C^{I_{f_0f_1}}(F_0;F_1)\otimes C^{I_{f_1f_2}}(F_1;F_2)\to C^J(F_0,F_2)$$

By definition, this operation is as follows.

Fix elements $\phi \in C^{I_{f_0 f_1}}(F_0; F_1)$; $\psi \in C^{I_{f_1 f_2}}(F_1; F_2)$. Fix a chain x in $C_0(J)$ by fixing objects $X_s \in C_0$, $0 \leqslant s \leqslant n+m$ and morphisms $h_s : X_s \to X_{s+1}$. We set

$$m_c(\phi, \psi)(x) := \psi(h_n, h_{n+1}, \cdots, h_{n+m-1}) \circ \phi(h_0, h_1, \cdots, h_{n-1}).$$

Example 2: Let now $\mathcal{U}$ (see (11) be a 2-ordinal such that $\mathcal{C}_{\mathcal{U}} = \{0, 1, 2\}$; $\mathcal{F}_{\overrightarrow{01}} = \{f_0 < f_1\}$; $\mathbf{globe}_{\overrightarrow{12}} := \{g_0 < g_1\}$. We then have $\mathcal{F}_{\mathcal{U}} = \{\overrightarrow{f_0 f_1}; \overrightarrow{g_0 g_1}\}$. Denote $f := \overrightarrow{f_0 f_1}$; $g := \overrightarrow{g_0 g_1}$:

$$\begin{tikzcd} 0 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_0"'] & 1 \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_0"'] & 2 \end{tikzcd} \tag{11}$$

Fix a coloring of $\mathcal{U}$ by setting $I_f := \{0_f < \cdots < n_f\}$; $I_g := \{0_g < \cdots < m_g\}$. Let $J := \{0_j < \cdots < (n+m-1)_j\}$. Given a number $R \in \{1, \ldots, m\}$ Let us define an element $s := s_R \in \mathbf{seq}(\mathcal{U})$ by

– fixing a total order on $\mathcal{I}$:

$$\{0_g < 1_g < \cdots < (R-1)_g < 0_f \cdots < n_f < R_g < (R+1)_g < \cdots < m_g\};$$

– fixing a map $W : \mathcal{I} \to J$ as a unique monotone surjective map such that $W((R-1)_g) = W(0_f)$ and $W(n_f) = W(R_g)$. More specifically:

$$0_g \mapsto 0_j;\ 1_g \mapsto 1_j; \cdots; (R-1)_g \mapsto (R-1)_j;$$
$$0_f \mapsto (R-1)_j;\ 1_f \mapsto R_j; \cdots n_f \mapsto (R-1+n)_j;$$
$$R_g \mapsto (R-1+n)_j;\ (R+1)_g \mapsto (R+n)_j; \cdots; m_g \mapsto (m-1+n)_j.$$

Let us now fix a $\mathcal{U}$-diagram i.e. categories C_0, C_1, C_2 and functors $F_0, F_1 : C_0 \to C_1$ and $G_0, G_1 : C_1 \to C_2$. Let us construct the operation

$$m_{s_R} : \hom^{I_f}(F_0, F_1) \otimes \hom^{I_g}(G_0, G_1) \to \hom^J(G_0 F_0, G_1 F_1).$$

Let $h_F \in \hom^{I_f}(F_0, F_1)$ and $h_G \in \hom^{I_g}(G_0, G_1)$ and let $X_0, \ldots X_{n+m-1}$; $u_k : X_k \to X_{k+1}$ be a chain of objects and arrows in C_0. Let $X : J \to C_0 :\ k_j \mapsto X_k$. We then have a chain $u := u_0 \otimes \ldots u_{n+m-2} \in C_0(X)$. Let as construct the arrow

$$m_{s_R}(h_F, h_G)(u) : G_0 F_0(X_0) \to G_1 F_1(X_{n+m-1})$$

in C_2.

Applying h_F, we get the following chain in C_1:

$$F_0(X_0) \xrightarrow{F_0(u_0)} F_0(X_1) \xrightarrow{F_0(u_1)} \cdots \xrightarrow{F_0(u_{R-2})} F_0(X_{R-1}) \xrightarrow{h_F(u_{R-1}, u_R, \ldots, u_{R-2+n})} F_1(X_{R-1+n})$$
$$\xrightarrow{F_1(u_{R-1+n})} F_1(X_{R+n}) \xrightarrow{F_1(u_{R+n})} \cdots \xrightarrow{F_1(u_{R+n+m-2})} F_1(X_{n+m-1})$$

We apply h_G to this chain so as to get an arrow

$$h_G\Big(F_0(u_0); F_0(u_1); \cdots; F_0(u_{R-2}); h_F(u_{R-1}, u_R, \cdots, u_{R-2+n}); F_1(u_{R-1+n}); \cdots; F_1(u_{R+n+m-2})\Big)$$

which belongs to

$$\hom_{C_2}(G_0 F_0(X_0); G_1 F_1(X_{n+m-1})).$$

This arrow is $m_{s_R}(h_F, h_G)(u)$.

6.2.5 In Sec. 3.1.1 we have defined a map of cosimplicial complexes

$$\hom^\bullet(F, G) \to \mathbf{hom}^\bullet(F, G)$$

for every pair of functors $F, G : A \to B$. In this way, we get a map of $\mathcal{C}$-complexes

$$\hom^\bullet \to \mathbf{hom}^\bullet.$$

It is easy to check that $\hom^\bullet$ is a **seq**-subalgebra of $\mathbf{hom}^\bullet$. Furthermore, given a 2-ordinal $\mathcal{U}$ and an $\mathcal{U}$-diagram $\mathcal{D}$, for every $e \in \mathbf{seq}(\mathcal{U})$, the structure map

$$\bigotimes_{\vec{f} \in \vec{\mathcal{F}}_{\mathcal{U}}} \hom^{I_{\vec{f}}}(\mathcal{D}_{\vec{f}}) \xrightarrow{1 \mapsto e} \mathbf{seq}(\mathcal{U}) \otimes \bigotimes_{\vec{f} \in \vec{\mathcal{F}}_{\mathcal{U}}} \hom^{I_{\vec{f}}}(\mathcal{D}_{\vec{f}}) \longrightarrow \hom^J(p_* \mathcal{D})$$

is the same.

This can be formulated as follows. Let $\mathcal{T}$ be the trivial $\mathbb{N}$-colored 2-operad: for every $\mathbb{N}$-colored 2-ordinal $\mathcal{U}$, $\mathcal{T}(\mathcal{U}) := \{1\}$, and the structure maps are then uniquely defined. Let

$$\mathbf{seq} \to \mathcal{T} \tag{12}$$

be the obvious projection. We then have:

Proposition 6.3. *i)* $\hom^\bullet \subset \mathbf{hom}^\bullet$ *is a* **seq***-subalgebra;*

ii) the **seq***-action on* $\hom^\bullet$ *passes through the projection* (12).

6.2.6 *Cosimplicial structure* Let us recover the cosimplicial structure on $\mathbf{hom}^\bullet$ from the **seq**-structure.

Let $\mathbf{globe}, I, J$ be the globe colored by the ordinals I and J. By definition,

$$\mathbf{seq}(\mathbf{globe}, I, J) = \Delta(I, J)$$

is the space of all monotone maps.

The 2-operadic structure gives rise to associative maps

$$\mathbf{seq}(\mathbf{globe}, I, J) \times \mathbf{seq}(\mathbf{globe}, J, K) \to \mathbf{seq}(\mathbf{globe}, I, K)$$

thus giving rise to a category structure on $\mathbb{N}$ which is just given by composing the corresponding monotone maps. That is, this category is nothing else but the simplicial category Δ.

6.2.7 Given a colored 2-ordinal $\mathcal{U}'$ with the underlying 2-ordinal $\mathcal{U}$ and a coloring given by the family of ordinals $I_{\vec{f}}$, $\vec{f} \in \vec{\mathcal{F}}_{\mathcal{U}}$; J, write

$$\mathbf{seq}(\mathcal{U})^J_{\{I_{\vec{f}}\}_{\vec{f} \in \vec{\mathcal{F}}}} := \mathbf{seq}(\mathcal{U}').$$

As a function in $I_{\vec{f}}, J$, $\mathbf{seq}(\mathcal{U})$ becomes a functor

$$(\Delta^{\mathrm{op}})^{\vec{\mathcal{F}}} \times \Delta \to \mathbf{sets}.$$

6.3 Passing to complexes

Define the realization

$$\mathcal{O}(\mathcal{U}) := |\mathbf{seq}(\mathcal{U})| := \hom_\Delta(\mathcal{S}; k[\mathbf{seq}(\mathcal{U})] \otimes_{(\Delta^{\mathrm{op}})^{\vec{\mathcal{F}}}} (\mathcal{S})^{\boxtimes \vec{\mathcal{F}}}),$$

where $\mathcal{S}$ is as in (3.1). A similar construction can be found in [MS04]

It is immediate that these realizations form a dg-2-operad $\mathcal{O}$, and that this operad acts on the $\mathcal{C}$-complex $\mathbf{Rhom} = \hom_\Delta(\mathcal{S}, \mathbf{hom}^\bullet)$.

Our goal now is to check that this operad satisfies the theorem.

6.4 Contractibility of $\mathcal{O}$

First of all, let us construct a quasi-isomorphism

$$\mathcal{O} \to \mathbf{triv}.$$

It is easy to construct such a map: it is just the augmentation map. Let us show that this map is a quasi-isomorphism.

We will start with studying a multisimplicial set

$$\mathbf{seq}(\mathcal{U})^J_{\bullet,\ldots,\bullet} : \Delta^{F_\mathcal{U}} \to \mathbf{sets}, \tag{13}$$

where J is a fixed ordinal. Let $S(\mathcal{U}, J)$ be the topological space obtained by applying the standard topological realization functor to this multisimplicial set.

Proposition 6.4. *For each ordinal J, the topological space $S(U, J)$ is contractible.*

Proof. One can describe this realization explicitly. Let us so do: a point of $S(\mathcal{U}, J)$ is given by an equivalence class decompositions of a fixed segment $I := I_\mathcal{U}$ of length $|\overrightarrow{\mathcal{F}}|$ into a number of subsegments labelled by the elements from $\overrightarrow{\mathcal{F}}_\mathcal{U}$.

The labelling should satisfy:

a) if $\overrightarrow{f}_1, \overrightarrow{f}_2 \in \overrightarrow{\mathcal{F}}$ and a segment labelled by $\overrightarrow{f}_2$ lies between those labelled by $\overrightarrow{f}_1$, then $\pi(\overrightarrow{f}_1) > \pi(\overrightarrow{f}_2)$;

b) if $\pi(\overrightarrow{f}_1) = \pi(\overrightarrow{f}_2) = \overrightarrow{c}$ and $\overrightarrow{f}_1 < \overrightarrow{f}_2$ in $\overrightarrow{\mathcal{F}}_{\overrightarrow{c}}$, then all segments labelled by $\overrightarrow{f}_1$ are on the LHS of elements labelled by $\overrightarrow{f}_2$;

c) the total length of all segments labelled by the same element $\overrightarrow{f}$ is 1;

d) a monotone map $\overrightarrow{J} \to I$.

Two such points are equivalent if one is obtained from another by a number of operations of the following two types:

- adding into or deleting from our decomposition a number of labelled segments of length 0;
- joining two neighboring segments of our decomposition labelled by the same letter into one segment labelled by the same letter, or the inverse operation.

This space receives an obvious CW-structure. The proof that this space is indeed a realization is straightforward.

Let $S(\mathcal{U})$ be the space whose points are described by a)-c) (without d), and the equivalence relation is the same. We then get an obvious isomorphism

$$S(\mathcal{U}, J) = S(\mathcal{U}) \times \Delta^J.$$

Remarks. 1) One can prove that this is an isomorphism of cosimplicial spaces.

2) The topological realization

$$|\Sigma(\mathcal{U})| := |S(\mathcal{U}, \bullet)|$$

is then identified with the space $S(\mathcal{U}) \times R$, where R is the space of monotone maps of a unit segment into the segment I. The spaces $\Sigma(\mathcal{U})$ form a topological 2-operad. This operad acts on topological realizations of the topological version of $\mathbf{hom}^\bullet(F, G)$

Thus, it only remains to show that the space $S(\mathcal{U})$ is contractible.

For simplicity, let us identify $\mathcal{C}$ with the set $0 < \cdots < n$. Let $\mathcal{U}_m$ be a ball in $\mathcal{U}$ which is the preimage of $[m, n] \subset \mathcal{C}$. We then have a natural projection

$$P_m : S(\mathcal{U}_m) \to S(\mathcal{U}_{m+1});$$

this projection sends a point in $S(\mathcal{U}_m)$ into a point obtained from it by collapsing each segment labelled by elements from $\pi^{-1}\overrightarrow{m,m+1}$ to a point.

Conversely, given

- a point $x \in S(\mathcal{U}_{m+1})$, and
- a monotone map $U : \pi^{-1}(\overrightarrow{m,m+1}) \to I_{\mathcal{U}_{m+1}}$,

one can construct a point in $S(\mathcal{U}_m)$ by inserting unit segments labelled by $i \in \pi^{-1}(\overrightarrow{m,m+1})$ in place of the point $U(i)$.

It is clear that this way we get a bijection

$$S(\mathcal{U}_m) \cong S(U_{m+1}) \times \Delta^{\pi^{-1}m}.$$

This argument implies that the space $S(\mathcal{U})$ is homeomorphic to a product of simplices, hence is contractible. □

Remark. This proof is similar to McClure and Smith's proof of contractibility of cells of their operad D_2 (see [MS04]).

6.4.1 Let us now use this proposition in order to show that the augmentation map $\mathcal{O} \to \mathbf{triv}$ is a weak equivalence of dg-operads.

Consider the complex

$$\Sigma(\mathcal{U}, J) := k[\mathbf{seq}(\mathcal{U})^J_{\bullet,\cdots,\bullet}] \otimes_{\Delta^{\mathcal{F}_\mathcal{U}}} \mathcal{S}^{\boxtimes \mathcal{F}_\mathcal{U}},$$

which is the realization of the multisimplicial complex $k[\mathbf{seq}(\mathcal{U})^J_{\bullet,\cdots,\bullet}]$.

Let $\mathbf{p}t : (\Delta^{\mathrm{op}})^{\mathcal{F}_\mathcal{U}} \to \mathbf{sets}$ be the constant functor sending everything to the fixed one element set. We then have a natural augmentation map

$$k[\mathbf{seq}(\mathcal{U})^J_{\bullet,\cdots,\bullet}] \to k[\mathbf{p}t]$$

The total complex of $k[\mathbf{p}t]$ is isomorphic to k, and we have an induced map

$$\mathbf{aug} : \Sigma(\mathcal{U}, J) \to k.$$

Proposition 6.4 readily implies that

Corollary 6.5. *The map* **aug** *is a quasi-isomorphism of complexes.*

It is clear that $J \mapsto \Sigma(\mathcal{U}, J)$ is a cosimplicial complex. Let $k' : \Delta \to \mathbf{complexes}$ be the cosimplicial complex sending every object to k and every arrow to the identity of k. It is straightforward to check that the map **aug** gives rise to a map of cosimplicial complexes

$$\mathbf{aug}' : \Sigma(\mathcal{U}, \bullet) \to k'$$

which is a term-wise quasi-isomorpism.

Taking the realization of the cosimplicial complexes on both sides, we get the augmentation map

$$\mathcal{O}(\mathcal{U}) \to k,$$

which is automatically a quasi-isomorphism (because the realization functor preserves quasi-isomorphisms). This implies

Corollary 6.6. *The augmentation map $\mathcal{O} \to$* **triv** *is a weak equivalence of dg 2-operads.*

6.4.2 Thus, we have proven the assertion of Theorem 5.1 that a contractible 2-operad acts on **Rhom**. The remaining conditions 1) and 2) follow immediately from Proposition 6.3. This completes the proof of the Theorem.

7. Relation to Deligne's conjecture

Given a dg-category $\mathcal{A}$, we can consider a complex $\mathbf{Rhom}(\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}})$ This complex is called the Hochschild complex of the category $\mathcal{A}$. If the category $\mathcal{A}$ has only one object p, then its Hochschild complex coincides with that of the associative algebra $\mathrm{End}_{\mathcal{A}}(p)$.

Thus, we denote

$$\mathbf{Hoch}_{\mathcal{A}} := \mathbf{Rhom}(\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}}).$$

As a corollary of the just proven theorem, we have a certain structure on $\mathbf{Hoch}_{\mathcal{A}}$; before defining it, let us give it a name "an $\mathcal{O}$-algebra structure on $\mathbf{Hoch}_{\mathcal{A}}$"

The definition is as follows. Given a complex $\mathcal{K}$ (for example $\mathcal{K} := \mathbf{Hoch}_{\mathcal{A}}$) we define a 2-operad $\mathbf{full}_{\mathcal{K}}$ by setting

$$\mathbf{full}_{\mathcal{K}}(\mathcal{U}) := \hom_k(\mathcal{K}^{\otimes \mathcal{F}_{\mathcal{U}}}; \mathcal{K})$$

with the obvious insertion maps.

Remark. Of course, this construction is a particular case of the full 2-operad of a $\mathcal{C}$-complex, where $\mathcal{C}$ is the category with one object and one arrow so that there is only one globe in $\mathcal{C}$, and a $\mathcal{C}$-complex is the same as a usual complex, so that our complex $\mathcal{K}$ gives rise to a $\mathcal{C}$-complex and we can apply the construction of the full operad of a $\mathcal{C}$-complex. In this way, we get another construction of $\mathbf{full}_{\mathcal{K}}$.

Given a 2-operad $\mathcal{E}$, we define *an $\mathcal{E}$-algebra structure on $\mathcal{K}$* as a map of 2-operads

$$E \to \mathbf{full}_{\mathcal{K}}.$$

Theorem 5.1 immediately implies that:

Proposition 7.1. $\mathbf{Hoch}_{\mathcal{A}}$ *has a structure of algebra over the 2-operad $\mathcal{O}$, as in the statement of the Theorem.*

As $\mathcal{O}$ is a contractible 2-operad, a result from [Bat07] implies the following.

Corollary 7.2. *A certain operad which is homotopy equivalent to the chain operad of little disks acts on* $\mathbf{Hoch}_{\mathcal{A}}$.

This corollary is known as *Deligne's conjecture on Hochschild cochains.*

References

Bat98 M. A. Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Adv. Math. **136** (1998), no. 1, 39–103.

Bat07 ———, *Symmetrisation of n-operads and compactifion of real configuration spaces*, Adv. in Math. (2007), in press.

Ber02 Clemens Berger, *A cellular nerve for higher categories*, Adv. Math. **169** (2002), no. 1, 118–175.

BF04 Clemens Berger and Benoit Fresse, *Combinatorial operad actions on cochains*, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 1, 135–174.

Dri04 Vladimir Drinfeld, *DG quotients of DG categories*, J. Algebra **272** (2004), no. 2, 643–691.

Joy A. Joyal, *Disks, duality, and θ-categories*, preprint.

KS00 Maxim Kontsevich and Yan Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud., vol. 21, Kluwer Acad. Publ., Dordrecht, 2000, pp. 255–307.

MS02 James E. McClure and Jeffrey H. Smith, *A solution of Deligne's Hochschild cohomology conjecture*, Recent progress in homotopy theory (Baltimore, MD, 2000), Contemp. Math., vol. 293, Amer. Math. Soc., Providence, RI, 2002, pp. 153–193.

MS04 ______, *Cosimplicial objects and little n-cubes. I*, Amer. J. Math. **126** (2004), no. 5, 1109–1153.

Str00 Ross Street, *The petit topos of globular sets*, J. Pure Appl. Algebra **154** (2000), no. 1-3, 299–315, Category theory and its applications (Montreal, QC, 1997).

VDT D. Tamarkin V. Dolgushev and B. Tsygan, *Proof of Swiss Cheese conjecture*, in preparation.

Vor00 Alexander A. Voronov, *Homotopy Gerstenhaber algebras*, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 307–331.

Dmitry Tamarkin tamarkin@math.northwestern.edu
Department of Mathematics, Northwestern University