QUANTIZATION OF LIE BIALGEBRAS VIA THE FORMALITY OF THE OPERAD OF LITTLE DISKS

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ABSTRACT. We give a proof of Etingof-Kazhdan theorem on quantization of Lie bialgebras based on the formality of the chain operad of little disks and show that the Grotendieck-Teichmüller group acts non-trivially on the corresponding quantization functors.

1. Introduction

1.1. The present paper is an improved and enlarged version of [18]. We give a proof of Etingof-Kazhdan theorem on quantization of Lie bialgebras based on the formality of the chain operad of little disks.

Any known construction of such a formality involves multiple zeta values; in particular there is no canonical way to establish such a formality over $\mathbb{Q}$. For example, in the construction from [15] one needs to choose an associator over $\mathbb{Q}$. In [16] it is shown that different associators produce homotopically non-equivalent formalities of chain operad of little disks. Each of these formalities, in turn, produces a certain quantization procedure of Lie bialgebras and we prove that these procedures are not isomorphic. This can be considered as a step in studying the action of Grotendieck-Teichmüller group on quantization functors originated in [7].

1.2. **Idea of the construction of quantization.**

1.2.1. For simplicity, let us work in the category of A-modules, where A is a commutative $\mathbb{Q}$-algebra. Let $\mathfrak{g}$ be a Lie bialgebra with bracket $[,]$ and cobracket δ. Call $\mathfrak{g}$ *conilpotent* if for any $x \in \mathfrak{g}$ there exists an N such that any N-fold iteration of δ applied to x produces zero.

1.2.2. Let H be a Hopf algebra with product $\cdot$, coproduct Δ, unit 1, and counit ϵ (we do not assume that the antipode exists). Let $\Delta'(x) = \Delta(x) - 1 \otimes x - x \otimes 1$. Call H *conilpotent* if for any x such that $\epsilon(x) = 0$ there exists an N such that any N-fold iteration of Δ' applied to x produces zero.

1.2.3. Note that in any conilpotent H there exists an antipode map and it is uniquely defined.

1.2.4. We are going to construct a functor $\mathbf{Q}$ from the category of conilpotent Lie bialgebras to the category of conilpotent Hopf algebras

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1.2.5. *How this implies the Etingof-Kazhdan quantization theorem?* This theorem, given a Lie bialgebra $\mathfrak{g}$ over $\mathbb{Q}$, produces a deformed Hopf algebra structure on $U(\mathfrak{g})[[t]]$ ($U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$; t is a formal parameter; the Hopf algebra structure is in the symmetric monoidal category of topologically free and complete $k[[t]]$-modules; this Hopf structure reduces (mod t) to that on $U(\mathfrak{g})$.) In our language this will look as follows. Set $\mathfrak{a}_n := \mathbb{Q}[t]/(t^n)$ and let C_n be the category of $\mathfrak{a}_n$-modules. For every n we have an obvious map $p_n : \mathfrak{a}_n \to \mathfrak{a}_{n-1}$ which induces a functor $P_n : C_n \to C_{n-1}$ (and C_n, P_n form a projective system of categories so that the projective two-limit $\mathrm{liminv}_n C_n$ is the category of complete $\mathbb{Q}[[t]]$-modules and their morphisms.)

Let $\mathfrak{g}_n$ be the Lie bialgebra over $\mathfrak{a}_n$ defined as follows:

1) $\mathfrak{g}_n = \mathfrak{g} \otimes_{\mathbb{Q}} \mathfrak{a}_n$;

2) the bracket is induced by that on $\mathfrak{g}$;

3) the cobracket δ_n on $\mathfrak{g}_n$ is given by:

$$\delta_n(x \otimes a) = ta\delta(x),$$

where $x \in \mathfrak{g}$, $a \in \mathfrak{a}_n$, and δ is the cobracket in $\mathfrak{g}$.

Then $\mathfrak{g}_n$ is conilpotent and we have natural identifications $P_n(\mathfrak{g}_n) \cong \mathfrak{g}_{n-1}$.

Our quantization functor will then produce a conilpotent Hopf algebra $H_n := \mathbf{Q}(\mathfrak{g}_n)$ over $\mathfrak{a}_n$. As our quantization is functorial, we will have identifications $P_n(H_n) \to H_{n-1}$; the Etingof-Kazhdan quantized Hopf algebra will be given by $\mathrm{liminv}_n H_n$.

1.2.6. *Constructing the quantization.* Let $\mathfrak{g}$ be a conilpotent Lie bialgebra. Let $C^\bullet(\mathfrak{g})$ be its cochain complex with respect to the cobracket. This means that $C^\bullet(\mathfrak{g}) = S(\mathfrak{g}[-1])$ is a free graded commutative algebra equipped with a differential D defined on the space of generators $\mathfrak{g}[-1] \subset S(\mathfrak{g}[-1])$ by the cobracket $\delta : \mathfrak{g}[-1] \to S^2(\mathfrak{g}[-1])$.

$C^\bullet(\mathfrak{g})$ has a structure of Gerstenhaber algebra so that the bracket on $\mathfrak{g}[-1] \subset C^\bullet(\mathfrak{g})$ is defined by the bracket on $\mathfrak{g}$.

1.2.7. Let **ger** be the operad of Gerstenhaber algebras and **braces** be the operad of brace structures (see [9],[8], see also Sec. 5.1). Let **hoger** $\to$ **ger** be the standard resolution of **ger** (as defined in [9], see also [11]). It is shown [13] that the operads **braces** and **ger** are quasi-isomorphic. This means that there exists a quasi-isomorphic map of operads **hoger** $\to$ **braces**. Therefore, there is a way to construct a brace-algebra out of a Gerstenhaber algebra. Denote this way by W (it is a functor from the category of Gerstenhaber algebras to the category of brace algebras. Thus, $W(C^\bullet(\mathfrak{g}))$ is a brace algebra.

1.2.8. **Remark 1.** It is exactly on this step that the associators or integrals are being used.

Remark 2. One of the steps of the proof of the formality of **braces** in [13] is linking **braces** with the operad of singular chains of the operad of little disks (this step is purely "combinatorial",— it does not use transcendental methods). Thus, the formality of **braces** follows from the formality of the operad of singular chains of the operad of little disks (see. [12], [15]).

1.2.9. As follows from the definitions, given a brace algebra A, one has a canonical Hopf algebra structure on the co-free coalgebra

$$\bigoplus_{n=0}^{\infty} A[1]^{\otimes n}.$$

Denote this Hopf algebra by $H(A)$. Thus $h(\mathfrak{g}) = HWC^\bullet(\mathfrak{g})$ is a differential graded Hopf algebra. If $H^{\neq 0}h(\mathfrak{g}) = 0$, one gets an induced Hopf algebra structure on $H^0h(\mathfrak{g})$ which is the quantization $\mathbf{Q}(\mathfrak{g})$ of $\mathfrak{g}$.

This is just an idea of our approach. Actually, we will carry out this program on the so called universal level, similar to [6].

1.3. **Universal language.** As was explained in [6], a more appropriate way to deal with quantization of Lie bialgebras/dequantization of bialgebras is via the universal language of PROPs.

The language of PROPs is designed in order to describe algebraic structures on an object A which include maps $A^{\otimes m} \to A^{\otimes n}$ where m, n can be any non-negative integers. Recall that in a simpler situation, when all structure maps are of the type $A^{\otimes m} \to A$ a simpler language, namely that of operads, can be used. Unfortunately, bialgebras are clearly not of this type, that's why we have to use PROPs. Thus, there are PROP's $\mathbf{LBA}$, $\mathbf{BA}$ of Lie bialgebras and bialgebras.

We have an additional subtlety: we have to deal with conilpotent bialgerbas. We will see that the conilpotency can be adequately expressed in terms of certain completions of the PROPs $\mathbf{LBA}$ and $\mathbf{BA}$. A similar approach is used in [6].

More precisely, we define projective systems of PROP's

$$\cdots \to \mathbf{LBA}_n \to \mathbf{LBA}_{n-1} \to \cdots \to \mathbf{LBA}_1;$$

$$\cdots \to \mathbf{BA}_n \to \mathbf{BA}_{n-1} \to \cdots \to \mathbf{BA}_1$$

where $\mathbf{LBA}_n$ are quotients of $\mathbf{LBA}$ with respect to a decreasing chain of ideals $\mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots$;

$$\mathbf{LBA}_n = \mathbf{LBA}/\mathcal{I}_n,$$

and likewise for $\mathbf{BA}_n$ (see 3.1.5).

Next we use Ioneda's embeddings $\mathbf{LBA}_n \to \mathbf{LBA}_n^\wedge$; $\mathbf{BA}_n \to \mathbf{BA}_n^\wedge$, where $\mathbf{LBA}_n^\wedge$ is the category of finitely generated functors from the category $\mathbf{LBA}_n$ to the category of vector spaces, and likewise for $\mathbf{BA}_n$. The categories $\mathbf{LBA}_n^\wedge, \mathbf{BA}_n^\wedge$ inherit a symmetric monoidal structure from $\mathbf{LBA}_n^\wedge, \mathbf{BA}_n^\wedge$, and we can also construct projective systems of symmetric monoidal categories

$$\cdots \to \mathbf{LBA}_n^\wedge \to \mathbf{LBA}_{n-1}^\wedge \to \cdots \to \mathbf{LBA}_1^\wedge;$$

$$\cdots \to \mathbf{BA}_n^\wedge \to \mathbf{BA}_{n-1}^\wedge \to \cdots \to \mathbf{BA}_1^\wedge.$$

We show (Theorem 3.3) that these systems are equivalent: there exist symmetric monoidal equivalences

$$\mathbf{BA}_n^\wedge \to \mathbf{LBA}_n^\wedge$$

compatible with the functors $\mathbf{LBA}_n^\wedge \to \mathbf{LBA}_{n-1}^\wedge$; $\mathbf{BA}_n^\wedge \to \mathbf{BA}_{n-1}^\wedge$.

We also show that this theorem gives us a way to quantize conilpotent Lie algebras:

given a $\mathbf{k}$-linear SMC $\mathcal{C}$ satisfying certain restrictions (Sec. 4.1) , we make a definition of a conilpotent Lie bialgebra and a conilpotent bialgebra in $\mathcal{C}$ (see 4.1.2), and construct an equivalence of the category of conilpotent Lie bialgebras in $\mathcal{C}$ and the category of conilpotent bialgebras in $\mathcal{C}$.

In the case when $\mathcal{C}$ is the category of A-modules, where A is a commutative $\mathbf{k}$-algebra, the abstract definition of a conilpotent bialgebra and Lie bialgebra in $\mathcal{C}$ as in 4.1.2 is equivalent to that in 1.2.

After having established the quantization procedure, we investigate how it depends on the choice of a zigzag quasi-isomorphism between the operad of braces and an operad of Gerstenhaber algebras. In the last section of the paper we study this dependence, and show that it is essential:

we prove that the quantization functors produced using different quasi-isomorphisms between the operads of braces and Gerstenhaber algebras are isomorphic if and only if the quasi-isomorphisms are homotopy equivalent (Theorem 8.1). Furthermore, we show that given a pair q_1, q_2 of homotopy inequivalent quasi-isomorphisms between the operads of braces and Gerstenhaber algebras, there exists a co-nilpotent Lie bialgebra in a certain symmetric monoidal category C such that its quantizations using q_1 and q_2 produce non-isomorphic bialgebras in C.

Lastly we investigate the relationship with associators. It turns out that given an associator, there is a natural way to construct a zigzag quasi-isomorphism between the operad of Gerstenhaber algebras and the operad of braces, hence a quantization procedure. We show that different associators produce homotopy non-equivalent quasi-isomorphisms between the operads, hence non-isomorphic quantization procedures. This concludes the paper.

1.4. **Plan of the paper.** In Section 3 we provide all the necessary material in order to formulate the universal quantization Theorem 3.3. This includes a discussion of PROP's of bialgebras and Lie bialgebras. Next, we define projective systems formed by the SMC $\mathbf{LBA}_n, \mathbf{BA}_n$ and their Ioneda's completions. This allows us to formulate the quantization theorem 3.3 in the universal language.

We postpone the proof of this result, showing instead that this universal quantization result allows one to quantize conilpotent Lie bialgebras (Section 4). We make an abstract definition of a conilpotent Lie bialgebra and a bialgebra and show that Theorem 3.3 defines a quantization in this setting.

Next, we prove Theorem 3.3. The proof occupies Sections 5-7. A plan of the proof can be found in Sec. 5, so we do not discuss it here, instead only making a remark that, essentially, we translate the sketch in Sec. 1.2 into the universal language.

The last Section 8 deals with dependence of the quantization functor on the choice of a quasi-isomorphism between the operads of braces and Gerstenhaber algebras.

There are three Appendices. in Appendix 1 we collect some categorical constructions which are used throughout the paper. In Appendix 2 we study the automorphism group of the operad **hoger** in the derived category of dg-operads. In Appendix 3 we collected all the information on the Grothendieck-Teichmuller group which is used in this paper.

2. Notation

We fix a ground field $\mathbf{k}$ of characteristic 0.

Throughout the paper we use abbreviations SM for "symmetric monoidal" and SMC for "symmetric monoidal category.

Mostly, operads are denoted by words in lower case boldface, for example:

lie is the operad of Lie algebras;

ger is the operad of Gerstenhaber algebras;

hoger $\to$ **ger** is the standard cofibrant resolution of **ger**;

braces is the operad of brace algebras.

PROPs are mostly denoted unsing the upper case bold face:

ASSOC is the PROP of associative algebras;

$\mathbf{COASS} = \mathbf{ASSOC}^{\mathrm{op}}$ is the PROP of co-associative co-algebras;

LIE is the PROP of Lie algebras; $\mathbf{COLIE} = \mathbf{LIE}^{\mathrm{op}}$ is the PROP of Lie coalgebras;

GER is the PROP of Gerstenhaber algebras;

HOGER is the PROP generated by the operad **hoger**;

BRACES is the PROP of brace algebras;

LBA is the PROP of Lie bialgebras;

BA is a PROP such that a **BA**-algebra structure on an object $\mathfrak{m}$ in a SMC $\mathcal{C}$ is equivalent to a bialgebra structure on $\mathbf{1} \oplus m$ with the standard unit and counit map. Here $\mathbf{1}$ is the tensor unit on $\mathcal{C}$, and we assume that $\mathbf{1} \oplus \mathfrak{m}$ exists.

Objects of PROPs are denoted by $< n >$, where $n = 1, 2, 3, \ldots$ so that $< m > \otimes < n >=< m + n >$. The reason for using the curly brackets is that the square brackets are used to denote the shift in cohomological degree.

Given an object X in a SMC $\mathcal{C}$ we denote by $\mathbf{full}(X)$ its full operad and by $\mathbf{FULL}(X)$ its full PROP.

3. Formulating the universal statement

3.1. Prop of bialgebras.

3.1.1. Let C be a dg SM category with unit $\mathbf{1}$ and finite direct sums. We say that $X \in C$ has a structure of bialgebra if C has structure of associative algebra with unit and a coassociative coalgebra with counit such that the coproduct map $X \to X \otimes X$ and the counit map $X \to \mathbf{1}$ are morphisms of unital associative algebras.

3.1.2. We say that $m \in C$ has a **BA**-structure if $X := \mathbf{1} \oplus m$ exists and we have a bialgebra structure on X with the natural inclusion of $\mathbf{1}$ being the unit and the natural projection onto $\mathbf{1}$ being the counit.

3.1.3. There exists a PROP **BA** of **BA**-algebras. **BA** is uniquely specified by the condition that **BA**-structures on $\mathfrak{m} \in C$ are in 1-1 correspondence with the maps of PROPs $\mathbf{BA} \to \mathbf{FULL}(\mathfrak{m})$

Note that we can now talk about **BA** -algebras in any SMC regardless of existence of direct sums.

3.1.4. A **BA**-structure on $\mathfrak{m}$ implies structures of associative algebra without unit and of a coassociative algebra without unit on $\mathfrak{m}$. Let **ASSOC** (resp. **COASS**) be the PROPs describing associative algebra (resp. coassociative coalgebra) structure. We thus have maps of PROPs $i : \mathbf{ASSOC} \to \mathbf{BA}$ and $j : \mathbf{COASS} \to \mathbf{BA}$. The composition defines a map

$$\phi_y : \mathbf{ASSOC}(< y >, < z >) \otimes_{S_y} \mathbf{COASS}(< x >, < y >) \to \mathbf{BA}(< x >, < z >).$$

Therefore, we have a map

$$\phi := \sum \phi_n : \bigoplus_{n=x}^{\infty} \mathbf{ASSOC}(< n >, < z >) \otimes_{S_n} \mathbf{COASS}(< x >, < n >) \\ \to \mathbf{BA}(< x >, < z >)$$

PROPOSITION 3.1. *([7], [14]) ϕ is an isomorphism*

3.1.5. Let $\mathcal{I}_n \subset \mathbf{BA}$ be the double-sided categorical ideal generated by $\mathrm{Id}_{<m>}$ for all $m > n$. Let $\mathbf{BA}_n := \mathbf{BA}/\mathcal{I}_n$. This readily implies that $\hom_{\mathbf{BA}_n}(< p >; < q >) = 0$ as long as $p > n$ or $q > n$ so that all $< p >$ with $p > n$ are isomorphic to the zero-object.

If $p, q \le n$, then we have:

$$(1) \quad \mathbf{BA}_n(< p >, < q >)$$

$$\cong \bigoplus_{r=\max(p,q)}^{n} \mathbf{COASS}(< p >, < r >) \otimes_{S_r} \mathbf{ASSOC}(< r >, < q >).$$

Note that the ideals $\mathcal{I}_n$ are actually tensor ideals meaning that whenever $f \in \mathcal{I}_n$ and g is any arrow, $f \otimes g \in \mathcal{I}_n$. Therefore, we have an induced tensor structure on $\mathbf{BA}_n$.

Since the ideals $\mathcal{I}_n$ form a decreasing chain, the quotients form a projective system

$$\mathbf{BA} \to \cdots \to \mathbf{BA}_n \to \mathbf{BA}_{n-1} \to \cdots \to \mathbf{BA}_1$$

in which every arrow is a symmetric monoidal functor.

We will use the following notation for these arrows:

$$P_{nm} : \mathbf{BA}_n \to \mathbf{BA}_m; \ P_n : \mathbf{BA} \to \mathbf{BA}_n.$$

We can form a topological PROP $\mathrm{liminv}_n \mathbf{BA}_n$ so that the map

$$\mathbf{BA} \to \mathrm{liminv}_n \mathbf{BA}_n$$

can be viewed as a completion of $\mathbf{BA}$. However, our results will be formulated on the level of the projective system of PROPs $\mathbf{BA}_n$ without passing to the projective limit; we thus won't discuss this projective limit in detail.

3.1.6. *Ioneda's completions.* Given a finite k-linear category $\mathcal{C}$ (see Appendix 1) we denote by $\mathcal{C}^\wedge$ the abelian category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{vect}_{\mathrm{fin}}$, where $\mathbf{vect}_{\mathrm{fin}}$ is the category of finite-dimensional $\mathbf{k}$-vector spaces.

Suppose that $\mathcal{C}$ is a SMC. We then have an induced SM-structure on $\mathcal{C}^\wedge$ (see Appendix 1).

We have functors

$$P_{nm}^{-1} : \mathbf{BA}_m^\wedge \to \mathbf{BA}_n^\wedge;$$

$$P_{nm!} : \mathbf{BA}_n^\wedge \to \mathbf{BA}_m^\wedge,$$

the latter functor has a symmetric monoidal structure (these functors are defined in the Appendix).

3.2. A similar story takes place in the world of Lie bialgebras. We have a PROP **LBA** of Lie bialgebras.

3.2.1. Let **LBA** be the PROP of Lie bialgebras, **LIE** be the PROP of Lie algebras and **COLIE** be the PROP of Lie coalgebras. We have maps $i : \mathbf{LIE} \to \mathbf{LBA}$ and $j : \mathbf{COLIE} \to \mathbf{LBA}$.

PROPOSITION 3.2. *The natural map* $\phi : \oplus_{n \ge |Y|} \mathbf{COLIE}(< y >, < n >) \otimes_{S_n} \mathbf{LIE}(< n >, < x >) \to \mathbf{LBA}(< y >, < x >)$ *is an isomorphism.*

3.2.2. Let $\mathcal{I}_n \subset \mathbf{LBA}$ be the double-sided ideal generated by $\mathrm{Id}_{<N>}$ for all $N > n$. We then see that this ideal is also a symmetric monoidal ideal.

We set $\mathbf{LBA}_n := \mathbf{LBA}/\mathcal{I}_n$. We then have

$$\mathbf{LBA}_n(< k >, < l >) = 0$$

as long as $k > n$ or $l > n$.

In the case $k, l \leq n$, we have

$$\mathbf{LBA}_n(< k >, < l >) \cong \bigoplus_{m=k}^{n} \mathbf{COLIE}(< k >, < m >) \otimes_{S_m} \mathbf{LIE}(< m >, < l >).$$

The symmetric monoidal categories $\mathbf{LBA}_n$ form a projective system

$$\mathbf{LBA} \to \cdots \to \mathbf{LBA}_n \to \mathbf{LBA}_{n-1} \cdots \to \mathbf{LBA}_1,$$

where all arrows have a natural symmetric monoidal structure.

We denote the arrows in this system as follows:

$$P_{nm} : \mathbf{LBA}_n \to \mathbf{LBA}_m; \; P_n : \mathbf{LBA} \to \mathbf{LBA}_n.$$

We can form a completion

$$\mathbf{LBA} \to \mathrm{liminv}_n \mathbf{LBA}_n,$$

but mostly, we won't pass to the limit.

We have functors

$$P_{nm}^{-1} : \mathbf{LBA}_m^{\wedge} \to \mathbf{LBA}_n^{\wedge}$$

and

$$P_{nm!} : \mathbf{LBA}_n^{\wedge} \to \mathbf{LBA}_m^{\wedge},$$

the latter functor has a symmetric monoidal structure (Appendix 1).

3.3. **Universal quantization theorem.** We have a pair of projective systems of SMC: the first one is

$$\cdots \to \mathbf{BA}_n^{\wedge} \to \mathbf{BA}_{n-1}^{\wedge} \to \cdots ;$$

the second one is

$$\cdots \to \mathbf{LBA}_n^{\wedge} \to \mathbf{LBA}_{n-1}^{\wedge} \to \cdots .$$

We are going to formulate the theorem saying that the two systems are equivalent. Let us first define the notion of equivalence of projective systems of SMC.

3.3.1. *Equivalence of projective systems of SMC.* We define *a projective system of SMC* as a collection of SMC $\mathcal{C}_n$, $n = 1, 2, \cdots$, and symmetric monoidal functors

$$p_n : \mathcal{C}_n \to \mathcal{C}_{n-1}, \; n > 1.$$

In order to denote projective systems of SMC, we will use underlined symbols or words. For example, let us denote the projective system we have just introduced by $\underline{\mathcal{C}}$.

Let

$$\cdots \xrightarrow{p'_{n+1}} \mathcal{C}'_n \xrightarrow{p'_n} \cdots$$

be another projective system of SMC, and denote it by $\underline{\mathcal{C}'}$. We define *a functor* $F : \underline{\mathcal{C}} \to \underline{\mathcal{C}'}$ as

— a collection of SM functors $F_n : \mathcal{C}_n \to \mathcal{C}'_n$;

— a collection of isomorphisms of SM functors

$$p'_n F_n \to F_{n-1} p_n$$

for all $n > 1$.

We say that F is *an equivalence* if each F_n is an equivalence of categories.

3.3.2. Let $\underline{\mathbf{BA}}$; $\underline{\mathbf{BA}}^\wedge$; $\underline{\mathbf{LBA}}$; $\underline{\mathbf{LBA}}^\wedge$ be the projective systems formed by $\mathbf{BA}_n$, $\mathbf{BA}_n^\wedge$, $\mathbf{LBA}_n$, and $\mathbf{LBA}_n^\wedge$ respectively.

3.3.3. *2-category.* This subsection won't be needed for the formulating of our result.

Given a pair of functors $F, G : \underline{\mathcal{C}} \to \underline{\mathcal{C}'}$ we can define the notion of a map $\alpha : F \to G$ as a collection of SM natural transformations

$$\alpha_n : F_n \to G_n$$

which commute with p_n, p'_n in the obvious way.

We thus get a set $\hom(F, G)$ of all maps $F \to G$. It is clear that this way the functors $\mathcal{C} \to \mathcal{C}'$ form a category, and all projective systems of SMC form a 2-category.

3.3.4. *Invertibility of equivalences.* Given a SM equivalence $F : C_1 \to C_2$ of SMC C_1, C_2, one can construct an inverse one $G : C_2 \to C_1$ such that the compositions $FG : C_2 \to C_2$; $GF : C_1 \to C_1$ are isomorphic to the Identity. Given another inverse to F, say G_1, we have a canonical isomorphism between G and G_1.

Same holds true for equivalences of projective systems of SMC.

3.3.5. *Universal quantization theorem.*

THEOREM 3.3. *There exists an equivalence*

$$\mathbf{Q} : \underline{\mathbf{BA}}^\wedge \to \underline{\mathbf{LBA}}^\wedge.$$

Note that, equivalently, one can say that there exists an equivalence in the opposite direction (according to 3.3.4).

Etingof-Kazhdan call $\mathbf{Q}$ "quantization", and the inverse equivalence "dequantization".

4. Theorem 3.3 produces a quantization of conilpotent Lie bialgebras

The plan is as follows:

1) We make a definition of a conilpotent Lie bialgebra/$\mathbf{BA}$-algebra in a SMC $\mathcal{C}$ (provided that $\mathcal{C}$ satisfies certain conditions). Our ultimate goal is to construct an equivalence of the categories of conilpotent Lie bialgebras and $\mathbf{BA}$-algebras in $\mathcal{C}$.

2) We relate the notions of conilpotent Lie bialgebra/ $\mathbf{BA}$-algebra with the projective systems of SMC $\underline{\mathbf{LBA}}^\wedge$ and $\underline{\mathbf{BA}}^\wedge$. We do it as follows:

a) Let $\mathbf{LBA}_n^{\mathbf{pro}} \subset \mathbf{BA}_n^\wedge$ be the full SM- subcategory of finitely-generated projective objects; let $\mathbf{BA}_n^{\mathbf{pro}}$ be a similar thing, it is clear that $\mathbf{LBA}_n^{\mathbf{pro}}$; $\mathbf{BA}_n^{\mathbf{pro}}$ form projective sub-systems of SMC: $\underline{\mathbf{LBA}}^{\mathbf{pro}} \subset \underline{\mathbf{LBA}}^\wedge$; $\underline{\mathbf{BA}}^{\mathbf{pro}} \subset \underline{\mathbf{BA}}^\wedge$ and that the quantization functor induces an equivalence of these sub-systems

$$\mathbf{Q} : \underline{\mathbf{BA}}^{\mathbf{pro}} \xrightarrow{\sim} \underline{\mathbf{LBA}}^{\mathbf{pro}}.$$

Indeed, any equivalence of abelian categories preserves the class of projective objects.

We define SM categories lim-$\underline{\mathbf{LBA}}$:= $\mathrm{liminv}_n \mathbf{LBA}_n^{\mathbf{pro}}$; lim-$\underline{\mathbf{BA}}$:= $\mathrm{liminv}_n \mathbf{BA}_n^{\mathbf{pro}}$; it follows that $\mathbf{Q}$ induces an equivalence between these categories

b) We show that the category of conilpotent Lie bialgebras in $\mathcal{C}$ is equivalent to the category of direct sum preserving SM-functors lim-$\underline{\mathbf{LBA}} \to \mathcal{C}$; likewise, the category of $\mathbf{BA}$-algebras is equivalent to the category of direct sum preserving functors lim-$\underline{\mathbf{BA}} \to \mathcal{C}$.

3) As the equivalence $\mathbf{Q}$: lim-$\underline{\mathbf{BA}} \to$ lim-$\underline{\mathbf{LBA}}$ happens to be direct sum preserving, it induces an equivalence functor from the category of conilpotent Lie bialgebras in $\mathcal{C}$ to the category of conilpotent $\mathbf{BA}$-algebras in $\mathcal{C}$.

4.1. Conilpotent Lie- and BA-algebras.

4.1.1. *Conditions on the SMC we will work in.* Let $\mathcal{C}$ be a k-linear symmetric monoidal category. We assume it possesses the following features:

1) Countable direct sums exist in $\mathcal{C}$ and are compatible with the tensor product: the natural map

$$\bigoplus_{(i,j)\in I\times J} X_i \otimes Y_j \to \left(\bigoplus_{i\in I} X_i\right) \otimes \left(\bigoplus_{j\in J} Y_j\right)$$

must be an isomorphism, where I, J are at most countable sets, $X_i, i \in I$; $Y_j, j \in J$ are arbitrary objects in $\mathcal{C}$.

2) We demand that for every object $Z \in \mathcal{C}$ and any at most countable family $\{X_i\}_{i\in I}$ in $\mathcal{C}$, the natural map

$$\hom_{\mathcal{C}}(Z, \bigoplus_{i\in I} X_i) \to \prod_{i\in I} \hom_{\mathcal{C}}(Z, X_i)$$

must be injective.

3) $\mathcal{C}$ must be closed under kernels of projectors.

We fix such a category $\mathcal{C}$.

4.1.2. *Definition of a conilpotent Lie bialgebra in $\mathcal{C}$.* Let $\mathfrak{g}$ be Lie bialgebra in $\mathcal{C}$. We then have natural maps $\mathbf{LBA}(< k >; < l >) \to \hom_{\mathcal{C}}(\mathfrak{g}^{\otimes k}; \mathfrak{g}^{\otimes l})$.

Let us now make a definition of *a conilpotent Lie bialgebra.* To this end, we need to introduce some notation. First of all, since finite direct sums exist in $\mathcal{C}$, given an object $X \in \mathcal{C}$ and a finite dimensional $\mathbf{k}$-vector space V we have a well defined notion of an object $V^* \otimes X$ it is defined as the representing object for the functor $h_{V^*\otimes X} : \mathcal{C}^{\mathrm{op}} \to$ vect: $Y \mapsto \hom_{\mathbf{k}}(V; \hom_{\mathcal{C}}(Y; X))$. It is clear that $V^* \otimes X$ is isomorphic to the direct sum of $\dim V$ number of copies of X.

Next, we have a natural map of PROPs j : $\mathbf{COLIE} \to \mathbf{LBA}$ (same as in Sec. 3.1.4), hence a map

$$\mathbf{COLIE}(< k >; < l >) \to \hom_{\mathbf{k}}(\mathfrak{g}^{\otimes k}; \mathfrak{g}^{\otimes l}).$$

Since the vector space on the LHS is finite dimensional, we get an induced element

$$\delta_{kl} \in \hom(\mathfrak{g}^{\otimes k}; \mathbf{COLIE}^*(< k >; < l >) \otimes \mathfrak{g}^{\otimes l})$$

where $\mathbf{COLIE}^*(< k >; < l >)$ is the dual vector space.

Furthermore, the symmetric group S_l acts on the object $\mathbf{COLIE}^*(< k >; < l >) \otimes \mathfrak{g}^{\otimes l}$ by automorphisms. Let $P \in \mathbf{k}[S_l]$ be the standard averaging projector in the group algebra of S_l:

$$P = \frac{1}{l!} \sum_{\sigma\in S_l} \sigma$$

Let $(\mathbf{COLIE}^*(< k >; < l >) \otimes \mathfrak{g}^{\otimes l})^{S_l}$ be the kernel of this projection The map δ_{kl} passes through this kernel so that we have a natural map

$$\delta_{kl} \in \hom(\mathfrak{g}^{\otimes k}; [\mathbf{COLIE}^*(< k >; < l >) \otimes \mathfrak{g}^{\otimes l}]^{S_l}).$$

We are now ready to make a definition of a conilpotent Lie bialgebra:

DEFINITION 4.1. *A Lie bialgebra $\mathfrak{g} \in F^1\mathcal{C}$ is called* conilpotent *if for every k there exists an element*

$$\delta_k \in \hom\left(\mathfrak{g}^{\otimes k}; \bigoplus_{l=k}^{\infty} (\mathbf{COLIE}(< k >; < l >)^* \otimes \mathfrak{g}^{\otimes l})^{S_l}\right)$$

whose natural projections onto

$$\hom(\mathfrak{g}^{\otimes k}; (\mathbf{COLIE}^*(< k >; < l >) \otimes \mathfrak{g}^{\otimes l})^{S_l})$$

are δ_{kl}.

Remarks 1. Such an element δ_k, if exists, must be unique because of condition 2) from Sec. 4.1.

2. One can prove that it is sufficient to demand that δ_1 exists, the existence of $\delta_k, k > 1$ will then follow.

4.1.3. *Conilpotent* **BA**-*algebras.* Conilpotent **BA**-algebras in $\mathcal{C}$ are defined along the same lines as conilpotent Lie bialgebras.

Let $\mathfrak{m}$ be a **BA**-algebra in $\mathcal{C}$. We have natural maps

$$\mathbf{COASS}(< k >; < l >) \to \mathbf{BA}(< k >; < l >) \to \hom_{\mathcal{C}}(\mathfrak{m}^{\otimes k}; \mathfrak{m}^{\otimes l})$$

which can be rewritten as

$$\Delta_{k,l} : \hom_{\mathcal{C}}(\mathfrak{m}^{\otimes k}; (\mathbf{COASS}^*(< k >; < l >) \otimes \mathfrak{m}^{\otimes l})^{S_l})$$

We then say that $\mathfrak{m}$ *is conilpotent* if for every k there exists an element

$$\Delta_k \in \hom_{\mathcal{C}}(\mathfrak{m}^{\otimes k}; \bigoplus_l (\mathbf{COASS}^*(< k >; < l >) \otimes \mathfrak{m}^{\otimes l})^{S_l})$$

such that its projection onto

$$\hom_{\mathcal{C}}(\mathfrak{m}^{\otimes k}; (\mathbf{COASS}^*(< k >; < l >) \otimes \mathfrak{m}^{\otimes l})^{S_l})$$

is δ_{kl}. It actually suffices to only check this condition for $k = 1$.

4.2. **Projective limits of $\mathbf{LBA}_n^{\mathbf{pro}}$; $\mathbf{BA}_n^{\mathbf{pro}}$.** As explained above, we denote by

$$\mathbf{LBA}_n^{\mathbf{pro}} \subset \mathbf{LBA}_n^{\wedge}; \quad \mathbf{BA}_n^{\mathbf{pro}} \subset \mathbf{BA}_n^{\wedge}$$

the full subcategories of projective objects (they are automatically finitely generated).

The functors $P_{nm!} : \mathbf{LBA}_n^{\wedge} \to \mathbf{LBA}_m^{\wedge}$ preserve the class of projective objects: indeed, they do clearly preserve free objects, hence they take any retraction of a free object to a retraction of a free object. Same is true for $P_{nm!} : \mathbf{BA}_n^{\wedge} \to \mathbf{BA}_m^{\wedge}$. One also sees that the class of projective objects is preserved by the tensor product in $\mathbf{LBA}_n, \mathbf{BA}_n$.

Therefore, the subcategories $\mathbf{LBA}_n^{\mathbf{pro}}$ form a projective subsystem of $\underline{\mathbf{LBA}}^{\wedge}$, denote this subsystem by $\underline{\mathbf{LBA}}^{\mathbf{pro}} \subset \underline{\mathbf{LBA}}^{\wedge}$. In the same way, we get a subsystem $\underline{\mathbf{BA}}^{\mathbf{pro}} \subset \underline{\mathbf{LBA}}^{\mathbf{pro}}$.

The functors $\mathbf{Q}_n : \mathbf{BA}_n^\wedge \to \mathbf{LBA}_n^\wedge$ being equivalences of abelian categories must preserve the class of projective objects, hence we have an induced equivalence of systems:

$$\mathbf{Q} : \underline{\mathbf{BA}}^{\mathbf{pro}} \to \underline{\mathbf{LBA}}^{\mathbf{pro}}$$

4.2.1. We are going to define a symmetric monoidal category lim-$\underline{\mathbf{LBA}}$ as the projective 2-limit of the projective system of SM-categories $\mathbf{LBA}_n^{\mathbf{pro}}$. The construction is the same as the construction of $\mathbb{Z}_l$-sheaves.

The definition is as follows.

An object $\underline{A}$ of lim-$\underline{\mathbf{LBA}}$ is

a collection of objects $\underline{A}_n \in \mathbf{LBA}_n^{\mathbf{pro}}$, $n = 1, 2, \ldots$, and a collection of isomorphisms

$$i_n : P_{n,n-1!}\underline{A}_n \to \underline{A}_{n-1}, \quad n = 2, 3, \ldots$$

Given $\underline{A}, \underline{B} \in$ lim-$\underline{\mathbf{LBA}}$, we define

$$\hom(\underline{A}; \underline{B}) := \operatorname{liminv}_n \hom_{\mathbf{LBA}_n}(\underline{A}_n; \underline{B}_n),$$

where the spaces $\hom_{\mathbf{LBA}_n}(A_n, B_n)$ form a projective system with the structure maps

$$\hom_{\mathbf{LBA}_n}(A_n, B_n) \to \hom_{\mathbf{LBA}_{n-1}}(\underline{A}_{n-1}, \underline{B}_{n-1})$$

defined by:

$$\hom_{\mathbf{LBA}_n}(\underline{A}_n, \underline{B}_n) \overset{P_{n,n-1!}}{\longrightarrow} \hom_{\mathbf{LBA}_{n-1}}(P_{n,n-1!}\underline{A}_n; P_{n,n-1!}\underline{B}_n) = \hom_{\mathbf{LBA}_{n-1}}(\underline{A}_{n-1}; \underline{B}_{n-1}).$$

This way, lim-$\underline{\mathbf{LBA}}$ is a category.

Define the tensor product on lim-$\underline{\mathbf{LBA}}$ componentwise:

$$(\underline{A} \otimes \underline{B})_n := \underline{A}_n \otimes \underline{B}_n.$$

We see that lim-$\underline{\mathbf{LBA}}$ is then naturally a SMC.

4.2.2. In a similar way, a SMC lim-$\underline{\mathbf{BA}}$ can be defined from the projective system $\underline{\mathbf{BA}}$. The equivalence $\mathbf{Q}$ induces an SM-equivalence lim-$\underline{\mathbf{BA}} \to$ lim-$\underline{\mathbf{LBA}}$.

4.3. Technical Lemmas concerning the categories lim-$\underline{\mathbf{BA}}$, lim-$\underline{\mathbf{LBA}}$.

4.3.1. *Direct sums in lim-$\underline{\mathbf{BA}}$, lim-$\underline{\mathbf{LBA}}$.* let $\mathbf{L}$ be either lim-$\underline{\mathbf{BA}}$ or lim-$\underline{\mathbf{LBA}}$. For an object $\underline{A} \in \mathbf{L}$ let $|\underline{A}|$ be the minimal n such that $\underline{A}_n \neq 0$. Call a family $\{\underline{A}^i\}_{i\in I}$ of objects in $\mathbf{L}$ *admissible* if for each N there are only finitely many objects in this family with $|\underline{A}^i| < N$. In particular, our family must be at most countable.

LEMMA 4.2. *There exist direct sums of admissible families. We have*

$$(\bigoplus_{i\in I} \underline{A}^i)_n = \bigoplus_{i\in I} \underline{A}^i_n,$$

where almost all terms on the RHS are zeros.

Proof. Clear $\square$

Call direct sums of the specified type *admissible direct sums.*

LEMMA 4.3. *Admissible direct sums are compatible with the tensor structure in the following sense: let* $\{\underline{A}^i\}_{i\in I}$*;* $\{\underline{B}^j\}_{j\in J}$ *be admissible families in* $\mathbf{L}$*. Then the family*

$$\{\underline{A}^i \otimes \underline{B}^j\}_{(i,j)\in I\times J}$$

is also admissible and the natural map

$$\bigoplus_{(i,j)\in I\times J} \underline{A}^i \otimes \underline{B}^j \to \left(\bigoplus_{i\in I} \underline{A}^i\right) \otimes \left(\bigoplus_{j\in J} \underline{B}^j\right) \tag{2}$$

is an isomorphism.

Proof. Indeed, admissibility of the family $\{\underline{A}^i \otimes \underline{B}^j\}_{(i,j)\in I\times J}$ follows from the equality $|\underline{A}\otimes\underline{B}| = \max(|\underline{A}|, |\underline{B}|)$. The isomorphism follows from the comparison of the n-th components of both sides of (2). □

LEMMA 4.4. *The equivalence* $\mathbf{Q}$: *lim-*$\underline{\mathbf{BA}}$ $\to$ *lim-*$\underline{\mathbf{LBA}}$ *preserves admissible direct sums: given an admissible family* $\{\underline{A}^i\}_{i\in I}$ *in lim-*$\underline{\mathbf{BA}}$ *the family* $\{\mathbf{Q}(\underline{A}^i)\}_{i\in I}$ *is also admissible, and the natural map*

$$\bigoplus_{i\in I} \mathbf{Q}(\underline{A}^i) \to \mathbf{Q}(\bigoplus_{i\in I} \underline{A}^i) \tag{3}$$

is an isomorphism

Proof. We have $|\mathbf{Q}(\underline{A})| = |\underline{A}|$, whence admissibility of $\{\mathbf{Q}(\underline{A}^i)\}_{i\in I}$. The isomorphism (3) can be easily checked by looking at the components of both sides. □

4.3.2.

LEMMA 4.5. *Let* $p : \underline{R} \to \underline{P}$ *be a component-wise surjective map of objects in lim-*$\underline{\mathbf{LBA}}$ *or lim-*$\underline{\mathbf{BA}}$*. Then* p *has a splitting* $i : \underline{P} \to \underline{R}$*.*

Proof. We will prove Lemma for the category lim-$\underline{\mathbf{LBA}}$; the proof for lim-$\underline{\mathbf{BA}}$ is similar and omitted.

Let us construct the splittings $i_n \underline{P}_n \to \underline{R}_n$ by induction. Choose i_1 to be any splitting of the map p_1. Next, suppose that we have splittings $i_m : \underline{P}_m \to \underline{R}_m$, $m \le n$, such that $P_{m,m-1} i_m = i_{m-1} P_{m,m-1}$ for all $m \le n$. Let us construct the splitting i_{n+1}. The splitting i_n defines an identification $\underline{P}_n = \underline{R}_n \oplus \underline{K}_n$, where $\underline{K}_n := \mathrm{Ker} p_n$. Choose any splitting of p_{n+1} so that we can identify $\underline{P}_{n+1} = \underline{R}_{n+1} \oplus \underline{K}_{n+1}$.

Let us denote $\Pi := P_{n+1,n}$. The isomorphism $\Pi_! \underline{P}_{n+1} \to \underline{P}_n$ induces an isomorphism $k : \Pi_! \underline{K}_{n+1} \to \underline{K}_n$ and a map $j : \Pi_! \underline{R}_{n+1} \to \underline{R}_n \oplus \underline{K}_n$. Let $j_R : \Pi_! \underline{R}_{n+1} \to \underline{R}_n$; $j_K : \Pi_! \underline{R}_{n+1} \to \underline{K}_n$ be its components. It follows that j_R is the structure isomorphism of the inverse system $\underline{R}$. We need to change the splitting $\underline{P}_{n+1} = \underline{R}_{n+1} \oplus \underline{K}_{n+1}$ so as to get rid of j_K. Equivalently, we are to find a map $\varepsilon : \underline{R}_{n+1} \to \underline{K}_{n+1}$ so that

$$k(\Pi_! \varepsilon) = j_K \tag{4}$$

Let us use the conjugacy property:

$$C : \hom(\Pi_! X; Y) \xrightarrow{\sim} \hom(X; \Pi^{-1} Y)$$

for any $X \in \mathbf{LBA}^\wedge_{n+1}$ and $Y \in \mathbf{LBA}^\wedge_n$. Let us apply C to the equation (4):

$$C(k)\varepsilon = C(j_K).$$

This equality means that ε has to be a lifting of the following diagram

(5)
$$\begin{tikzcd} & \underline{R}_{n+1} \arrow[d, "C(j_K)"] \\ \underline{K}_{n+1} \arrow[r, "C(k)"] & \Pi^{-1}\underline{K}_n \end{tikzcd}$$

The existence of such a lifting follows from the projectivity of $\underline{R}_{n+1}$ and surjectivity of the map $C(k)$, which follows from the isomorphism $\Pi_! \underline{K}_{n+1} \to \underline{K}_n$. $\square$

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4.3.3. *Free systems.* Given a positive integer $< k >$ we have an object $\underline{S}^k$ in lim-$\underline{\mathbf{LBA}}$, lim-$\underline{\mathbf{BA}}$ such that $\underline{S}^k_n := h_{<k>}$ with the obvious isomorphisms $P_{n,n-1!}h_{<k>} = h_{<k>}$.

Call an object $\underline{A}$ *free* if it is a direct sum of an admissible family of the form $\{h_{k_i}\}_{i\in I}$

It follows that such a family is admissible iff for every N the set $\{i|k_i < N\}$ is finite. So, we will call a family $\{k_i\}_{i\in I}$ of positive integers admissible if it satisfies this condition. Given an admissible family of numbers $\mathbf{k} := \{k_i\}_{i\in I}$ we denote by $\underline{S}^{\mathbf{k}}$ the corresponding free object.

LEMMA 4.6. *Every object $\underline{P}$ in lim-$\underline{\mathbf{BA}}$, lim-$\underline{\mathbf{LBA}}$ is a retraction of a free object. In other words, there is a free system $\underline{F}$ and an isomorphism $\underline{F} = \underline{P} \oplus \underline{K}$, where $\underline{K}$ is another projective system.*

Proof. We will only prove it for lim-$\underline{\mathbf{LBA}}$, as the proof for lim-$\underline{\mathbf{BA}}$ is similar.

Because of Lemma 4.5, it suffices to find a free system that surjects onto $\underline{P}$. Such a free system can be constructed by induction as long as we establish the following fact:

Let $\underline{F}_n \in \mathbf{LBA}_n^\wedge$ be a free finitely generated object

$$\underline{F}_n = \bigoplus_{i=1}^{M} h_{<k_i>}$$

Let $G \in \mathbf{LBA}_{n+1}^\wedge$ be a free object generated by "the same elements" as $\underline{F}_n$:

$$G = \bigoplus_{i=1}^{M} h_{<k_i>},$$

so that we have an isomorphism

$$i : \Pi_! G = \underline{F}_n,$$

where $\Pi := P_{n+1,n}$. Next, suppose we are given a surjection $q_n : \underline{F}_n \to \underline{P}_n$.

The statement is then as follows:

SUBLEMMA 4.7. *There exist:*

— a finitely generated free object $\Phi \in \mathbf{BA}_{n+1}$ which is a direct sum of finitely many copies of $h_{<n+1>}$;

— a surjection $q_{n+1} : G \oplus \Phi \to \underline{P}_{n+1}$ such that the map

$$\Pi_!(q_{n+1}) : \Pi_!(G \oplus \Phi) \to \Pi_! \underline{P}_{n+1}$$

is equal to:

$$\Pi_!(G \oplus \Phi) = \Pi_! G = \underline{F}_n \xrightarrow{p_n} \underline{P}_n = \Pi_! \underline{P}_{n+1}.$$

It is clear that this sub-Lemma implies the Theorem, so let us prove the sub-Lemma.

We have the following diagram

$$\begin{tikzcd} \underline{P}_{n+1} \arrow[r] & \Pi^{-1}\underline{P}_n \\ & \Pi^{-1}\underline{F}_n \arrow[u] \\ & G \arrow[u] \end{tikzcd}$$

where all vertical arrows are surjective, therefore, we have a lifting $l : G \to \underline{P}_{n+1}$. Cokernel C of this map is a surjective image of the kernel K of the upper horizontal arrow $\underline{P}_{n+1} \to \Pi^{-1}\underline{P}_n = \Pi^{-1}\Pi_!\underline{P}_{n+1}$. The target of this arrow is, by definition, the quotient of the source by its submodule generated by $\underline{P}_{n+1}(< n+1 >)$. Therefore, this submodule, which is just K, is generated by elements in $K(< n+1 >)$. Therefore, C, being a quotient of K, is also generated by $C(< n+1 >)$, hence we have a surjection from a finite direct sum of a sufficiently large number of copies of $h_{<n+1>}$ to C. Let us denote the direct sum by Φ and the surjection by $\pi : \Phi \to C$. We have a lifting of π to a map $\pi' : \Phi \to \underline{P}_{n+1}$. It then follows that

$$l \oplus \pi' : G \oplus \Phi \to \underline{P}_{n+1}$$

is a surjection. Let us set $q_{n+1} := l \oplus \pi'$. This surjection satisfies all the requirements. $\square$

4.3.4. Free systems form full SM-subcategories lim-$\underline{\mathbf{BA}}_{\mathbf{free}} \subset$ lim-$\underline{\mathbf{BA}}$; lim-$\underline{\mathbf{LBA}}_{\mathbf{free}} \subset$ lim-$\underline{\mathbf{LBA}}$. Let us describe the tensor product of free systems and the complex of homomorphisms of free systems. Our results imply that lim-$\underline{\mathbf{BA}}$ (resp. lim-$\underline{\mathbf{LBA}}$) is a Karoubian completion of lim-$\underline{\mathbf{BA}}_{\mathbf{free}}$ (resp. lim-$\underline{\mathbf{LBA}}_{\mathbf{free}}$).

Let $\mathbf{k} = \{k_i\}_{i\in I}$; $\mathbf{l} = \{l_j\}_{j\in J}$. Let $\mathbf{k}+\mathbf{l} := \{k_i + l_j\}_{(i,j)\in I\times J}$. It is clear the the family $\mathbf{k}+\mathbf{l}$ is admissible and that

$$\underline{S}^{\mathbf{k}} \otimes \underline{S}^{\mathbf{l}} \cong \underline{S}^{\mathbf{k}+\mathbf{l}}.$$

Let us calculate

$$\hom_{\text{lim-}\underline{\mathbf{LBA}}}(\underline{S}^{\mathbf{k}^1}; \underline{S}^{\mathbf{k}^2})$$

First of all, we see that

$$\hom(\underline{S}_k; \underline{S}_l) \cong \mathrm{liminv}_n \mathbf{LBA}_n(< k >; < l >).$$

Next, we have a natural projection

$$\hom(\underline{S}^{\mathbf{k}^1}; \underline{S}^{\mathbf{k}^2}) \to \prod_{ij} \hom(S_{k^1_i}; S_{k^2_j})$$

One can check that

LEMMA 4.8. *this map is an isomorphism.*

Proof. Let us construct the inverse map.

We have natural projections

$$P_n : \hom(S_{k^1_i}; S_{k^2_j}) \to \mathbf{LBA}_n(< k >; < l >).$$

This projection vanishes as long as $k_i^1 > n$ or $k_j^2 > n$, therefore P_n vanishes for almost all i,j, hence the maps P_n extend to a map

$$P_n : \prod_{ij} \hom(S_{k_i^1}; S_{k_j^2}) \to \bigoplus_{ij} \mathbf{LBA}_n(< k_i^1 >; < k_j^2 >) = \mathbf{LBA}_n(S_n^{\mathbf{k}^1}; S_n^{\mathbf{k}^2})$$

It is immediate that, altogether, the maps P_n define a map

$$\prod_{ij} \hom(S_{k_i^1}; S_{k_j^2}) \to \hom(S^{\mathbf{k}^1}; S^{\mathbf{k}^2}).$$

□

Same result is true for $\hom_{\text{lim-}\underline{\mathbf{BA}}}(\underline{S}^{\mathbf{k}}; \underline{S}^{\mathbf{l}})$.

4.4. **Conilpotent bialgebras as SM functors lim-$\underline{\mathbf{BA}}$, lim-$\underline{\mathbf{LBA}}$ $\to \mathcal{C}$.** Let $\mathcal{C}$ be a category satisfying the conditions from Sec. 4.1. Call a SM-functor F : lim-$\underline{\mathbf{LBA}} \to \mathcal{C}$ *admissible* if it takes admissible direct sums in lim-$\underline{\mathbf{LBA}}$ to direct sums in $\mathcal{C}$. More precisely, given an admissible family $\{\underline{A}^i\}_{i\in I}$ in lim-$\underline{\mathbf{LBA}}$, the natural map

$$\bigoplus_{i\in I} F(\underline{A}^i) \to F(\bigoplus_{i\in I} \underline{A}^i)$$

must be an isomorphism.

The notion of an admissible functor lim-$\underline{\mathbf{BA}} \to \mathcal{C}$ is defined in the same way.

Let $\mathbf{L}$ be either lim-$\underline{\mathbf{LBA}}$ or lim-$\underline{\mathbf{BA}}$. Admissible functors are uniquely determined by their restriction onto the full SM subcategory $\mathbf{L}_h$ of $\mathbf{L}$ formed by the objects $S^{<k>}$. However, not every SM functor $\mathbf{L}_h \to \mathcal{C}$ extends to $\mathbf{L}$.

On the other hand, given an admissible functor $F : \mathbf{L}_{\mathbf{free}} \to \mathcal{C}$, it uniquely extends to an admissible functor $\mathbf{L} \to \mathcal{C}$.

Given an admissible functor F : lim-$\underline{\mathbf{LBA}} \to \mathcal{C}$ (or, equivalently, F : lim-$\underline{\mathbf{LBA}}_{\mathbf{free}} \to \mathcal{C}$, the object $F(h_{<1>})$ is naturally a conilpotent Lie bialgebra in $\mathcal{C}$. Indeed, $h_{<1>}$ is a Lie bialgebra in lim-$\underline{\mathbf{LBA}}$; the family $\{\mathbf{COLIE}^*(< k >; < l >) \otimes h_{<1>}^{\otimes l}\}_{l=1,2,...}$ is admissible, and we have a map

$$\delta_k \in \hom_{\text{lim-}\underline{\mathbf{LBA}}} \left(h_{<1>}^{\otimes k}; \bigoplus_{l=1}^{\infty} \mathbf{COLIE}^*(< k >; < l >) \otimes h_{<1>}^{\otimes l} \right)$$

whose natural projections onto

$$\hom_{\text{lim-}\underline{\mathbf{LBA}}} \left(h_{<1>}^{\otimes k}; \mathbf{COLIE}^*(< k >; < l >) \otimes \mathfrak{h}_{<1>}^{\otimes l} \right)$$

are δ_{kl}.

Since F preserves direct sums, $F(h_{<1>})$ is automatically a conilpotent Lie bialgebra.

Analogously, given an admissible functor F : lim-$\underline{\mathbf{BA}} \to \mathcal{C}$, $F(h_{<1>})$ is a conilpotent $\mathbf{BA}$-algebra.

It turns out that an admissible functor $F : \mathbf{L} \to \mathcal{C}$ can be recovered up-to a unique isomorphism from the (Lie)-bialgebra $F(h_{<1>})$. This is done as follows. Let $\mathfrak{g}$ be a conilpotent Lie bialgebra in $\mathcal{C}$. Let us construct an admissible functor $\mathcal{G} := F_{\mathfrak{g}}$: lim-$\underline{\mathbf{LBA}}_{\mathbf{free}} \to \mathcal{C}$.

Given a free object $\underline{S}^{\mathbf{k}}$ set

$$\mathcal{G}(\underline{S}^{\mathbf{k}}) := \bigoplus_i \mathfrak{g}^{\otimes k_i}.$$

Let us now construct a map

$$\hom_{\text{lim-}\underline{\mathbf{LBA}}}(\underline{S}^{\mathbf{k}}; \underline{S}^{\mathbf{l}}) \to \hom_{\mathcal{C}}(\mathcal{G}(\underline{S}^{\mathbf{k}}); \mathcal{G}(\underline{S}^{\mathbf{l}})).$$

The LHS is identified with

$$\prod_{ij} \mathrm{liminv}_n \mathbf{LBA}_n(<k_i>;<l_j>) = \prod_{ijn} \mathbf{COLIE}(<k_i>;<n>) \otimes_{S_n} \mathbf{LIE}(<n>;<l_j>).$$

Let us now work with the RHS. The precomposition with δ_k gives rise to a map

$$\prod_i \hom(\bigoplus_n (\mathbf{COLIE}(<k_i>;<n>)^* \otimes \mathfrak{g}^{\otimes n})^{S_n}; \mathcal{G}(\underline{S}^{\mathbf{l}})) \to \hom(\mathcal{G}(\underline{S}^{\mathbf{k}}); \mathcal{G}(\underline{S}^{\mathbf{l}})).$$

The LHS is isomorphic to

$$\prod_{n,i} \mathbf{COLIE}(<k_i>;<n>) \otimes_{S_n} \hom(\mathfrak{g}^{\otimes n}; \bigoplus_j \mathfrak{g}^{\otimes l_j})$$

Next we have natural maps

$$\mathbf{LIE}(<n>;<l_j>) \to \hom(\mathfrak{g}^{\otimes n}; \mathfrak{g}^{\otimes l_j}).$$

Since $\mathbf{LIE}(<n>;<l>) = 0$ for all $l > n$, we actually have a map

$$\prod_{n,i,j} \mathbf{COLIE}(<k_i>;<n>) \otimes_{S_n} \mathbf{LIE}(<n>;<l_j>)$$
$$\to \prod_{n,i} \mathbf{COLIE}(<k_i>;<n>) \otimes_{S_n} \hom(\mathfrak{g}^{\otimes n}; \bigoplus_j \mathfrak{g}^{\otimes l_j}).$$

This results in a map

$$\prod_{n,i,j} \mathbf{COLIE}(<k_i>;<n>) \otimes_{S_n} \mathbf{LIE}(<n>;<l_j>) \to \hom(\mathcal{G}(\underline{S}^{\mathbf{k}}); \mathcal{G}(\underline{S}^{\mathbf{l}}))$$

Note that the LHS is identified with $\hom(\underline{S}^{\mathbf{k}}; \underline{S}^{\mathbf{l}})$. Thus, we have constructed a map

$$\hom(\underline{S}^{\mathbf{k}}; \underline{S}^{\mathbf{l}}) \to \hom(\mathcal{G}(\underline{S}^{\mathbf{k}}); \mathcal{G}(\underline{S}^{\mathbf{l}}))$$

One can easily check that this way $\mathcal{G}$ is an admissible functor lim-$\underline{\mathbf{LBA}}_{\mathbf{free}} \to \mathcal{C}$. As lim-$\underline{\mathbf{LBA}}$ is a Karoubian closure of lim-$\underline{\mathbf{LBA}}_{\mathbf{free}}$, we have a canonical extension of $\mathcal{G}$:

$$F_{\mathfrak{g}} : \text{lim-}\underline{\mathbf{LBA}} \to \mathcal{C}$$

which is also admissible.

Analogously, given a conilpotent $\mathbf{BA}$-algebra $\mathfrak{m}$ in $\mathcal{C}$, one constructs an admissible functor

$$F_{\mathfrak{m}} : \text{lim-}\underline{\mathbf{BA}} \to \mathcal{C}$$

One can easily check that *we have constructed mutually inverse equivalences between the category of conilpotent Lie bialgebras (resp.* $\mathbf{BA}$*-algebras) in* $\mathcal{C}$ *and the category of admissible functors* lim-$\underline{\mathbf{LBA}} \to \mathcal{C}$ (resp. lim-$\underline{\mathbf{BA}} \to \mathcal{C}$).

Taking into account the SM-equivalence

$$\mathbf{Q} : \text{lim-}\underline{\mathbf{BA}} \to \text{lim-}\underline{\mathbf{LBA}}$$

which is compatible with admissible direct sums, we immediately get an equivalence between the categories of admissible functors lim-$\underline{\mathbf{LBA}} \to \mathcal{C}$ and lim-$\underline{\mathbf{BA}} \to \mathcal{C}$.

In particular, given a conilpotent Lie bialgebra $\mathfrak{g} \in \mathcal{C}$ we can convert it to an admissible functor $F_{\mathfrak{g}}$: lim-$\underline{\mathbf{LBA}} \to \mathcal{C}$, then to a functor $F'_{\mathfrak{g}} := F_{\mathfrak{g}}\mathbf{Q}$: lim-$\underline{\mathbf{BA}} \to \mathcal{C}$, and lastly to a conilpotent $\mathbf{BA}$-algebra $\mathbf{Q}(\mathfrak{g}) := F'_{\mathfrak{g}}(S^{<1>})$.

We have proven:

THEOREM 4.9. *The functor* $\mathbf{Q}$ *is an equivalence of the category of conilpotent Lie bialgebras in* $\mathcal{C}$ *and conilpotent* $\mathbf{BA}$*-algebras in* $\mathcal{C}$.

5. Plan of the proof of the universal quantization theorem 3.3

We will link the projective systems $\underline{\mathbf{BA}}^\wedge$ and $\underline{\mathbf{LBA}}^\wedge$ by introducing a series of other projective systems of SMC and establishing a zigzag of equivalences of these projective systems. All these mediating projective systems will be constructed as cores with respect to t-structures [1] on some dg-categories.

Let us list the mediating projective systems we are going to introduce.

1) We replace the categories $\mathbf{BA}_n^\wedge$, $\mathbf{LBA}_n^\wedge$ with dg-categories $\mathbf{BA}_n^{\mathbf{proj}}$; $\mathbf{LBA}_n^{\mathbf{proj}}$ which are just the categories of finite complexes of finitely generated projective objects in $\mathbf{BA}_n^\wedge; \mathbf{LBA}_n^\wedge$; a precise definition is given in Appendix 1.

2) We will exploit the well known link between the bialgebras and brace-algebras. We will explain this link in detail in Section 5.1 Brace algebras are controlled by a certain dg-operad **braces**, hence we can construct a PROP **BRACES** generated by **braces** and obtain a projective system of SMC $\mathbf{BRACES}_n$ which is obtained from **BRACES** by taking a quotient with respect to the ideal generated by $\mathrm{Id}_{<N>}$ for all $N > n$, in the same way as we obtained the projective systems $\underline{\mathbf{BA}}$, $\underline{\mathbf{LBA}}$. We will then establish SM weak equivalences (see Appendix 1 for the definition) between $\mathbf{BA}_n^{\mathbf{proj}}$ and $\mathbf{BRACES}_n^{\mathbf{proj}}$ which will define a weak equivalence of the corresponding projective systems.

3) We will use a quasi-isomorphism of operads **hoger** $\to$ **braces**. We can construct a projective system of SMC $\underline{\mathbf{HOGER}}$ in the same way as above: by taking a quotient by the ideal generated by $\mathrm{Id}_{<N>}$, $N > n$. We then automatically have weak equivalences $\underline{\mathbf{HOGER}} \to \underline{\mathbf{BRACES}}$; $\underline{\mathbf{HOGER}}^{\mathbf{proj}} \to \underline{\mathbf{BRACES}}^{\mathbf{proj}}$.

4) We will use standard functors which convert Lie bialgebras into Gerstenhaber algebras and vise versa in order to construct weak equivalences (in both directions) between the projective systems $\underline{\mathbf{LBA}}^{\mathbf{proj}}$ and $\underline{\mathbf{HOGER}}^{\mathbf{proj}}$. This way we get a chain of weak equivalences linking $\underline{\mathbf{BA}}^{\mathbf{proj}}$ and $\underline{\mathbf{LBA}}^{\mathbf{proj}}$.

5) We need to get back to projective systems of SM abelian categories $\underline{\mathbf{BA}}^\wedge, \underline{\mathbf{LBA}}^\wedge$.

We endow the categories $\mathbf{BA}_n^{\mathbf{proj}}$, $\mathbf{LBA}_n^{\mathbf{proj}}$, $\mathbf{BRACES}_n^{\mathbf{proj}}$, $\mathbf{HOGER}_n^{\mathbf{proj}}$, $\mathbf{GER}_n^{\mathbf{proj}}$ with t-structures and show that the weak equivalences between these categories that we have constructed are actually exact t-functors. One also easily checks that the tensor product and the projections $\mathbf{BA}_n^{\mathbf{proj}} \to \mathbf{BA}_{n-1}^{\mathbf{proj}}$, $\mathbf{LBA}_n^{\mathbf{proj}} \to \mathbf{LBA}_{n-1}^{\mathbf{proj}}$, $\mathbf{BRACES}_n^{\mathbf{proj}} \to \mathbf{BRACES}_{n-1}^{\mathbf{proj}}$, etc. are right exact with respect to the t-structure.

We also check that the cores of $\mathbf{BA}_n^{\mathbf{proj}}, \mathbf{LBA}_n^{\mathbf{proj}}$ are equivalent to $\mathbf{BA}_n^\wedge$; $\mathbf{LBA}_n^\wedge$.

We thus obtain induced equivalences of the cores and, thereby, the quantization theorem.

5.1. Bialgebras and brace algebras.

The notion of brace algebra is defined in [9] see also [8]. We will reproduce the definition using a slightly different language.

5.1.1. *Category of complexes.* Let $\mathcal{C}$ be a dg SMC -category closed under finite direct sums. We can form SM categories $\mathbf{Com}_-\mathcal{C}$ (resp. $\mathbf{Com}_+\mathcal{C}$) of complexes in $\mathcal{C}$ bounded from above (resp. the category of complexes bounded from below) in the following slightly non-standard way. Objects of $\mathbf{Com}_-\mathcal{C}$ (resp. $\mathbf{Com}_+\mathcal{C}$) are sequences of objects $X^n \in \mathcal{C}$; and elements $d_{nm} \in Z^1 \hom(X_n, X_m)$, $m > n$ such that

1) for all $n < m$:

$$d(d_{nm}) + \sum_{n<k<m} d_{km} d_{nk} = 0; \tag{6}$$

note that d is the differential in the complex $\hom_{\mathcal{C}}(X_n; X_m)$

2) $X_n = 0$ for all $n >> 0$ (resp. $n << 0$).

Define the complex of homomorphisms.

For each n, let $H^N := \prod_k \hom(X^k, Y^{N+k})$. The differentials d_{mn} on X, Y give rise to maps

$$D_{MN} : H^M \to H^N$$

for all $N > M$. The elements D_{MN} satisfy (6).

Let

$$Z := \mathrm{limdir}_M \prod_{N \geq M} H^N$$

the sum of all D_{MN} is a well defined map $D : Z \to Z$ and we assign

$$\hom(X^\bullet, Y^\bullet) := (Z, d_Z + D).$$

The tensor product in both $\mathbf{Com}_+\mathcal{C}, \mathbf{Com}_-\mathcal{C}$ is given by the formula

$$(X \otimes Y)^n = \bigoplus_m X^m \otimes Y^{n-m},$$

where the direct sum is actually finite, and the differentials on $X \otimes Y$ are defined by the formula:

$$d_{nm} = \sum_i d^X_{n-i,m-i} \otimes \mathrm{Id}_{Y_i} \pm \mathrm{Id}_{X_i} \otimes d^Y_{n-i,m-i}$$

Thus, $\mathbf{Com}_-\mathcal{C}$, $\mathbf{Com}_+\mathcal{C}$ are dg SMC.

5.1.2. *Definition of brace algebras.* Let $V \in \mathcal{C}$. Let $W^{-k} := T^k V, k = 0, 1, 2, \ldots$; let $W^{\geq 0} = 0$.

We define *a brace structure on V* as

1) a collection of maps $d_{-n,-m} : W^{-n} \to W^{-m}$, $-n < -m$ so that $(W^\bullet, \{d_{nm}\})$ is a complex in $\mathbf{Com}_-\mathcal{C}$. Denote this complex by $\mathfrak{H}(V)$.

We have the standard coproduct $\Delta \in \hom_{\mathbf{Com}_-\mathcal{C}}(\mathfrak{H}(V); \mathfrak{H}(V) \otimes \mathfrak{H}(V))$

given by the fromula

$$\Delta|_{V^{\otimes n}} = \sum_{k=0}^{n} \Delta_k,$$

where

$$\Delta_k : V^{\otimes n} \xrightarrow{\sim} (V)^{\otimes k} \otimes (V)^{\otimes n-k} \to \mathfrak{H}(V) \otimes \mathfrak{H}(V).$$

We demand that Δ must be compatible with the differential

2) an associative bialgebra structure on the complex TV such that:

—the coproduct is as defined above, and

—The unit and counit are the standard ones:

$$\mathbf{k} \to (V[1])^{\otimes 0} \to \mathfrak{H}(V) \to (V[1])^{\otimes 0} = \mathbf{k}.$$

It is assumed that all the structure maps (the product, the co-product, the unit, the counit) are compatible with the differential;

3) Let $p : \mathfrak{H}(V) \to V[1]$ be the natural projection and $m_{k,l}$ be the following components of the product:

$$V[1]^{\otimes k} \otimes V[1]^{\otimes l} \to \mathfrak{H}(V) \otimes \mathfrak{H}(V) \xrightarrow{m} \mathfrak{H}(V) \xrightarrow{p} V[1]$$

we then demand that $m_{k,l} = 0$ for all $k \neq 1$. The components $m_{1,n} : V[1] \otimes V[1]^{\otimes n} \to V[1]$ are called *the n-brace operations.*

Note that in order to specify a brace algebra one has to prescribe the n-braces and also to prescribe the components of the differential

$$M_n : (V[1])^{\otimes n} \to \mathfrak{H}(V) \xrightarrow{d} \mathfrak{H}(V) \to V[1]$$

for all $n > 1$ (the component M_1 is determined by the pre-existing differential on $V[1]$).

The operations M_n are called *higher cup-products.*

This description implies that there exists a dg-operad controlling the structure of a brace algebra. We denote this operad by **braces**

We then have a tautological statement that $V \mapsto \mathfrak{H}(V)$ is a functor from the category of brace algebras in $\mathcal{C}$ to the category of **BA**-algebras in $\mathbf{Com}_{-}\mathcal{C}$.

5.1.3. Let $\mathbf{M}$ be a (non-symmetric) dg monoidal category enriched over a SMC $\mathcal{C}$ and let A be a unital associative monoid in $\mathcal{C}$.

We then have the Hochschild complex $C^{\bullet}(A, A) \in \mathbf{Com}_{+}\mathcal{C}$ defined in the standard way.

Assume that the unital map $\mathbf{1} \to A$ splits so that $A = \mathbf{1} \oplus \mathfrak{m}$. We then have a notion of the reduced Hochschild complex of A, $\overline{C}^{\bullet}(A, A)$.

It is well known that both $C^{\bullet}(A, A)$ and $\overline{C}^{\bullet}(A, A)$ are brace algebras in $\mathbf{Com}_{+}\mathcal{C}$, see [9], [8].

5.2. **Constructing a brace algebra out of a BA-algebra.** In order to produce a natural definition we need a little bit of a categorical nonsense

5.2.1. Let $\mathcal{C}^{-}$ be the category of all dg functors $\mathcal{C}^{\mathrm{op}} \to$ **complexes**. We have a SM-structure on $\mathcal{C}^{-}$ defined in the same way as it is defined on $\mathcal{C}^{\wedge \mathrm{dg}}$ in Appendix 1.

The category $\mathcal{C}$ is naturally enriched over $\mathcal{C}^{-}$: given $X, Y \in \mathcal{C}$, define $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{C}^{-}$ by setting

$$\underline{\mathrm{Hom}}(X, Y)(U) := \mathrm{hom}_{\mathcal{C}}(X \otimes U, Y)$$

for all $U \in \mathcal{C}$.

5.2.2. Let $\mathfrak{m} \in \mathcal{C}$ be a **BA**-algebra. Let $H := \mathbf{1} \oplus \mathfrak{m}$ so that $\mathcal{H}$ is an associative bialgebra. In particular, H is an associative algebra in $\mathcal{C}$, and we can consider the category $\mathbf{M}$ of left H-modules in $\mathcal{C}$. It is a dg-category enriched over $\mathcal{C}^{-}$.

The coproduct on H makes $\mathbf{M}$ into a monoidal category : given $X, Y \in \mathbf{M}$, their $\mathcal{C}$-tensor product has a natural left H-module structure. The counit map in H endows the object $\mathbf{1}$ with the left H-action, we denote the corresponding object in $\mathbf{M}$ by $\mathbf{1}'$.

Let $H' \in \mathbf{M}$ be H viewed as a left H-module. The coproduct on H makes H' into a co-associative co-unital co-monoid in $\mathbf{M}$, hence H' is an associative unital monoid in $\mathbf{M}^{\mathrm{op}}$. The unit map produces an H-module splitting

$$H' = \mathbf{1}' \oplus \mathfrak{m}',$$

where $\mathfrak{m}'$ is $\mathfrak{m}$ viewed as a left H-module.

Consider the reduced Hochschild complex $\overline{C}^{\bullet}(H'; H')$; it is automatically a brace algebra in $\mathcal{C}^{-}$. Denote this algebra by $\mathfrak{B}(\mathfrak{m})$. Let us give a concrete description of $\mathfrak{B}(\mathfrak{m})$.

We have
$$\overline{C}^n(H' : H') = \hom_{\mathbf{M}}(H'; (\mathfrak{m}')^{\otimes n})$$
It is clear that H' is a free H-module. In other words, for every $T \in \mathbf{M}$, $\hom_{\mathbf{M}}(H';T) = h_T$, where the RHS is an object of C^- which represents T viewed as an object of $\mathcal{C}$.

Therefore, we have an isomorphism
$$\overline{C}^n(H';H') = h_{\mathfrak{m}^{\otimes n}}$$

We can identify $\mathcal{C}$ with the full subcategory of $\mathcal{C}^-$ consisting of representable functors, we then see that $\overline{C}^{\bullet}(H',H')$ is a complex from $\mathbf{Com}_+\mathcal{C}$. As mentioned above, it is naturally a brace algebra. Let us denote this brace algebra by $\mathfrak{B}(\mathfrak{m})$ and give its explicit description.

1) As a complex, $\mathfrak{B}(\mathfrak{m})$ looks as follows:
$$0 \to \mathfrak{m} \to \mathfrak{m}^{\otimes 2} \to \mathfrak{m}^{\otimes 3} \to \cdots,$$
where $\mathfrak{m}$ is in the first degree and the differential is the co-bar differential induced by the coproduct on $\mathfrak{m}$.

The cup-products $M_n = 0$ for $n > 2$ and M_2 is just a free associative product

The brace operations are uniquely determined by demanding:

— the composition
$$\mathfrak{m}[-1] \otimes \mathfrak{B}(\mathfrak{m})^{\otimes N} \to \mathfrak{B}(\mathfrak{m}) \otimes \mathfrak{B}(\mathfrak{m})^{\otimes N} \stackrel{m_{1;N}}{\to} \mathfrak{B}(\mathfrak{m})$$
should vanish when $N > 1$;

— the composition
$$\mathfrak{m} \otimes (\mathfrak{m}[-1])^{\otimes n}[1] \stackrel{m_{1,1}}{\to} \mathfrak{B}(\mathfrak{m})[1] \otimes \mathfrak{B}(\mathfrak{m})[1] \to \mathfrak{B}(\mathfrak{m})[1]$$
should be equal to:
$$\mathfrak{m} \otimes (\mathfrak{m}[-1])^{\otimes n}[1] \xrightarrow{\Delta^n} H^{\otimes n} \otimes (\mathfrak{m}[-1])^{\otimes n}[1] \stackrel{\mu^{\otimes n}}{\to} (\mathfrak{m}[-1])^{\otimes n}[1] \to \mathfrak{B}(\mathfrak{m})[1],$$
where H is the Hopf algebra $\mathbf{1} \oplus \mathfrak{m}$; $\Delta^n : H \to H^{\otimes n}$ is the n-fold coproduct on H and $\mu : H \otimes \mathfrak{m} \to \mathfrak{m}$ is the map induced by the product on H.

5.2.3. Let $\mathfrak{m}$ be a $\mathbf{BA}$-algebra in $\mathcal{C}$ satisfying the following condition: there exists an $N > 0$ such that $\mathfrak{m}^{\otimes N} = 0$. No matter how exotic this condition is it will be the case in our situation, where $\mathfrak{m}$ will be the $\mathbf{BA}$-algebra $h_{<1>} \in \mathbf{BA}_n$,so that $N = n+1$).

Provided this condition is the case, $\mathfrak{B}(\mathfrak{m})$ is just a finite complex in $\mathcal{C}$. Let us denote by $\mathrm{Com}(\mathcal{C})$ the SMC of finite complexes in $\mathcal{C}$ so that $\mathfrak{B}(\mathfrak{m})$ is a brace algebra in $\mathrm{Com}(\mathcal{C})$. Since $\mathfrak{B}(\mathfrak{m})$ is composed of non-zero powers of $\mathfrak{m}$, it follows that the N-th tensor power of $\mathfrak{B}(\mathfrak{m})$ is zero, therefore, $\mathfrak{H}(\mathfrak{B}(\mathfrak{m}))$ is a finite complex of finite complexes in $\mathcal{C}$ so that we can take the total complex of this double-complex obtaining this way a $\mathbf{BA}$-algebra $\mathfrak{H}(\mathfrak{B}(\mathfrak{m}))$ in $\mathrm{Com}(\mathcal{C})$.

It turns out that in this case we have a natural map of $\mathbf{BA}$-algebras $\alpha : \mathfrak{m} \to \mathfrak{H}(\mathfrak{B}(\mathfrak{m}))$ defined as follows:

as α must be compatible with the co-product it suffices to define its restriction
$$\alpha_r : \mathfrak{m} \to (\mathfrak{B}\mathfrak{m})[1]$$
we define it to be the identical map
$$\mathfrak{m} \stackrel{\mathrm{Id}}{\to} \mathfrak{m} \subset (\mathfrak{B}\mathfrak{m})[1].$$

The map α_r extends uniquely to a map α of coalgebras. It only remains to show it respects the product and the differential, which is straightforward.

5.3. **Gerstenhaber algebras and Lie bialgebras.** Throughout this section we assume that $\mathcal{C}$ is closed under kernels of projectors.

5.3.1. Let now A be a Gerstenhaber algebra in $\mathcal{C}^-$. We can construct Harrison's complex of A, $\mathrm{Harr}(A) \in \mathbf{Com}_-\mathcal{C}$. It is of the form

$$\cdots \mathrm{Harr}(A)^{-n} \to \mathrm{Harr}(A)^{-n+1} \to \cdots \mathrm{Harr}(A)^{-1} \to 0.$$

where $\mathrm{Harr}(A)^{-n}[n] := (\mathbf{lie}(n)^* \otimes k[1]^{\otimes n}) \otimes_{S_n} A^{\otimes n}$, and the differential is induced by the product on A in the standard way. We have a cofree Lie algebra structure on $\mathrm{Harr}(A)$, and the differential is uniquely determined by the condition that it should be compatible with the Lie co-bracket and that the differential

$$S^2(A) = \mathrm{Harr}(A)^{-2} \to \mathrm{Harr}(A)^{-1} = A$$

is just the commutative product on A.

The Gerstenhaber bracket $\{,\}$ on A naturally extends to $\mathrm{Harr}(A)$: let us define a map

$$[,] : \mathrm{Harr}(A)^{\otimes 2} \to \mathrm{Harr}(A)$$

by demanding that:

1)

$$\delta([x,y]) = [\delta(x), y] + [x, \delta(y)];$$

2) The co-restriction

$$\mathrm{Harr}(A) \otimes \mathrm{Harr}(A) \xrightarrow{[,]} \mathrm{Harr}(A) \xrightarrow{r} A[1]$$

should coincide with the map

$$\mathrm{Harr}(A) \otimes \mathrm{Harr}(A) \xrightarrow{r\otimes r} A[1] \otimes A[1] = A \otimes A[2] \xrightarrow{\{,\}} A[-1][2] = A[1],$$

where $\{,\}$ is the bracket on the Gerstenhaber algebra A.

One can easily check that there is a unique map $[,]$ satisfying 1),2) and that $(\mathrm{Harr}(A); d; [,]; \delta)$ is a dg Lie bialgebra. We denote this Lie bialgebra by $\mathfrak{L}(A)$.

5.3.2. Let us construct the inverse map. Let $\mathfrak{g}$ be a Lie bialgebra in C. Consider its Chevalley-Eilenberg cochain complex $C^\bullet(\mathfrak{g})$ with respect to the cobracket. This complex is an object of $\mathbf{Com}_+\mathcal{C}$; we have $C^n(\mathfrak{g}) = \Lambda^n(\mathfrak{g})$ and the differential is defined by the cobracket δ on $\mathfrak{g}$. It is well known that $C^\bullet(\mathfrak{g})$ is a commutative algebra in $\mathbf{Com}_+\mathcal{C}$.

We can define the Gerstenhaber bracket

$$\{,\} : C^\bullet(\mathfrak{g})^{\otimes 2} \to C^\bullet(\mathfrak{g})[1]$$

by asserting that

1) the bracket should obey the Leibnitz rule;

2) the restriction

$$\mathfrak{g}[-1]^{\otimes 2} \to C^\bullet(\mathfrak{g})^{\otimes 2} \xrightarrow{\{,\}} C^\bullet(\mathfrak{g})[-1]$$

should coincide with the map

$$\mathfrak{g}[-1]^{\otimes 2} \xrightarrow{[,]} \mathfrak{g}[-1][-1] \to C^\bullet(\mathfrak{g})[-1].$$

It is easy to check that this way, $C^\bullet(\mathfrak{g})$ is a Gerstenhaber algebra in $\mathbf{Com}_+\mathcal{C}$. Denote this Gerstenhaber algebra by $\mathfrak{G}(\mathfrak{g})$

5.3.3. Given a Lie bialgebra $\mathfrak{g}$ in C satisfying $\mathfrak{g}^{\otimes N}=0$ for N large enough, define a map

$$\mathfrak{g} \to \mathfrak{L}\mathfrak{G}(\mathfrak{g}), \tag{7}$$

where the RHS is viewed as a finite complex in $\mathcal{C}$.

As the RHS in (7) is cofree as a Lie coalgebra, it suffices to define the composition

$$\mathfrak{g} \to \mathfrak{L}\mathfrak{G}(\mathfrak{g}) \to \mathfrak{G}(\mathfrak{g})[1]$$

We set this composition to be the obvious inclusion

$$i : \mathfrak{g} \to \mathfrak{G}(\mathfrak{g})[1] = C^\bullet(\mathfrak{g})[1]$$

onto the term $C^1(\mathfrak{g}) = \mathfrak{g}$.

One can easily check that this way we indeed get a Lie bialgebra map. Note that the map i can be extended to a map of Lie bialgebras beacause the cobracket on $\mathfrak{g}$ is conilpotent.

5.3.4. Let A be a Gerstenhaber algebra in $\mathcal{C}$ satisfying $A^{\otimes N}=0$ for N large enough.

Let us construct a map of Gerstenhaber algebras

$$\mathfrak{G}\mathfrak{L}(A) \to A, \tag{8}$$

where the LHS is viewed as a finite complex in $\mathcal{C}$. As the LHS in (8) is free, it suffices to prescribe the restriction of this map onto the generators:

$$\mathfrak{L}(A)[-1] = \mathrm{Harr}(A)[-1] \to A$$

which we set to be the obvious projection onto $\mathrm{Harr}^{-1}(A) = A$.

It is easy to check that this way we indeed get a map of Gerstenhaber algebras.

5.4. **Bialgebras and brace algebras: translation into the universal language.** We use freely the notations from the Appendix 1.

5.4.1. We have a universal brace algebra $<1>\in \mathbf{BRACES}$. The projections $P_N : \mathbf{BRACES} \to \mathbf{BRACES}_N$ endow

$$P_N <1> = <1> \in \mathbf{BRACES}_N$$

with a brace-algebra structure.

Let $h_{<1>} \in \mathbf{BRACES}_N^{\mathbf{free}}$ be Ioneda's image of $<1>$

$$h_{<1>}(<k>) := \hom_{\mathbf{BRACES}_N}(<k>;<1>).$$

We can apply the functor $\mathfrak{H}$ to the brace algebra $h_{<1>} \in \mathbf{BRACES}_N^{\mathbf{free}}$ so that we get an object

$$\mathfrak{H}_N := \mathfrak{H}(h_{<1>})$$

which is actually a finite complex of objects in $\mathbf{BRACES}_N^{\mathbf{free}}$, hence, this complex naturally defines an object in $\mathbf{BRACES}_N^{\mathbf{free}}$. We denote this object by the same symbol $\mathfrak{H}(h_{<1>})$. We have a $\mathbf{BA}$-structure on $\mathfrak{H}(h_{<1>})$.

This implies that we have a SM functor

$$\mathfrak{H}'_N : \mathbf{BA} \to \mathbf{BRACES}_N^{\mathbf{free}}$$

such that $\mathfrak{H}'(<m>) = (\mathfrak{H}_N)^{\otimes m}$.

One can easily check that $\mathfrak{H}_N^{\otimes m} = 0$ for all $m > N$. This implies that we actually have a SM functor

$$\mathfrak{H}_N : \mathbf{BA}_N \to \mathbf{BRACES}_N^{\mathbf{free}}.$$

As explained in Appendix 1, given any such a functor, we have its canonical SM extensions

$$\mathfrak{H}_N : \mathbf{BA}_N^{\mathbf{free}} \to \mathbf{BRACES}_N^{\mathbf{free}};$$

$$\mathfrak{H}_N : \mathbf{BA}_N^{\mathbf{proj}} \to \mathbf{BRACES}_N^{\mathbf{proj}}.$$

It is easy to see that the SM functors $\mathfrak{H}_N$ are compatible with the projections

$$P_{NM} : \mathbf{BA}_N \to \mathbf{BA}_M;$$

$$P_{NM} : \mathbf{BRACES}_N \to \mathbf{BRACES}_M$$

so that we get a map of projective systems

$$\underline{\mathfrak{H}} : \underline{\mathbf{BA}}^{\mathbf{free}} \to \underline{\mathbf{BRACES}}^{\mathbf{free}};$$

$$\underline{\mathfrak{H}} : \underline{\mathbf{BA}}^{\mathbf{proj}} \to \underline{\mathbf{BRACES}}^{\mathbf{proj}}.$$

5.4.2. Let us now construct a functor in the opposite direction

$$\underline{\mathfrak{B}} : \underline{\mathbf{BRACES}}^{\mathbf{free}} \to \underline{\mathbf{BA}}^{\mathbf{free}}$$

using as a prototype the construction from Sec. 5.1.2

We have **BA**-algebras $< 1 > \in \mathbf{BA}_N$ and $h_{<1>} \in \mathbf{BA}_N^{\mathbf{free}}$. Therefore, we have a brace algebra $\mathfrak{B}(h_{<1>})$ which is a finite complex of objects from $\mathbf{BA}_N^{\mathbf{free}}$, hence, it defines an object in $\mathbf{BA}_N^{\mathbf{free}}$ which is a brace algebra. We denote this brace algebra in $\mathbf{BA}_N^{\mathbf{free}}$ by $\mathfrak{Br}_N$.

We thus get SM functors

$$\mathfrak{B}'_N : \mathbf{BRACES} \to \mathbf{BA}_N^{\mathbf{free}},$$

where $\mathfrak{B}'_N(< m >) := (\mathfrak{Br}_N)^{\otimes m}$.

One checks that $\mathfrak{B}'_N(< m >) = 0$ for all $m > N$ so that we automatically get functors

$$\mathfrak{B}'_N : \mathbf{BRACES}_N \to \mathbf{BA}_N^{\mathbf{free}}$$

which canonically extend to SM functors

$$\mathfrak{B}_N : \mathbf{BRACES}_N^{\mathbf{free}} \to \mathbf{BA}_N^{\mathbf{free}};$$

$$\mathfrak{B}_N : \mathbf{BRACES}_N^{\mathbf{proj}} \to \mathbf{BA}_N^{\mathbf{proj}}.$$

These maps produce maps of systems

$$\underline{\mathfrak{B}} : \underline{\mathbf{BRACES}}^{\mathbf{free}} \to \underline{\mathbf{BA}}^{\mathbf{free}};$$

$$\underline{\mathfrak{B}} : \underline{\mathbf{BRACES}}^{\mathbf{proj}} \to \underline{\mathbf{BA}}^{\mathbf{proj}}.$$

The formulas from 5.2.3 are applicable in our setting so that we have a SM natural transformation

$$\alpha : \mathrm{Id} \to \underline{\mathfrak{B}}\underline{\mathfrak{H}}.$$

LEMMA 5.1. *This map is a quasi-isomorophism of SM-functors.*

Proof. It suffices to check that the maps

$$\alpha_N(h_{<n>}) : h_{<n>} \to \mathfrak{B}_N \mathfrak{H}_N(h_{<n>})$$

are quasi-isomorphisms in $\mathbf{BA}_N^{\mathbf{free}}$. Since all the functors involved are strong SM, it suffices to consider the case $n = 1$ and to check that the map

$$h_{<1>} \to \mathfrak{B}_N \mathfrak{H}_N(h_{<1>})$$

is a quasi-isomorphism.

We have

$$\mathfrak{H}_N(h_{<1>}) = 0 \to h_{<N>} \to h_{<N-1>} \to \cdots h_{<1>} \to 0 \tag{9}$$

where the differential is the bar differential induced by the associative algebra structure on $h_{<1>} \in \mathbf{BRACES}_N^{\mathbf{free}}$ and the term $h_{<1>}$ is in the co-homological degree -1.

Analogously,

$$\mathfrak{B}_N(h_{<1>}) = (0 \to h_{<1>} \to h_{<2>} \to \cdots \to h_{<N>} \to 0) \tag{10}$$

where the differential is just the co-bar differential induced by the co-associative coalgebra structure on $\mathfrak{h}_{<1>} \in \mathbf{BA}_N^{\mathbf{proj}}$.

The composition $\mathfrak{B}_N \mathfrak{H}_N h_{<1>}$ is isomorphic to the bar construction of the co-bar construction applied to $h_{<1>} \in \mathfrak{H}_N^{\mathbf{proj}}$ viewed as a co-associative co-algebra. Let us temporarily forget the differential and view $\mathfrak{B}_N \mathfrak{H}_N h_{<1>}$ as a functor from $\mathfrak{H}_N^{\mathrm{op}}$ to the category of finite-dimensional graded vector spaces. We then have an isomorphism

$$\mathfrak{B}_N \mathfrak{H}_N h_{<1>} \cong T(T(h_{<1>}[-1])[1]), \tag{11}$$

where TX stands for $\bigoplus_{n=1}^{\infty} X$. Since $(h_{<1>})^{\otimes > N} = 0$, the RHS in (11) is a finite direct sum of tensor powers of X.

The RHS of (11) has an additional grading Gr by the number of tensor factors. Introduce a filtration

$$F^M T(T(h_{<1>}[-1])[1]) := \bigoplus_{K \geq M} \mathrm{Gr}^K T(T(h_{<1>}[-1])[1]).$$

We see that the differential on $\mathfrak{B}_N \mathfrak{H}_N h_{<1>}$ preserves this filtration. Let us pass to the associated graded quotient of $\mathfrak{B}_N \mathfrak{H}_N h_{<1>}$ with respect to this filtration. The M-th graded piece of this quotient is isomorphic to

$$\mathrm{Gr}^M T(T(h_{<1>}[-1])[1])$$

with the bar differential coming from the free associative algebra structure on $T(h_{<1>}[-1])$. It is well known that these complexes are acyclic for all $M > 1$.

We also have $\mathrm{Gr}^1 T(T(h_{<1>}[-1])[1]) = h_{<1>}$. The Proposition now follows easily ☐

5.4.3. *The composition $\underline{\mathfrak{H}}\underline{\mathfrak{B}}$ is a weak equivalence.* We will prove

PROPOSITION 5.2. *The compositions;*

$$\underline{\mathfrak{H}}\underline{\mathfrak{B}} : \underline{\mathbf{BRACES}}^{\mathbf{free}} \to \underline{\mathbf{BRACES}}^{\mathbf{free}};$$

$$\underline{\mathfrak{H}}\underline{\mathfrak{B}} : \underline{\mathbf{BRACES}}^{\mathbf{proj}} \to \underline{\mathbf{BRACES}}^{\mathbf{proj}}$$

are weak equivalences.

We denote $E := \underline{\mathfrak{H}\mathfrak{B}}$. The Proposition will follow from a series of Lemmas.

LEMMA 5.3. *The object $E_N(h_{<1>})$ is weakly equivalent to $h_{<1>} \in \mathbf{BRACES}_N^{\mathbf{free}}$.*

Proof. Similar to Lemma 5.1. The object $E_N(h_{<1>})$ is isomorphic to the co-bar construction of the bar-construction applied to the homotopy associative algebra $h_{<1>} \in \mathbf{BRACES}_N^{\mathbf{free}}$. Up-to the differential, we have an isomorphism

$$E_N(h_{<1>}) = T(T(h_{<1>}[1])[-1])$$

We then introduce the grading Gr by the number of the tensor factors and set

$$F^M T(T(h_{<1>}[1])[-1]) := \bigoplus_{K \le M} \mathrm{Gr}^K T(T(h_{<1>}[1])[-1].$$

We then see that the differential preserves this filtration and pass to the associated graded quotients. These quotients are acyclic in all gradings greater than 1 and the quotient of grading 1 is isomorphic to $h_{<1>}$. $\square$

5.4.4. Let $X := \mathfrak{B}_N \mathfrak{h}_{<1>}$, $N \ge 2$. We have a natural map

$$\mathbf{BRACES}(<2>;<1>) \to \hom_{\mathbf{BA}_N^{\mathbf{free}}}(X^{\otimes 2}; X) \tag{12}$$

LEMMA 5.4. *This map is a quasi-isomorphism.*

Proof. Let $j : \mathbf{ASSOC}_N = \mathbf{COASS}_N^{\mathrm{op}} \to \mathbf{BA}_N^{\mathrm{op}}$ be the natural inclusion of PROPs. We have $X = j_! Y$, where

$$Y = 0 \to h_{<1>} \to h_{<2>} \to h_{<3>} \to \cdots \to h_{<N>} \to 0$$

is the cobar-complex of the coalgebra $h_{<1>}$, same as in (10).

Therefore, $X^{\otimes 2} = j_! Y^{\otimes 2}$ and, by the conjugacy property,

$$\hom_{\mathbf{BA}_N^{\mathbf{free}}}(X^{\otimes 2}; X) \cong \hom_{\mathbf{COASS}_N^{\mathbf{free}}}(Y^{\otimes 2}; j^{-1}X)$$

Decomposition (1) implies that

$$j^{-1} h_{<m>} \cong \bigoplus_{r=m}^{N} \mathbf{ASSOC}(<r>;<m>) \otimes_{S_r} h_{<r>}. \tag{13}$$

Next, we have

SUBLEMMA 5.5.

$$H^\bullet(\hom_{\mathbf{COASS}_N^{\mathbf{free}}}(Y^{\otimes 2}, h_{<r>})) = 0$$

if $r \neq 2$;

$$\hom(Y^{\otimes 2}, h_{<2>}) \cong \mathbf{COASS}(<2>;<2>)[2] \cong (\mathbf{k} \oplus \mathbf{k})[2].$$

Proof. Let V_r be the r-dimensional vector space with the fixed basis $e_1, e_2, \ldots, e_r$; one can view it as a space with r gradings so that the i-th grading of e_j equals the Kronecker symbol δ_{ij}. Given any vector space U with r gradings let $|U|$ be its homogeneous part whose all gradings are 1.

Let $A_r := TV_r$ be the free associative algebra generated by V_r. We then have

$$h_{<r>}(<k>) = \mathbf{COASS}(<k>;<r>) = \mathbf{ASSOC}(<r>;<k>) = |A_r^{\otimes k}|,$$

where $h_{<r>} \in \mathbf{COASS}_N^{\mathbf{free}}$ is the functor represented by $<r>$.

The complex $\hom(Y^{\otimes 2}; h_{<r>})$ is isomorphic to

$$|(\mathbb{B}A_r)^{\otimes 2}|,$$

where $\mathbb{B}A_r$ is the standard bar complex of A_r. We have a natural quasi-isomorphism

$$\mathbb{B}A_r \to V_r[1],$$

hence a quasi-isomorphism

$$|(\mathbb{B}A_r)^{\otimes 2}| \to |(V_r[1])^{\otimes 2}|$$

the complex on the RHS is 0 unless $r \neq 2$ in which case the complex on the RHS is a 2-dimensional space canonically identified with $\mathbf{COASS}(< 2 >; < 2 >)[2]$ $\square$

This sublemma implies that the map

$$\mathbf{ASSOC}(< 2 >; < n >) \otimes_{S_2} h_{<2>} \to j^{-1}h_{<n>}$$

induces a quasi-isomorphism

$$\mathbf{ASSOC}(< 2 >; < n >)[2] = \hom(Y^{\otimes 2}; \mathbf{ASSOC}(< 2 >; < n >) \otimes_{S_2} \mathfrak{h}_{<2>})$$
$$\to \hom(Y^{\otimes 2}; j^{-1}h_{<n>})$$

In particular, these spaces are homotopy equivalent to 0 if $n > 2$.

Taking into account (13) and the sub-Lemma, we get a quasi-isomorphism

$$\hom(Y^{\otimes 2}; j^{-1}X) \to \hom(Y^{\otimes 2}; \mathrm{Cone}[h_{<1>} \to h_{<2>}])[-1]$$

The latter complex is quasi-isomorphic to the complex

$$0 \to \mathbf{ASSOC}(< 2 >; < 1 >) \to \mathbf{ASSOC}(< 2 >; < 2 >) \to 0,$$

where the term $\mathbf{ASSOC}(< 2 >; < 2 >)$ is in the cohomological degree 0.

Let us now compute the trough map

$$(14) \quad H^\bullet\mathbf{braces}(2) \to H^\bullet \hom(X^{\otimes 2}; X)$$
$$= H^\bullet(Y^{\otimes 2}; j^{-1}X)$$
$$\to H^\bullet[(\mathbf{ASSOC}(< 2 >; < 1 >) \to \mathbf{ASSOC}(< 2 >; < 2 >)]$$

The cohomology $H^\bullet\mathbf{braces}(2)$ has 2 generators: $\cup \in H^0$ and $b \in H^{-1}$.

The cup product is represented by the map $X^{\otimes 2} \to X$ given by the free tensor product $h_{<k>}[-k] \otimes h_{<l>}[-l] \to h_{<k+l>}[-k-l]$. This map produces the identity map in $\mathbf{ASSOC}(< 2 >; < 2 >)$.

The bracket b is the anti-symmetrization of the brace operation

$$X \otimes X \to X$$

We only need to know its restriction onto $h_{<1>} \otimes h_{<1>}$ which is induced by the commutator

$$m_{12} - m_{21} :< 1 > \otimes < 1 > \to < 1 >$$

where $m_{12} \in \mathbf{ASSOC}(< 2 >; < 1 >)$ is the associative product.

We now see that the map (14) is an isomorphism. This proves the statement.

$\square$

Let us consider the map of PROP's

$$\mathbf{BRACES}_N \to \mathbf{FULL}(E_N(h_{<1>}))$$

naturally induced by the functor E_n.

LEMMA 5.6. *This map is a quasi-isomorphism of PROPs*

Proof. As was shown above, the object $E_N(h_{<1>})$ is weakly equivalent to $h_{<1>}$. Therefore, the PROP $\mathbf{FULL}(E_N(h_{<1>}))$ is weakly equivalent to $\mathbf{BRACES}_N$. Therefore, we have an isomorphism

$$H^\bullet\mathbf{FULL}(E_N(h_{<1>})) \to H^\bullet\mathbf{BRACES}$$

As the operad **braces** is weakly equivalent to the operad **ger**, so are the PROPs generated by these operads and we have an isomorphism

$$H^\bullet\mathbf{BRACES} \cong \mathbf{GER}.$$

Thus we have a chain of maps

$$\mathbf{GER} \cong H^\bullet\mathbf{BRACES} \to H^\bullet\mathbf{FULL}(E_n(h_{<1>})) \cong \mathbf{GER}$$

and it suffices to prove that this through map is an isomorphism. Since the PROP **GER** is generated by its binary operations i.e. hom$(<2>;<1>)$, it suffices to check that our map is an isomorphism when restricted onto the two-dimensional space hom$([2];[1])$.

we have a chain of maps

$$\begin{aligned}\mathbf{ger}(2) &\xrightarrow{1} H^\bullet \hom_{\mathbf{BA}_N}(X^{\otimes 2};X)\\ &\xrightarrow{2} H^\bullet\mathbf{FULL}(E_N(h_{<1>}))(<2>;<1>)\\ &\xrightarrow{3} H^\bullet \hom_{\mathbf{BA}_N}(\mathfrak{B}\mathfrak{H}(X)^{\otimes 2};\mathfrak{B}\mathfrak{H}(X))\end{aligned}$$

The arrow 1 is induced by the map (12);

the arrow 2 is induced by the functor $\mathfrak{H}_N$: note that we have an isomorphism $E_N h_{<1>} \cong \mathfrak{H}_N(X)$;

the arrow 3 is induced by the functor $\mathfrak{B}_N$.

The arrow 1 is an isomorphism (Lemma 5.4); the composition of arrows 2,3 is an isomorphism because it is induced by the composition $\mathfrak{B}_N\mathfrak{H}_N$ which is a weak equivalence by Lemma 5.1. Taking into account that the dimensions of all spaces involved are equal and finite, we must conclude that all arrows in this sequence are isomorphisms, whence the statement. □

5.4.5. *proof of the Proposition*

Let us prove that $E : \underline{\mathbf{BRACES}}^{\mathbf{free}} \to \underline{\mathbf{BRACES}}^{\mathbf{free}}$ is a weak equivalence. The statement for $\underline{\mathbf{BRACES}}^{\mathbf{proj}}$ will then follows automatically as $\mathbf{BRACES}_n^{\mathbf{proj}}$ is the Karoubian closure of $\mathbf{BRACES}_n^{\mathbf{free}}$.

Lemma 5.6 implies the following: given free finitely generated objects F, G in $\mathbf{BRACES}_n^{\mathbf{free}}$ (that is each of F, G is a finite direct sum of the form $\bigoplus_i h_{<k_i>}[n_i]$), we have

$$E_n : \hom_{\mathbf{BRACES}_N^{\mathbf{free}}}(F,G) \to \hom_{\mathbf{BRACES}_N^{\mathbf{free}}}(E_N(F), E_N(G))$$

is a quasi-isomorphism. The standard argument then implies that the same holds true if F, G are finite complexes of finitely generated free objects, i.e. of F, G are any objects in $\mathbf{BRACES}_N^{\mathbf{free}}$. It only remains to prove that E_N is essentially surjective. Let us show by induction that *every length M complex of finitely generated free objects in $\mathbf{BRACES}_n^{\mathbf{free}}$ is quasi-isomorphic to an object from the image of E_n* Indeed, Lemma 5.3 implies that $h_{<k>}$ is quasi-isomorphic to $E_n h_{<k>}$; this readily implies that given any free finitely generated object $F \in \mathbf{BRACES}^{\mathbf{free}}$, $E_N(F)$ is quasi-isomorphic to F. This covers the case $M = 1$.

The transition goes as follows. Every complex of length M is isomorphic to a Cone$(f : K \to F)$, where K is a complex of length $M-1$ and F is free. The induction assumption implies that there is $K' \in \mathbf{BRACES}^{\mathbf{free}}$ and a quasi-isomorphism $k : E(K') \to K$. We also have a quasi-isomorphism

$\phi : E(F) \to F$. This implies that there is an arrow $f' : K' \to F$ such that $\phi E_N(f') - fk = d\gamma$, where $\gamma \in \hom^{-1}(E_N(K'); F)$. These data produce a quasi-isomorphism of $E_N(\mathrm{Cone}(f'))$ and $\mathrm{Cone}(f)$. $\square$

5.4.6. *Gerstenhaber algebras.* We have quasi-isomorphisms of operads

$$\begin{tikzcd} \mathbf{hoger} \arrow[r] \arrow[d] & \mathbf{braces} \\ \mathbf{ger} & \end{tikzcd}$$

Whence induced equivalences of projective systems of PROPs

$$\begin{tikzcd} \underline{\mathbf{HOGER}} \arrow[r] \arrow[d] & \underline{\mathbf{BRACES}} \\ \underline{\mathbf{GER}} & \end{tikzcd}$$

$$\begin{tikzcd} \underline{\mathbf{HOGER}}^{\mathbf{proj}} \arrow[r] \arrow[d] & \underline{\mathbf{BRACES}}^{\mathbf{proj}} \\ \underline{\mathbf{GER}}^{\mathbf{proj}} & \end{tikzcd}$$

5.4.7. *Lie bialgebras and Gerstenhaber algebras in the universal language.* The constructions of 5.3 give rise to maps of the projective systems

$$\begin{tikzcd} \underline{\mathbf{LBA}}^{\mathbf{proj}} \arrow[r, shift left, "\underline{\mathfrak{L}}"] & \underline{\mathbf{GER}}^{\mathbf{proj}} \arrow[l, shift left, "\underline{\mathfrak{G}}"] \end{tikzcd}$$

We have natural transformations

$$\mathrm{Id} \to \underline{\mathfrak{L}\mathfrak{G}};\ \mathrm{Id} \to \underline{\mathfrak{G}\mathfrak{L}}$$

These natural transformations are weak equivalences. Hence so are $\underline{\mathfrak{L}}$ and $\underline{\mathfrak{G}}$.

Let us prove all these statements.

5.4.8. *Constructing the maps $\underline{\mathfrak{L}}$, $\underline{\mathfrak{G}}$.* The construction is similar to that in 5.4. The major difference is that, since our construction involve tensoring with non-regular representations of the symmetric group, the resulting objects will be projective, not necessarily free.

The functor $\underline{\mathfrak{L}}$ We have a universal Gerstenhaber algebra $< 1 > \in \mathbf{GER}$. Its images $< 1 > \in \mathbf{GER}_n$; $h_{<1>} \in \mathbf{GER}_n^{\mathbf{free}}$ are Gerstenhaber algebras. We can apply the functor $\mathfrak{L}$ so as to get Lie bialgebras

$$\mathfrak{L}'_n := \mathfrak{L}(h_{<1>}) \in \mathbf{GER}_n^{\mathbf{proj}}.$$

These bialgebras can be interpreted as SM functors

$$\mathfrak{L}_n : \mathbf{LBA} \to \mathbf{GER}_n^{\mathbf{proj}}.$$

One sees that $\mathfrak{L}_n(< m >) = 0$ for all $m > n$ so that we get an SM-functor

$$\mathfrak{L}_n : \mathbf{LBA}_n \to \mathbf{GER}_n^{\mathbf{proj}}.$$

This functor canonically extends to a SM-functor

$$\mathfrak{L}_n : \mathbf{LBA}_n^{\mathbf{proj}} \to \mathbf{GER}_n^{\mathbf{proj}}.$$

The functors $\mathfrak{L}_n$ produce a map of projective systems

$$\underline{\mathfrak{L}} : \underline{\mathbf{LBA}}^{\mathbf{proj}} \to \underline{\mathbf{GER}}^{\mathbf{proj}}.$$

The map $\underline{\mathfrak{G}} : \underline{\mathbf{GER}}^{\mathbf{proj}} \to \underline{\mathbf{LBA}}^{\mathbf{proj}}$ is constructed in a similar way.

The natural transformations from 5.3.3,5.3.4 produce the natural transformations

$$\mathrm{Id} \to \underline{\mathfrak{L}\mathfrak{G}};\ \mathrm{Id} \to \underline{\mathfrak{G}\mathfrak{L}}$$

One can easily check that these transformations are weak equivalences. Therefore, the maps of projective systems of SMC $\underline{\mathfrak{L}}, \underline{\mathfrak{G}}$ are weak equivalences (because their compositions in both ways are weakly equivalent to the identity).

5.4.9. *Conclusion.* We have constructed a number of SM weak equivalences

$$\begin{tikzcd}
\underline{\mathbf{BA}}^{\mathbf{proj}} \arrow[r, shift left, "\underline{\mathfrak{H}}"] & \underline{\mathbf{BRACES}}^{\mathbf{proj}} \arrow[l, shift left, "\underline{\mathfrak{B}}"] & \underline{\mathbf{HOGER}}^{\mathbf{proj}} \arrow[l] \arrow[d] \\
 & & \underline{\mathbf{GER}}^{\mathbf{proj}} \arrow[d, shift left, "\underline{\mathfrak{G}}"] \\
 & & \underline{\mathbf{LBA}}^{\mathbf{proj}} \arrow[u, shift left, "\underline{\mathfrak{L}}"]
\end{tikzcd} \tag{15}$$

We have also constructed weak equivalences

$$\mathrm{Id} \to \underline{\mathfrak{B}\mathfrak{H}};\ \mathrm{Id} \to \underline{\mathfrak{G}\mathfrak{L}};\ \mathrm{Id} \to \underline{\mathfrak{L}\mathfrak{G}}.$$

6. t-STRUCTURES

We are going to endow each SMC in (15) with a t-strucrture [1]. Next, we will show that all the arrows in (15) are exact with respect to these t-structures.

6.0.10. Let us recall the definition of a t-structure on a dg-category $\mathcal{C}$. We assume that for every object $X \in \mathcal{C}$ and any integer n there exists an object $X[n]$ which represents the functor

$$U \mapsto \hom(U; X)[n].$$

It follows that we have a natural isomorphism

$$\hom(X[n]; Y[m]) \xrightarrow{\sim} \hom(X; Y)[m-n].$$

We define *a t-structure* on $\mathcal{C}$ as a collection of full subcategories $\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n} \subset \mathcal{C}$ for all integers n, $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$; $\mathcal{D}^{\geq n} = D^{\geq n}[-n]$. The following properties should be satisfied:

1 $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$;
2 $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$;
3 For any $X \in \mathcal{D}^{\leq -1}$ and $Y \in \mathcal{D}^{\geq 0}$, $H^0 \hom(X, Y) = 0$;
4 For any X in C there exist objects $A \in \mathcal{D}^{\leq -1}$, $B \in \mathcal{D}^{\geq 0}$ and arrows $A \to X \to B$ whose composition is zero and such that for every $U \in \mathcal{C}$ the complex

$$0 \to \hom(U, A) \to \hom(U, X) \to \hom(U, B) \to 0$$

is acyclic.

6.0.11. *The categories* $\mathbf{BA}_n^{\mathbf{proj}}$.

PROPOSITION 6.1. *Every object in* $\mathbf{BA}_n^{\wedge}$ *admits a finite projective resolution centered in degrees* ≤ 0.

Proof. Let $F \in \mathbf{BA}_n^{\wedge}$. We have a quasi-isomorphism

$$F \to \mathfrak{B}\mathfrak{H}(F)$$

On the other hand we have a finite projective resolution $K \to \mathfrak{H}(F)$ (it is easy to see that every finite complex in $\mathbf{BRACES}_n$ admits a finite projective resolution.) Thus we have a diagram

$$F \to \mathfrak{B}\mathfrak{H}(F) \leftarrow \mathfrak{B}(K).$$

The complex $\mathfrak{B}(K)$ is finite and free and its only non-vanishing cohomology is the zeroth, this zeroth cohomology is isomorphic to F. It then follows that the subcomplex

$$RF := \tau_{\leq 0}\mathfrak{B}(K) \subset \mathfrak{B}(K)$$

is finite and projective. Indeed, Let N be the maximal number for which $K^N \neq 0$. If $N = 0$ there is nothing to prove. If $N > 0$, then, since $H^N(K) = 0$, the map

$$d_N : K^{N-1} \to K^N$$

is surjective, hence splits, as all terms are projective. Let $K'_{N-1} := \mathrm{Ker} d_N$. We then know that K'_{N-1} is projective. If $N - 1 = 0$ we are done; if not, we repeat the same procedure. □

6.0.12. Set $\mathcal{D}^{\geq 0}\mathbf{BA}_n^{\mathbf{proj}}$ to consist of all objects whose cohomology is only in degrees ≥ 0 and $\mathcal{D}^{\leq 0}\mathbf{BA}_n^{\mathbf{proj}}$ to consist of all objects whose cohomology is only in degrees ≤ 0.

The axioms can be verified as follows.

1,2 — clear;

3 follows from the statement that any object from $\mathcal{D}^{\leq 0}$ admits a finite projective resolution whose all terms are in degree ≤ 0;

4 Given an object F let $G \to \tau_{\leq 0}F$ be a resolution. Then the cone of the map $G \to F$ is in $D^{>0}$. This is the required decomposition.

6.0.13. The core of this t-structure is $\mathbf{BA}_n^{\wedge}$.

6.0.14. Tensor products on $\mathbf{BA}_n$ and the projection maps $\mathbf{BA}_n \to \mathbf{BA}_{n-1}$ are right exact (i.e. preserve $\mathcal{D}^{\leq 0}$).

6.1. **t-structre on $\mathbf{LBA}_n$.** is introduced in the same way. Same results about the right exactness of the tensor product and projections do hold.

6.2. **t-structure on $\mathbf{GER}_n, \mathbf{BRACES}_n$.** Let C be a dg-category with $n+1$ objects $< 0 >, < 1 >, < 2 >, \ldots, < n >$. Assume that:

–hom($< j >, < i >$) $= 0$ if $j > i$;

—hom($< i >, < i >$) is a finite dimensional semi-simple algebra.

— if $j \leq i$, then hom($< j >, < i >$) is a finite complex whose all non-zero terms are in degrees from 0 to $j - i$ inclusive;

The categories $\mathbf{GER}_n^{\mathrm{op}}, \mathbf{BRACES}_n^{\mathrm{op}}$ are such.

Let us construct a t-structure on the category E of finite projective complexes of functors $C \to$ vect.

We set $\mathcal{D}^{\geq 0}E$ to consist of all objects X such that $H^jX(<i>) \neq 0$ only if $j \geq -i$. We set $\mathcal{D}^{\leq 0}$ to consist of all complexes of projective modules P generated by several elements from several complexes $P(<i>)$ in degree $\leq -i$.

Let us prove it is indeed a t-structure.

1,2,3 are obvious. Let us pass to 4. We will use the cone construction; in order to avoid an umbiguity, let us fix the agreements: given a map of objects $f : X \to Y$, $X, Y \in E$ We define an object $\mathrm{Cone}(f)$ by setting

$$\mathrm{Cone}(f)(<i>)^n := Y(<i>)^n \oplus X(<i>)^{n+1}.$$

Let $y \in Y(<i>)^n$; $x \in X(<i>)^{n+1}$ we then set

$$d(y \oplus x) = y' \oplus x',$$

where $y' = dy + (-1)^n f(x)$; $x' = dx$.

In particular we have natural maps

$$Y \to \mathrm{Cone}(f) \to X[1]$$

whose composition is zero.

Let us now proceed to the proof.

Let $E_i \subset E$ be the full subcategory consisting of all objects X with $X(<j>) = 0$ whenever $j + i < n$, so that $E_0 \subset E_1 \subset E_2 \subset \cdots E_n = E$.

Let us prove our statement by induction.

1) if $X \in E_0$, the statement is clear: indeed, the category E_0 is that of $\mathrm{End}(<n>)$-modules. $\mathcal{D}^{\geq 0}E \cap E_0$ consists of all modules in degrees $\geq -n$, and the category $D^{\leq 0}E \cap E_0$ consists of all modules in degrees $\leq -n$, so $\tau_{\leq -n-1}X \in \mathcal{D}^{<0}$; and $\tau_{\geq -n}X \in \mathcal{D}^{\geq 0}$.

2) Induction step. Let $X \in E_i$ so that $X(<j>) \neq 0$ only if $j \geq n - i$.

Consider the complex $X(<n-i>)$. This complex has a natural structure of a dg $\mathrm{End}(<n-i>)$-module. Let us decompose

$$\tau_{\leq i-n-1}X(<n-i>) \to X(<n-i>) \to \tau_{\geq i-n}X(<n-i>),$$

where the truncations are taken in the category of complexes of $\mathrm{End}(<n-i>)$-modules.

Let $L \in E_i$ be the object freely generated by the $\mathrm{End}(<n-i>)$-module $\tau_{\leq i-n-1}X(<n-i>)$. Let $W : C \to$ **complexes** be defined as follows:

— $W(<k>) = 0$ for all $k \neq n - i$;

— $W(<n-i>) = \tau_{\geq i-n}X(<n-i>)$.

The C-action is as follows: all complexes $\hom_C(<k_1>;<k_2>)$ act by zero except $\hom_C(<n-i>;<n-i>) = \mathrm{End}(<n-i>)$ which has a natural action on W.

We then have naturally defined maps

$$L \overset{i_L}{\to} X \overset{p_W}{\to} W.$$

The following properties are the case:

— $p_W i_L = 0$;

—$L \in \mathcal{D}^{<0}E$;

— Let W' be any object from E_i such that there exists a quasi-isomorphism $W' \to W$ in the category of dg functors $C \to$ **complexes**, then $W' \in \mathcal{D}^{\geq 0}$.

—The following sequence of complexes

$$0 \to L(< n-i >) \to X(< n-i >) \to W(< n-i >) \to 0$$

is short exact.

Let us define the functor $A : C \to$ **complexes** by setting $A(< k >)$ to be the total complex of the bi-complex

$$0 \to L(< k >) \to X(< k >) \to W(< k >) \to 0,$$

where L is in the horizontal cohomological degree 0.

We have natural maps $\pi_L : A \to L$ and $\iota_W : W[-2] \to A$ satisfying $\pi_L \iota_W = 0$. The object X is canonically quasi-isomorphic to

$$X' := \mathrm{Cone}[W[-2] \stackrel{\iota_W}{\to} \mathrm{Cone}(A \stackrel{\pi_L}{\to} L)] \tag{16}$$

Since $A(< n-i >)$ is acyclic, A is quasi-isomorphic to an object from E_{i-1}, hence, by the induction assumption, the statement of the Lemma is applicable to $A[1]$. Thus there is a $U \in \mathcal{D}^{<0}$ and a map $U \stackrel{\iota_U}{\to} A[1]$ such that the cone of $\iota_U : U \to A[1]$ is quasi-isomorphic to an object from $\mathcal{D}^{\geq 0}$. Consider the through map

$$U[-1] \stackrel{\iota_U}{\to} A \stackrel{\pi_L}{\to} L \tag{17}$$

Let V be the cone of this through map. Since $U[-1] \in \mathcal{D}^{\leq 0}$ and $L \in \mathcal{D}^{<0}$, we have $V \in \mathcal{D}^{<0}$.

We are going to define an arrow $\phi : V \to X'$ such that $\mathrm{Cone}\,\phi \in \mathcal{D}^{\geq 0}$. By doing so we will prove the statement.

The arrow ϕ will be constructed as a composition

$$\phi : V \to \mathrm{Cone}(A \to L) \to X' \tag{18}$$

The left arrow is defined by the diagram

$$\begin{tikzcd} A \arrow[r, "\pi_L"] & L \\ U[-1] \arrow[u, "\iota_U"] \arrow[r] & L \arrow[u, "\mathrm{Id}"'] \end{tikzcd}$$

where ι_U, π_L are as in (17). The arrow $\mathrm{Cone}(A \to L) \to X'$ is defined as the identity embedding onto $\mathrm{Cone}(A \to L)$.

It is not hard to see that $\mathrm{Cone}\,\phi$ is canonically quasi-isomorphic to the cone of the following composition:

$$W[-1] \stackrel{\iota_W}{\to} A[1] \to \mathrm{Cone}(U \to A[1])$$

.

Next, we know that $\mathrm{Cone}(U \to A[1])$ is quasi-isomorphic to an object from $\mathcal{D}^{\geq 0}$ and so is W, this implies that the cone is also quasi-isomorphic to an object from $\mathcal{D}^{\geq 0}$. Therefore, $\mathrm{Cone}(\phi) \in \mathcal{D}^{\geq 0}$ as we wanted.

PROPOSITION 6.2. *The natural projections* $\mathbf{GER}_n \to \mathbf{GER}_m$*;* $\mathbf{HOGER}_n \to \mathbf{HOGER}_m$*;* $\mathbf{BRACES}_n \to \mathbf{BRACES}_m$ *are right exact. The tensor products on these categories are right exact.*

Proof. Straightforward ∎

6.2.1.

PROPOSITION 6.3. *The weak equivalences $\mathfrak{H}_n; \mathfrak{B}_n, \mathfrak{L}_n, \mathfrak{G}_n$ are exact, i.e. preserve $\mathcal{D}^{\geq 0}$, $\mathcal{D}^{\leq 0}$.*

Proof. Since all the functors listed are weak equivalences, it suffices to check they preserve $\mathcal{D}^{\leq 0}$. As all the functors commute with shifts, $\mathcal{D}^{\leq i}$ will be preserved as well.

The categories $\mathcal{D}^{\geq i}$ are uniquely determined by the condition :

$X \in \mathcal{D}^{\geq i}$ *iff* $H^{<0} \hom(Y, X) = 0$ *for all* $Y \in \mathcal{D}^{\leq i}$.

Hence, $\mathcal{D}^{\geq i}$ *must be preserved by any weak equivalence as long as* $\mathcal{D}^{\leq 0}$ *is preserved*

The preservation of $\mathcal{D}^{\leq 0}$ follows from the definitions. Indeed, we have:

$X \in \mathcal{D}^{\leq 0}\mathbf{BA}_n^{\mathbf{proj}}$ iff X is a retraction of a finite complex of finitely generated free objects whose all generators have grading ≤ 0. Same definition works for $\mathcal{D}^{\leq 0}\mathbf{LBA}_n^{\mathbf{proj}}$;

$X \in \mathcal{D}^{\leq 0}\mathbf{BRACES}_n^{\mathbf{proj}}$ iff X is a retraction of a finite complex of finitely generated free objects whose all generators from $X(< k >)$ have grading $\leq -k$; same definition for $\mathcal{D}^{\leq 0}\mathbf{ger}_n^{\mathbf{proj}}$.

We can now check immediately that

$$\mathfrak{H}_n(h_{<1>}) \in \mathcal{D}^{\leq 0}\mathbf{BRACES}_n^{\mathbf{free}},$$

this follows from (9)

Therefore, $\mathfrak{H}_n(h_{<k>}) = \mathfrak{H}_n(h_{<1>})^{\otimes k} \in \mathcal{D}^{\leq 0}\mathbf{BRACES}_n^{\mathbf{free}}$ because the tensor product is right exact. This fact easily implies that $\underline{\mathfrak{H}}_n(\mathcal{D}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$.

Let us show that $\mathfrak{B}_n\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}$. Since the weak equivalence $\underline{\mathfrak{H}}_n$ is exact, it follows that the functor

$$\mathfrak{H}_n : \mathcal{D}^{\leq 0}\mathbf{BA}_n^{\mathbf{proj}} \to \mathcal{D}^{\leq 0}\mathbf{BRACES}_n^{\mathbf{proj}}$$

is essentially surjective. Therefore, it suffices to prove that

$$\mathfrak{B}_n\mathfrak{H}_n\mathcal{D}^{\leq 0}\mathbf{BA}_n^{\mathbf{proj}} \subset \mathcal{D}^{\leq 0}\mathbf{BA}_n^{\mathbf{proj}}$$

This fact follows immediately from the weak equivalence $\mathrm{Id} \to \mathfrak{B}_n\mathfrak{H}_n$ (see 5.1).

The proof for the functors $\underline{\mathfrak{L}}, \underline{\mathfrak{G}}$ goes along the same lines.

$\square$

7. PROOF OF THEOREM 3.3

The above proven propositions show that the equivalences $\mathfrak{H}_n; \mathfrak{B}_n, \mathfrak{L}_n, \mathfrak{G}_n$ induce equivalences of the cores of the corresponding projective systems of SMC. Given a dg-category C with a t-structure, let $[C]$ be its core. Given an exact dg SM-functor $F : C \to D$ between SM-categories with t-structures, let $[F] : [C] \to [D]$ be the induced equivalence of the cores.

Thus we have a chain of SM-equivalences

$$\begin{tikzcd}
{[\underline{\mathbf{HOGER}}^{\mathbf{proj}}]} \arrow[r] \arrow[d] & {[\underline{\mathbf{BRACES}}^{\mathbf{proj}}]} \arrow[r, shift left, "{[\mathfrak{B}]}"] & \underline{\mathbf{BA}}^{\wedge} \arrow[l, shift left, "{[\mathfrak{H}]}"] \\
{[\underline{\mathbf{GER}}^{\mathbf{proj}}]} \arrow[d, shift left, "{[\mathfrak{G}]}"] & & \\
\underline{\mathbf{BA}}^{\wedge} \arrow[u, shift left, "{[\mathfrak{L}]}"] & &
\end{tikzcd}$$

The arrows $[\underline{\mathfrak{B}}]$ and $[\underline{\mathfrak{H}}]$ are mutually quasi-inverse (i.e. their compositions are isomorphic to the identity), as well as $[\underline{\mathfrak{G}}]$ and $[\underline{\mathfrak{L}}]$.

This gives rise to a unique (up-to a unique isomorphism) equivalence

$$\mathbf{Q} : \underline{\mathbf{BA}}^{\wedge} \to \underline{\mathbf{LBA}}^{\wedge},$$

as stated in the theorem

8. Dependence on the choice of a quasi-isomorphism hoger $\to$ braces

Our construction of the equivalence of systems $\underline{\mathbf{BA}}^{\wedge} \to \underline{\mathbf{LBA}}^{\wedge}$ depends on the choice of a quasi-isomorphism of operads $U : \mathbf{hoger} \to \mathbf{braces}$. Denote the corresponding equivalence

$$\mathbf{Q}_U : \underline{\mathbf{BA}}^{\wedge} \to \underline{\mathbf{LBA}}^{\wedge}.$$

We will answer the following question: given two different quasi-isomorphisms $U_1, U_2 : \mathbf{hoger} \to \mathbf{braces}$, are the equivalences $\mathbf{Q}_{U_1}$ and $\mathbf{Q}_{U_2}$ isomorphic?

The answer to this question can be conveniently given in terms of the derived category of dg-operads coming from the closed model structure on it [10]. Denote this derived category by D(op)

We then have the set of isomorphisms $\mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$. The quasi-isomorphisms U_1, U_2 define elements $\overline{U}_1, \overline{U}_2 \in \mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$.

In order to formulate a precise result, we need a one more thing. Given $x \in \mathbf{k}^{\times}$, let $\iota_x : \mathbf{hoger} \to \mathbf{hoger}$ be the automorphism which acts on $\mathbf{hoger}(n)$ as the dilation by x^{1-n}. This way, we have a map

$$\mathbf{k}^{\times} \to \mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{ger}). \tag{19}$$

The latter group acts on $\mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$, hence the induced $\mathbf{k}^{\times}$-action on $\mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$. Let $\sim$ be the equivalence relation on $\mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$ generated by this $\mathbf{k}^{\times}$-action.

We will prove:

THEOREM 8.1. *1) If $\overline{U}_1 \sim \overline{U}_2$, then $\mathbf{Q}_{U_1}$ is isomorphic to $\mathbf{Q}_{U_2}$;*

2) If $\overline{U}_1 \nsim \overline{U}_2$ then $\mathbf{Q}_{U_1}$ and $\mathbf{Q}_{U_2}$ are not isomorphic. Furthermore, there exists an N such that the induced symmetric monoidal functors

$$\mathbf{Q}_{U_1,N}, \mathbf{Q}_{U_2,N} : \mathbf{BA}_N^{\wedge} \to \mathbf{LBA}_N^{\wedge}$$

are not isomorphic.

Remark 1 If $\mathbf{Q}_{U_1}$ and $\mathbf{Q}_{U_2}$ are isomorphic, then the set of isomorphisms $\mathbf{Q}_{U_1} \to \mathbf{Q}_{U_2}$ is pretty large and seems to be parameterized by the set of all homotpy classes of homotopies between the quasi-isomorphisms $U_1, U_2 : \mathbf{hoger} \to \mathbf{braces}$ which is a torsor over the pro-nilpotent group $\mathrm{Exp}(H^{-1}F^1\mathfrak{g})$, where $\mathfrak{g}$ is as in Appendix 2.

Remark 2 Using the statement 2) of the theorem we can show that given U_1, U_2 producing non-equivalent elements in $\mathrm{Iso}_{\mathrm{D(op)}}\,\mathbf{ger}$, one can construct a conilpotent Lie bialgebra in some SMC whose quantizations using $\mathbf{Q}_{U_1}$ and $\mathbf{Q}_{U_2}$ are non-isomorphic.

As an appropriate SMC, we take the category of all functors from $\mathbf{LBA}_N^{\mathrm{op}}$ to the category of vector spaces (it is larger than $\mathbf{LBA}_N^{\wedge}$; we have to take such a larger category in order to satisfy the conditions from 4.1.2). Denote this category by $\mathcal{C}_N$.

We see that the representing object $h_{<1>} \in \mathcal{C}_N$, of $<1> \in \mathbf{LBA}_N$, is a conilpotent Lie bialgebra because $h_{<1>}^{\otimes M} = 0$ for all $M > N$. Denote this conilpotent Lie bialgebra by $\mathfrak{a}$.

The quantization functors produce a pair of conilpotent $\mathbf{BA}$-algebras in $\mathcal{C}_n$: $\mathfrak{m}_i := \mathbf{Q}_{U_i}\mathfrak{a}$, $i = 1, 2$ Let us show this $\mathbf{BA}$-algebras are not isomorphic. Indeed, the opposite assumption implies that we have SM-monoidal isomorphism of the functors $\mathbf{Q}_{U_i,N}$, $i = 1, 2$, which, for N large enough, contradicts to the statement 2) of the theorem.

8.1. **proof of the first part of the Theorem.** Suppose that $U_1, U_2 : \mathbf{hoger} \to \mathbf{braces}$ produce equivalent elements in $\mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$. By definition, there exists an $x \in \mathbf{k}^\times$ such that U_1 and $U_2' := U_2\iota_x$ produce the same element in $\mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$.

Step 1 Let us first prove that $\mathbf{Q}_{U_2}$ is isomorphic to $\mathbf{Q}_{U_2\iota_x}$. Indeed, the induced SM functors

$$U_2, U_2' : \mathbf{HOGER} \to \mathbf{BRACES}$$

are isomorphic: the isomorphism $i : U_2 \to U_2'$ is defined by setting

$$i(< l >) : U_2(< l >) \to U_2'(< l >)$$

to be the dilation $x^l\mathrm{Id}$ on $< l >= U_2(< l >) = U_2'(< l >)$. This isomorphism induces a quasi-isomorphism $\mathbf{Q}_{U_2} \to \mathbf{Q}_{U_2'}$ in the obvious way.

Step 2 Let us now prove that the functors $\mathbf{Q}_{U_1}$ and $\mathbf{Q}_{U_2'}$ are isomorphic. We will write U_2 instead of U_2'. Our task then reduces to showing that $\mathbf{Q}_{U_1}$ and $\mathbf{Q}_{U_2}$ are isomorphic as long as U_1 and U_2 produce the same element in $\mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{braces})$.

According to Quillen, this means that there exists a map

$$P : \mathbf{hoger} \to \mathbf{braces}[t, dt]$$

such that $r_1P = U_1$; $r_2P = U_2$.

Since the operad $\mathbf{hoger}$ is concentrated in degrees ≤ 0, therefore, the map P takes values in the suboperad $\tau_{\leq 0}\mathbf{braces}[t, dt]$. Denote the latter suboperad by $\mathbf{o}$ so that we have a diagram

$$\mathbf{hoger} \longrightarrow \mathbf{o} \overset{r_1}{\underset{r_2}{\rightrightarrows}} \mathbf{braces}$$

Let $\mathbf{O}$ be the PROP generated by $\mathbf{o}$, let $\mathbf{O}_n$ be the quotient of $\mathbf{O}$ by the ideal generated by Id_N, $N > n$. Let $\underline{\mathbf{O}}$ be the corresponding projective system of PROPs.

Let $i : \mathbf{braces} \to \mathbf{braces}[t, dt]$ be the constant embedding so that $r_1i = r_2i$. We denote by the same letter the induced embedding $i : \mathbf{braces} \to \mathbf{o}$.

We then have an induced diagram of projective systems of PROPs:

$$\begin{tikzcd} \underline{\mathbf{HOGER}} \arrow[r, "P"] & \underline{\mathbf{O}} \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & \underline{\mathbf{BRACES}} \\ & \underline{\mathbf{BRACES}} \arrow[u, "i"] & \end{tikzcd}$$

Let us define a t-structure on $\mathbf{O}_N$ according to the recipe of Sec. 6.2 (note that the category $\mathbf{O}_N$ satisfies all the conditions therein). We then see that all the arrows are t-exact, hence induce an equivalence of cores. We have a natural isomorphism $[r_1i] = [r_2i]$. Since i is an equivalence, there is a quasi-inverse equivalence $j : [\underline{\mathbf{O}}] \to [\underline{\mathbf{BRACES}}]$, and we have equivalences $[r_1ijP] = [r_2ijP]$; $[r_1ijP] = [r_1P] = [U_1]$; $[r_2ijP] = [r_2P] = [U_2]$, so we have established an equivalence $[U_1] \to [U_2]$. This readily implies that $\mathbf{Q}_{U_1}$ and $\mathbf{Q}_{U_2}$ are isomorphic.

8.2. **proof of the second part of the theorem.** According to the Appendix 2 (Proposition 10.6, there exists an N such that the images of U_1, U_2 in $\mathrm{Iso}_{D(n\text{-op})}(\mathbf{ger}_{\leq N}; \mathbf{braces}_{\leq N})$ are not conjugated by any $\iota_x : \mathbf{hoger}_{\leq N} \to \mathbf{hoger}_{\leq N}$, $x \neq 0$. Let us prove that the SM-functors $\mathbf{Q}_{U_1,N}$ and $\mathbf{Q}_{U_2,N}$ are non-isomorphic. We need an auxiliary construction.

8.2.1. Let C be a dg SMC and X, Y be a pair of homotopy equivalent objects. This implies that there exist elements $\overline{f} \in H^0 \hom_C(X,Y)$; $\overline{g} \in H^0 \hom_C(Y,X)$ such that $\overline{f}\overline{g} = \overline{\mathrm{Id}_Y}$; $\overline{g}\overline{f} = \overline{\mathrm{Id}_X}$, in particular, $\overline{g}$ is uniquely determined by $\overline{f}$. Given such an $\overline{f}$, we will construct an isomorphism in $D(n\text{-op})$

$$I(\overline{f}) : \mathbf{full}(X)_{\leq n} \to \mathbf{full}(Y)_{\leq n}$$

Note that $I(\overline{f})$ will only depend on $\overline{f}$.

First of all let C' be the dg-category of finite complexes in C and let us identify C with the corresponding full subcategory in C'. The latter category is closed under taking cones.

Pick a representative $f : X \to Y$ whose class in $H^0 \hom(X,Y)$ is $\overline{f}$, and consider the following object in C':

$$K_f := K := \mathrm{Cone}(X \oplus Y \stackrel{f \oplus \mathrm{Id}_X}{\to} Y)[-1]$$

We have natural projections $p_X : K \to X$; $p_Y : K \to Y$. As f is a quasi-isomorphism, it easily follows that both p and q are split quasi-isomorphisms. (i.e. there exist $i_X : X \to K$; $i_Y : Y \to K$ such that $p_X i_X = \mathrm{Id}_X$; $p_Y i_Y = \mathrm{Id}_Y$. This, in turn, implies that there exist isomorphisms

$$X \oplus R_X \to K; \quad Y \oplus R_Y \to K;$$

such that R_X, R_Y are quasi-isomorhic to 0 and the maps p_X, p_Y, i_X, i_Y correspond under these isomorphisms to the natural embeddings and projections of X, Y as direct summands.

We will now define a pair of sub-operads of $\mathbf{full}(K)$ as follows.

Let $\mathbf{o}_X(n)$ be the kernel of the following arrow

$$A_X : \hom(K^{\otimes n}; K) \oplus \hom(X^{\otimes n}; X) \stackrel{A_X^1 - A_X^2}{\to} \hom(K^{\otimes n}; X)$$

whose components are defined as follows:

$$A_X^1 : \hom(K^{\otimes n}; K) \to \hom(K^{\otimes n}; X)$$

by post-composition with $p : K \to X$;

$$A_X^2 : \hom(X^{\otimes n}; X) \to \hom(K^{\otimes n}; X)$$

by pre-composition with p.

We then have natural maps of operads

$$\mathbf{full}(X) \leftarrow \mathbf{o}_X \to \mathbf{full}(K)$$

which are quasi-isomorphisms as easily follows from the isomorphism $K = X \oplus R_X$ and exactness of the tensor product.

In the same way, we construct a diagram

$$\mathbf{full}(Y) \leftarrow \mathbf{o}_Y \to \mathbf{full}(K),$$

where $\mathbf{o}_Y(n)$ is defined as the kernel of

$$A_Y : \hom(K^{\otimes n}; K) \oplus \hom(Y^{\otimes n}; Y) \overset{A^1_Y - A^2_Y}{\to} \hom(K^{\otimes n}; Y),$$

where

$$A^1_Y : \hom(K^{\otimes n}; K) \to \hom(K^{\otimes n}; Y);$$
$$A^2_X : \hom(Y^{\otimes n}; Y) \to \hom(K^{\otimes n}; Y)$$

are given by post-composing and pre-composing with p_Y.

We then have a diagram of quasi-isomorphisms

$$\mathbf{full}(X) \to \mathbf{full}(K) \to \mathbf{full}(Y)$$

which defines an isomorphism

$$I_f : \mathbf{full}(X)_{\leq n} \to \mathbf{full}(Y)_{\leq n}$$

in $D(n\text{-op})$. Let $f' : X \to Y$ be another representative of $\overline{f}$. Therefore, we have $f' - f = d\gamma$, where

$$\gamma \in \hom^{-1}(X, Y).$$

We can construct an isomorphism $K_f \to K_{f'}$ defined as follows:

$$\begin{tikzcd}
X \oplus Y \arrow[r, "f\oplus \mathrm{Id}"] \arrow[d, "\mathrm{Id}"] \arrow[dr, "\gamma\oplus 0"] & Y \arrow[d, "\mathrm{Id}"] \\
X \oplus Y \arrow[r, "f'\oplus \mathrm{Id}"] & Y
\end{tikzcd}$$

This isomorphism commutes with the projections $K_f \to X, Y$; $K_{f'} \to X, Y$, therefore, it induces a commutative diagram

$$\begin{tikzcd}
\mathbf{full}(X) \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathbf{full}(K) \arrow[d] & \mathbf{full}(Y) \arrow[l] \arrow[d, "\mathrm{Id}"] \\
\mathbf{full}(X) \arrow[r] & \mathbf{full}(K') & \mathbf{full}(Y) \arrow[l]
\end{tikzcd}$$

in which all arrows are quasi-isomorphisms.

This diagram implies that $I_f = I_{f'}$.

8.2.2. Let $\mathcal{P}, \mathcal{C}$ be dg SMC. Suppose that we are given two symmetric monoidal maps

$$\begin{tikzcd}
\mathcal{P} \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & \mathcal{C}
\end{tikzcd}$$

and a quasi-isomorphism $i_1(X) \to i_2(X)$ for some $X \in \mathcal{P}$. We then get a couple of arrows in $D(n\text{-op})$

$$\mathbf{full}_{\mathcal{P}}(X)_{\leq n} \rightrightarrows \mathbf{full}_{\mathcal{C}}(i_2(X))_{\leq n}.$$

Indeed, we have maps of operads

$$\mathbf{full}_{\mathcal{P}}(X)_{\leq n} \to \mathbf{full}_{\mathcal{C}}(i_1(X))_{\leq n};\ \mathbf{full}_{\mathcal{P}}(X)_{\leq n} \to \mathbf{full}_{\mathcal{C}}(i_2(X))_{\leq n}$$

and an isomorphism in $D(n\text{-op})$ $\mathbf{full}_{\mathcal{C}}(i_1(X))_{\leq n} \to \mathbf{full}_{\mathcal{C}}(i_2(X))_{\leq n}$, whence a pair of arrows as promised.

8.2.3. Suppose we have a diagram

$$\begin{tikzcd}
\mathcal{P}_2 \arrow[r, shift left, "j_1"] \arrow[r, shift right, "j_2"'] & \mathcal{C}_2 \\
\mathcal{P}_1 \arrow[u, "P"] \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & \mathcal{C}_1 \arrow[u, "Q"']
\end{tikzcd}$$

of SMC and SM functors; let us also assume that we are given isomorphisms of SM-functors:

$$j_1P \xrightarrow{\sim} Qi_1;$$

$$j_2P \xrightarrow{\sim} Qi_2.$$

Let $X_1 \in \mathcal{P}_1$ and $X_2 := P(X_1)$. Suppose we have a homotopy equivalence $\xi : i_1(X_1) \to i_2(X_1)$. We then have an induced homotopy equivalence $Q(\xi) : j_1(X_2) \to j_2(X_2)$.

It is clear that all these data produce a commutative diagram in $D(n\text{-op})$:

(20)
$$\begin{tikzcd}
\mathbf{full}(X_2)_{\leq n} \arrow[r, shift left] \arrow[r, shift right] & \mathbf{full}(j_1(X_2))_{\leq n} \\
\mathbf{full}(X_1)_{\leq n} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & \mathbf{full}(i_1(X_1))_{\leq n} \arrow[u]
\end{tikzcd}$$

Assume P, Q are homotopy equivalences, then the top horizontal arrows in (20) are equivalent iff so are the bottom horizontal arrows.

8.2.4. As explained above, we have SM maps

$$\begin{tikzcd}
\mathbf{HOGER}_n^{\mathbf{proj}} \arrow[r, shift left, "U_1"] \arrow[r, shift right, "U_2"'] \arrow[d] & \mathbf{BRACES}_n^{\mathbf{proj}} \arrow[r, "\mathfrak{B}_n"] & \mathbf{BA}_n^{\mathbf{proj}} \\
\mathbf{LBA}_n^{\mathbf{proj}} & &
\end{tikzcd}$$

Denote the composition of the horizontal arrows by

$$\mathbb{H}_{U_1}, \mathbb{H}_{U_2} : \mathbf{HOGER}_n^{\mathbf{proj}} \to \mathbf{BA}_n^{\mathbf{proj}};$$

denote the vertical arrow by

$$\mathbb{L} : \mathbf{HOGER}_n^{\mathbf{proj}} \to \mathbf{LBA}_n^{\mathbf{proj}}.$$

As was explained above, this diagram induces a pair of SM equivalences of the cores:

$$\mathbf{Q}_{U_1}, \mathbf{Q}_{U_2} : \mathbf{BA}_n^{\wedge} \to \mathbf{LBA}_n^{\wedge}.$$

Let us make these equivalences more expicit. To this end let us choose $Z_1, Z_2 \in \mathbf{HOGER}_n^{\mathbf{proj}}$ such that

(21)
$$\mathbb{H}_i(Z_i) \cong h_{<1>}$$

for $i = 1, 2$.

here $\cong$ means "quasi-isomorphic". Such objects Z_i do exist because $\mathbb{H}_i$ are weak equivalences.

Since $\mathbb{H}_i$ are exact functors, $Z_i \in \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$, therefore, Z_i define objects $[Z_i]$ in the core of $\mathbf{HOGER}_n^{\mathbf{proj}}$. We know that $\mathbb{H}_i$ induce equivalences

$$[\mathbb{H}_i] : [\mathbf{HOGER}_n^{\mathbf{proj}}] \to [\mathbf{BA}_n^{\mathbf{proj}}],$$

and we have isomorphisms $[\mathbb{H}_i][Z_i] \cong [h_{<1>}]$ induced by (21). As $[h_{<1>}]$ ia a $\mathbf{BA}_n$-algebra, we have an induced $\mathbf{BA}_n$-structure on $[Z_i]$.

By definition, we have a canonical isomorphism in $[\mathbf{LBA}^{\mathbf{proj}}]$:

$$\mathbf{Q}_{U_i,n}(h_{<1>}) \cong [\mathbb{L}]([Z_i])]. \tag{22}$$

The $\mathbf{BA}_n$-algebra structure on $h_{<1>}$ induces $\mathbf{BA}_n$-structures on $\mathbf{Q}_{U_i,n}(h_{<1>})$, $i = 1, 2$. For each $i = 1, 2$, these structures, , upon the identification (22), coincide with the $\mathbf{BA}_n$-structures induced by $[\mathbb{L}]$ from the $\mathbf{BA}_n$-structures on $[Z_i]$.

On the other hand, by our assumption, the SM functors $\mathbf{Q}_{U_i,n}$, $i = 1, 2$ are isomorophic. This means that the $\mathbf{BA}_n$-algebras $\mathbf{Q}_{U_i,n}(h_{<1>})$, $i = 1, 2$ are also isomorophic, hence, the $\mathbf{BA}_n$-algebras $[\mathbb{L}][Z_i]$ are isomorphic as well. Since $[\mathbb{L}]$ is a SM equivalence, the $\mathbf{BA}_n$-algebras $[Z_i]$ are also isomorphic. Let $f : [Z_1] \to [Z_2]$ be the isomorphism.

Let us paraphrase this statement. Let

$$\mathbf{FULL}([Z_1], [Z_2]) \subset [\mathbf{HOGER}_n^{\mathbf{proj}}]$$

be the full SM subcategory consisting of all tensor powers $[Z_1]^{\otimes K_1} \otimes [Z_2]^{\otimes K_2}$ (for all K_1, K_2).

Let

$$I_k : \mathbf{FULL}([Z_k]) \to \mathbf{FULL}([Z_1], [Z_2])$$

be the obvious embeddings.

Using the isomorphism f we can construct the SM equivalence

$$P : \mathbf{FULL}([Z_1], [Z_2]) \to \mathbf{FULL}([Z_1])$$

according to the following rules:

1) $PI_1 = \mathrm{Id}_{\mathbf{FULL}([Z_1])}$;
2) $P([Z_2]) = [Z_1]$;
3) $P(f) = \mathrm{Id}_{[Z_1]}$

The functors $\mathbb{H}_k$ produce isomorphisms

$$J_k : \mathbf{FULL}([Z_k]) \to \mathbf{FULL}(h_{<1>}) = \mathbf{BA}_n$$

The fact that f is an isomorphism of $\mathbf{BA}_n$-algebras simply means that

$$J_1 P I_k J_k^{-1} = \mathrm{Id}_{\mathbf{BA}_n} \tag{23}$$

for $k = 1, 2$.

Next, we observe that there exists an element $\overline{T} \in \mathrm{Aut}_{D(n\text{-op})}(\mathbf{ger}_{\leq n})$ such that $\overline{U_1} = \overline{U_2 T}$. Let $T : \mathbf{hoger}_{\leq N} \to \mathbf{hoger}_{\leq N}$ be a representative of $\overline{T}$. It follows that U_1 and $U_2 T$ produce the same element in $\mathrm{Iso}_{D(n\text{-op})}(\mathbf{ger}, \mathbf{braces})$ hence, without loss of generality one can assume $U_1 = U_2 T$. It follows that that $\overline{T} \neq \overline{\iota_x}$ for all $x \in \mathbf{k}^\times$. Let $\mathcal{T} : \mathbf{HOGER}_n^{\mathbf{proj}} \to \mathbf{HOGER}_n^{\mathbf{proj}}$ be the isomorphism induced by T.

It follows that one can choose $Z_2 = \mathcal{T}(Z_1)$. Indeed, in this case

$$\mathbb{H}_{U_2}(\mathcal{T}(Z_1)) = \underline{\mathfrak{B}}_n U_2 \mathcal{T}(Z_1) = \underline{\mathfrak{B}}_n U_1(Z_1) = \mathbb{H}_{U_1}(Z_1) \cong h_{<1>}$$

It also follows that the $\mathbf{BA}_n$-structure on $[\mathcal{T}(Z_1)]$ is induced by $\mathcal{T}$ from that on $[Z_1]$ so that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{FULL}([Z_1]) \arrow[r, "{[\mathcal{T}]}"] \arrow[dr, "J_1"] & \mathbf{FULL}([\mathcal{T}(Z_1)]) \arrow[d, "J_2"] \\
& \mathbf{BA}_n
\end{tikzcd}$$

Taking into account (23) we get the following commutative diagram

$$\begin{tikzcd}
\mathbf{BA}_n \arrow[r, "\mathrm{Id}"] & \mathbf{BA}_n \\
\mathbf{FULL}([Z_1]) \arrow[u, "J_1"] \arrow[r, shift left, "I_1"] \arrow[r, shift right, "{I_2[\mathcal{T}]}"'] & \mathbf{FULL}([Z_1],[Z_2]) \arrow[u, "P"']
\end{tikzcd}$$

We can expand this diagram as follows

(24)
$$\begin{tikzcd}
\mathbf{BA}_n \arrow[r, "\mathrm{Id}"] & \mathbf{BA}_n \\
\mathbf{FULL}([Z_1]) \arrow[u, "J_1"] \arrow[r, shift left, "I_1"] \arrow[r, shift right, "{I_2[\mathcal{T}]}"'] & \mathbf{FULL}([Z_1],[Z_2]) \arrow[u, "P"'] \\
\tau_{\leq 0}\mathbf{FULL}(Z_1) \arrow[u] \arrow[r, shift left, "I_1"] \arrow[r, shift right, "I_2\mathcal{T}"'] & \tau_{\leq 0}\mathbf{FULL}(Z_1, Z_2) \arrow[u]
\end{tikzcd}$$

where the lower vertical arrows are just natural arrows that map $\tau_{\leq 0}$ of a complex to its zeroth cohomology. The composition of the left vertical arrows is a quasi-isomorphism: indeed, this composition equals to another composition:

$$\tau_{\leq 0}\mathbf{FULL}(Z_1) \to \tau_{\leq 0}\mathbf{FULL}(\mathbb{H}(Z_1)) \to \mathbf{FULL}([\mathbb{H}(Z_1)])$$
$$\cong \mathbf{FULL}_{\mathbf{BA}_n^{\wedge}}(h_{<1>}) = \mathbf{BA}_n.$$

in which all arrows are SM weak equivalences .

It then easily follows that all the arrows in (24) are quasi-isomorphisms.

Next, we can produce the following diagram

$$\begin{tikzcd}
\mathbf{BA}_n^{\mathbf{proj}} \arrow[r, "\mathrm{Id}"] & \mathbf{BA}_n^{\mathbf{proj}} \\
(\tau_{\leq 0}\mathbf{FULL}(Z_1))^{\mathbf{proj}} \arrow[u, "\mathcal{P}"] \arrow[d, "Q"] \arrow[r, shift left, "I_1"] \arrow[r, shift right, "I_2\mathcal{T}"'] & (\tau_{\leq 0}\mathbf{FULL}(Z_1;\mathcal{T}(Z_1)))^{\mathbf{proj}} \arrow[u, "\mathcal{P}'"'] \arrow[d, "Q_1"] \\
\mathbf{HOGER}_n^{\mathbf{proj}} \arrow[r, shift left, "\mathrm{Id}"] \arrow[r, shift right, "\mathcal{T}"'] & \mathbf{HOGER}_n^{\mathbf{proj}}
\end{tikzcd}$$

Where the maps Q, Q_1 are obvious embeddings which send Z_1 and $\mathcal{T}(Z_1)$ to themselves. Let us show that Q is an equivalence. It is clear that Q, Q_1 induce a quasi-isomorphism on each complex of homomorphisms, we only need to check the essential surjectivity. As the image of Q_1 contains the image of Q, it suffices to check that Q is an equivalence. It suffices to check that

$$\mathbb{H}Q : \mathbf{FULL}(Z_1)^{\mathbf{proj}} \to \mathbf{FULL}(\mathbb{H}(Z_1))^{\mathbf{proj}} \to \mathbf{BA}_n^{\mathbf{proj}}$$

is an equivalence. This easily follows from the fact that $\mathbb{H}(Z_1) \cong h_{<1>}$. The argument is similar to that in (5.4.5)

Let $\mathcal{H} \in [\tau_{\leq 0}\mathbf{FULL}(Z)]^{\mathbf{proj}}$ be such that

$$Q\mathcal{H} \cong h_{[1]}.$$

Such an $\mathcal{H}$ exists due to the fact that Q is an equivalence.

The objects $\mathcal{P}'I_1\mathcal{H} \cong \mathcal{P}\mathcal{H}$ and $\mathcal{P}'I_2\mathcal{T}(\mathcal{H}) \cong \mathcal{P}\mathcal{H}$ are weakly equivalent, hence so are $I_1\mathcal{H}$ and $I_2\mathcal{T}(\mathcal{H})$. Let us choose this weak equivalence so that its image under $\mathcal{P}'$ be the canonical isomorphism $\mathcal{P}'I_1\mathcal{H} \cong \mathcal{P}'I_2\mathcal{T}(\mathcal{H}) \cong \mathcal{P}(\mathcal{H})$.

Consider the following collection of data as in Sec. 8.2.2:

— the pair of maps $I_1, I_2\mathcal{T}$,

— the object $\mathcal{H}$ and the weak equivalence $I_1\mathcal{H} \to I_2\mathcal{T}(\mathcal{H})$.

These data produce a pair of arrows in $D(n\text{-op})$. Actually these arrows coincide. Indeed, it suffices to check this for the data obtained by applying the functors $\mathcal{P}, \mathcal{P}'$: we will get

— an object $\mathcal{P}(\mathcal{H}) \in \mathbf{BA}_n^{\mathbf{free}}$;

— two coincident functors $\mathrm{Id} = \mathrm{Id} : \mathbf{BA}_n^{\mathbf{free}} \to \mathbf{BA}_n^{\mathbf{free}}$

— the identity quasi-isomorphism $\mathcal{P}(\mathcal{H}) \sim \mathrm{Id} \to \mathcal{P}(\mathcal{H})$.

These data do clearly produce a pair of coincident identity arrows in $D(n\text{-op})$.

Therefore, the following data produce coincident arrows in $D(n\text{-op})$:

$$\mathbf{HOGER}_n^{\mathbf{proj}} \underset{\mathcal{T}}{\overset{\mathrm{Id}}{\rightrightarrows}} \mathbf{HOGER}_n^{\mathbf{proj}};$$

the object $Q(\mathcal{H}) \in \mathbf{HOGER}_n^{\mathbf{proj}}$;

the weak equivalence $Q(\mathcal{H}) \to \mathcal{T}(Q(\mathcal{H}))$ induced by the equivalence $I_1\mathcal{H} \to I_2\mathcal{T}(\mathcal{H})$.

Next, we have a quasi-isomorphism $h_{<1>} \to Q(\mathcal{H})$, hence a chain of quasi-isomorphisms

$$h_{<1>} \to Q(\mathcal{H}) \to \mathcal{T}Q(\mathcal{H}) \to \mathcal{T}h_{<1>} = h_{\mathcal{T}(<1>)} = h_{<1>}.$$

The composition of these arrows produces an element

$$x \in H^0 \hom_{\mathbf{HOGER}_n}(<1>, <1>) = \mathbf{k}.$$

This element must be invertible (i.e $x \neq 0$). This implies that the following data produce a pair of coincident arrows in $D(n\text{-op})$:

$$\mathbf{hoger}_{\leq n} \underset{\mathcal{T}}{\overset{\mathrm{Id}}{\rightrightarrows}} \mathbf{hoger}_{\leq n}$$

and a quasi-isomorphism

$$<1> \to \mathcal{T}(<1>) = <1>$$

which is the multiplication by x.

This implies that the arrows $T : \mathbf{hoger}_{\leq n} \to \mathbf{hoger}_{\leq n}$ and

$$\iota_x : \mathbf{hoger}_{\leq n} = \mathbf{full}_{\mathbf{HOGER}}(<1>)_{\leq n} \to \mathbf{full}_{\mathbf{HOGER}}(<1>)_{\leq n} \to \mathbf{hoger}_{\leq n}$$

coincide in $D(n\text{-op})$. This is a contradiction with our original assumption.

8.3. **Dependence on the choice of an associator.** As explained in Appendix 3, given an associator Φ, we get an isomorphism $A_\Phi \in \mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{braces}, \mathbf{ger})$. Using the inverse isomorphism $(A_\Phi)^{-1}$ we then get a quantization functor $\mathbf{Q}_\Phi := \mathbf{Q}_{(A_\Phi)^{-1}}$.

Question From [6] we can get another construction of a quantization functor, also using an associator. Does this Etingof-Kazhdan construction produce the quantization functor isomorphic to $\mathbf{Q}_\Phi$? □

Theorem 11.4 from Appendix 3 implies that given different associators Φ_1 and Φ_2, we get $A_{\Phi_1} \not\sim A_{\Phi_2}$. Theorem 8.1 then readily implies

COROLLARY 8.2. *Given different associators $\Phi_1 \neq \Phi_2$, the corresponding quantization functors $\mathbf{Q}_{\Phi_1}$, $\mathbf{Q}_{\Phi_2}$ are non-isomorphic.*

9. Appendix 1: Categories

9.0.1. *Finite* $\mathbf{k}$*-linear categories.* A k-linear category is a category enriched over the category of $\mathbf{k}$-vector spaces.

Call such a category *finite* if the set of isomorphism classes of its objects is finite and the vector spaces of homomorphisms are finite dimensional for every pair of objects.

9.0.2. Given a finite category $\mathcal{C}$, let $\mathcal{C}^\wedge$ be the abelian category of $\mathbf{k}$-linear functors from $\mathcal{C}^{\mathrm{op}}$ to the category of finite dimensional $\mathbf{k}$-vector spaces.

We have Ioneda's embedding $h : \mathcal{C} \to \mathcal{C}^\wedge$ defined by the formula $X \mapsto h_X$, where

$$h_X(Y) := \mathrm{hom}_{\mathcal{C}}(Y, X).$$

9.1. **DG-categories.** Call a dg-category $\mathcal{C}$ *finite* if the set of isomorphism classes of its objects is finite and each complex $\mathrm{hom}_{\mathcal{C}}(X, Y)$ is finite (i.e. is bounded in both directions and each of its spaces is finitely dimensional).

Every finite k-linear category can be naturally viewed as a finite dg-category in which all hom-complexes are concentrated in degree 0.

Let $\mathcal{C}^{\wedge\mathrm{dg}}$ be the category of functors from$\mathcal{C}^{\mathrm{op}}$ to the category of finite complexes. Given an $X \in \mathcal{C}$ we have $h_X \in \mathcal{C}^{\wedge\mathrm{dg}}$:

$$h_X(Y) := \mathrm{hom}_{\mathcal{C}}(Y, X),$$

whence Ioneda's embedding $\mathcal{C} \to \mathcal{C}^{\wedge\mathrm{dg}}$.

9.1.1. *Complexes.* Given any dg-category $\mathcal{D}$ with a zero object, *a complex in* $\mathcal{D}$ is, by definition, a collection of objects $X_n, n \in \mathbb{Z}$ and elements $d_n :\in Z^1 \mathrm{hom}(X_n, X_{n+1})$ such that $d_{n+1} d_n = 0$. A complex is called *bounded* if almost all of X_n are zeros.

Given a complex $X^\bullet$ in $\mathcal{D}$ and an object $U \in \mathcal{D}$, we have a bi-complex of $\mathbf{k}$-vector spaces

$$\cdots \mathrm{hom}(U, X^n) \xrightarrow{d_n} \mathrm{hom}(U, X^{n+1}) \overset{d_{n+1}}{\mathrm{hom}} (U, X^{n+2}) \to \cdots$$

Denote by $h_{X^\bullet}(U)$ the total complex of this bicomplex. This way we get a functor $h_{X^\bullet} : \mathcal{D}^{\mathrm{op}} \to$ **complexes**. If this functor is representable we denote the representing object by $|X^\bullet|$ and call it the *realization* of $X^\bullet$. Instead of saying that $h_{X^\bullet}$ is representable, we will say that *the complex* $X^\bullet$ *has a realization.*

One sees that every bounded complex in $\mathcal{C}^{\wedge \mathrm{dg}}$ has a realization.

9.1.2. We say that an $F \in \mathcal{C}^{\wedge \mathrm{dg}}$ is *finitely generated and free* if it is isomorphic to a finite direct sum of objects of the form $h_{X_i}[n_i]$, where $X_i \in \mathcal{C}$ and $[n_i]$ denotes a degree shift.

Let $\mathcal{C}^{\mathbf{free}} \subset \mathcal{C}^{\wedge \mathrm{dg}}$ be the full sub-category consisting of all objects isomorphic to realizations of finite complexes of free finitely generated objects. Let $\mathcal{C}^{\mathbf{proj}} \subset \mathcal{C}^{\wedge \mathrm{dg}}$ be the full subcategory consisting of all objects which are retractions of objects from $\mathcal{C}^{\mathbf{free}}$. Alternatively, $\mathcal{C}^{\mathbf{proj}}$ can be defined as the Karoubian closure of $\mathcal{C}^{\mathbf{free}}$.

9.1.3. *Induced functors.* Let $\mathcal{C}, \mathcal{D}$ be a pair of finite dg categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor.

The pre-composition with F gives rise to a functor

$$F^{-1} : \mathcal{D}^{\wedge \mathrm{dg}} \to \mathcal{C}^{\wedge \mathrm{dg}}.$$

This functor has a left adjoint

$$F_! : \mathcal{C}^{\wedge \mathrm{dg}} \to \mathcal{D}^{\wedge \mathrm{dg}}$$

which is defined as follows. Define a functor

$$H : \mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}} \to \mathbf{complexes};$$

$$H(X, Y) = \hom_{\mathcal{D}}(Y, F(X)).$$

Given $R \in \mathcal{C}^{\wedge \mathrm{dg}}$, that is $R : \mathcal{C}^{\mathrm{op}} \to \mathbf{complexes}$, we set

$$F_! R := F \otimes_{\mathcal{C}} R.$$

9.1.4.

PROPOSITION 9.1. *The functor $F_!$ takes $\mathcal{C}^{\mathbf{free}}$ to $\mathcal{D}^{\mathbf{free}}$ and $\mathcal{C}^{\mathbf{proj}}$ to $\mathcal{D}^{\mathbf{proj}}$.*

Proof. Indeed the functor $F_!$ takes the free functor h_X into $h_{F(X)}$, hence it takes any finitely generated free functor to a finitely generated free functor and any finite complex of finitely generated free modules to a finite complex of finitely generated free modules. Next $F_!$ takes a retract $P \to G \to P$ of finite complex G of finitely generated free modules to a retract of $F_! G$, hence $F_! P \in \mathcal{D}^{\mathbf{proj}}$. $\square$

9.1.5. Let us generalize the above construction as follows. Let $\mathcal{C}, \mathcal{D}$ be finite dg-categories; let F be a functor $\mathcal{C} \to \mathcal{D}^{\wedge \mathrm{dg}}$. Define a functor $F_! : \mathcal{C}^{\wedge \mathrm{dg}} \to \mathcal{D}^{\wedge \mathrm{dg}}$ as follows. Let $K_F : \mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}} \to \mathbf{complexes}$; $K_F(X, Y) = F(X)(Y)$. For $U \in \mathcal{C}^{\wedge \mathrm{dg}}$, set $F_! U := F \otimes_{\mathcal{C}} K_F$.

9.1.6. One can prove that if $F : \mathcal{C} \to \mathcal{D}^{\mathbf{free}}$ (resp. $F : \mathcal{C} \to \mathcal{D}^{\mathbf{free}}$), then $F_!(\mathcal{C}^{\mathbf{free}}) \subset \mathcal{D}^{\mathbf{free}}$ (resp. $F_! \mathcal{C}^{\mathbf{proj}} \subset \mathcal{D}^{\mathbf{proj}}$).

9.1.7. All these definitions make sense in the world of finite **k**-linear categories. Let $\mathcal{C}$, $\mathcal{D}$ be k-linear categories and $F:\mathcal{C}\to\mathcal{D}$ be a k-linear functor. The functors

$$F^{-1}:\mathcal{D}^\wedge\to\mathcal{C}^\wedge$$

and

$$F_!:\mathcal{C}^\wedge\to\mathcal{D}^\wedge$$

can be defined in the same way as in the setting of dg-categories.

We define *a finitely generated free object* in $\mathcal{C}^\wedge$ as any object isomorphic to a finite direct sum of objects of the form h_{X_i}. We define *a projective object* as a retraction of a free object.

We then form full subcategories $\mathcal{C}^f\subset\mathcal{C}^{\mathbf{pro}}\subset\mathcal{C}^\wedge$ consisting of free and projective objects respectively. In the same way as above, we see that the subcategories of free and projective objects are preserved by the functor $F_!$.

9.2. **Symmetric monoidal categories.** We will see that the above defined constructions work in the setting of symmetric monoidal categories.

9.2.1. We will use the notion of exterior product of dg-categories. Given a finite family of dg-categories $\mathcal{C}_i, i\in I$, we define their exterior product

$$\boxtimes_{i\in I}\mathcal{C}_i$$

by setting its objects to be arbitrary families $\boxtimes_{i\in I}X_i$, where $X_i\in\mathcal{C}_i$. Morphisms are defined as follows:

$$\hom(\boxtimes_{i\in I}X_i;\boxtimes_{i\in I}Y_i):=\bigotimes_{i\in I}\hom_{\mathcal{C}_i}(X_i;Y_i),$$

the composition of morphisms is defined in the obvious way.

We have natural functors

$$\boxtimes_{i\in I}\mathcal{C}_i^{\wedge\mathrm{dg}}\to(\boxtimes_{i\in I}\mathcal{C}_i)^{\wedge\mathrm{dg}};$$

$$\boxtimes_{i\in I}\mathcal{C}_i^{\wedge\mathrm{dg}}\to(\boxtimes_{i\in I}\mathcal{C}_i)^{\wedge\mathrm{dg}};$$

$$\boxtimes_{i\in I}\mathcal{C}_i^{\mathbf{free}}\to(\boxtimes_{i\in I}\mathcal{C}_i)^{\mathbf{free}};$$

$$\boxtimes_{i\in I}\mathcal{C}_i^{\mathbf{proj}}\to(\boxtimes_{i\in I}\mathcal{C}_i)^{\mathbf{proj}};$$

All these functors send a family of functors $F_i:\mathcal{C}_i^{\mathrm{op}}\to$ **complexes** to the following functor

$$F:\boxtimes_{i\in I}\mathcal{C}_i\to\textbf{complexes}:$$

$$F(\boxtimes_{i\in I}X_i):=\bigotimes_{i\in I}F_i(X_i).$$

9.2.2. *Induced SM structure on* $\mathcal{C}^{\mathbf{free}}, \mathcal{C}^{\mathbf{proj}}, \mathcal{C}^{\wedge dg}$*.* Let $\mathcal{C}$ be a symmetric monoidal category. The tensor products give rise to functors

$$T^I : \mathcal{C}^{\boxtimes I} \to \mathcal{C} :$$
$$\boxtimes_{i\in I} X_i \mapsto \bigotimes_{i\in I} X_i.$$

Whence induced functors

$$(\mathcal{C}^{\wedge \mathrm{dg}})^{\boxtimes I} \to (\mathcal{C}^{\boxtimes I})^{\wedge \mathrm{dg}} \xrightarrow{T^I_!} \mathcal{C}^{\wedge \mathrm{dg}}$$

$$(\mathcal{C}^{\mathbf{free}})^{\boxtimes I} \to (\mathcal{C}^{\boxtimes I})^{\mathbf{free}} \xrightarrow{T^I_!} \mathcal{C}^{\mathbf{free}}$$

$$(\mathcal{C}^{\mathbf{proj}})^{\boxtimes I} \to (\mathcal{C}^{\boxtimes I})^{\mathbf{proj}} \xrightarrow{T^I_!} \mathcal{C}^{\mathbf{proj}}$$

It is straightforward to check that these maps define an SM-structure on $\mathcal{C}^{\wedge \mathrm{dg}}, \mathcal{C}^{\mathbf{free}}, \mathcal{C}^{\mathbf{proj}}$.

The tensor product on $\mathcal{C}^{\mathbf{free}}$ admits a more explicit description. Recall that any object of $\mathcal{C}^{\mathbf{free}}$ is a realization of a bounded complex of finitely generated and free objects. Let us start with describing the tensor product of finitely generated and free objects. It is easy to see that we have

$$(\bigoplus_{a\in A} h_{X_a}[n_a]) \otimes (\bigoplus_{b\in B} h_{X_b}[n_b]) \cong \bigoplus_{(a,b)\in A\times B} X_a \otimes Y_b[n_a+n_b].$$

Next, given finite complexes of finitely generated and free objects, $X^\bullet$ and $Y^\bullet$, we see that

$$|X^\bullet| \otimes |Y^\bullet| = |Z^\bullet|,$$

where

$$Z^n = \bigoplus_m X^m \otimes Y^{n-m}$$

and the differential on $Z^\bullet$ is naturally induced by those on $X^\bullet, Y^\bullet$.

The tensor product on $\mathcal{C}^{\mathbf{proj}}$ is uniquely defined by that on $\mathcal{C}^{\mathbf{free}}$ and by the condition that the tensor product of kernels of projectors $P_1 : X \to X$ and $P_2 : Y \to Y$ is the kernel of the projector

$$P_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes P_2 : X \otimes Y \to X \otimes Y.$$

9.2.3. Let $\mathcal{C}, \mathcal{D}$ be SMC and let $F : \mathcal{C} \to \mathcal{D}$ be a SM-functor. We will define a SM-structure on the induced functors

$$F_! : \mathcal{C}^{\wedge \mathrm{dg}} \to \mathcal{D}^{\wedge \mathrm{dg}};$$
$$F_! : \mathcal{C}^{\mathbf{free}} \to \mathcal{D}^{\mathbf{free}};$$
$$F_! : \mathcal{C}^{\mathbf{proj}} \to \mathcal{D}^{\mathbf{proj}}.$$

Let us first of all reformulate the SM-structure on F in a way convenient for us. Let $T^2_{\mathcal{C}} : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$; $T^2_{\mathcal{D}} : \mathcal{D} \boxtimes \mathcal{D} \to \mathcal{D}$ be the tensor products. Part of the tensor structure on F is an isomorphism

$$T^2_{\mathcal{D}}(F \boxtimes F) \cong F T^2_{\mathcal{C}}. \tag{25}$$

Let us now proceed to defining an SM-structure on $F_!$.

Given $X, Y \in \mathcal{C}^{\wedge \mathrm{dg}}$, we are supposed to define isomorphisms

$$F_!(X) \otimes F_!(Y) \to F_!(X \otimes Y). \tag{26}$$

The LHS is isomorphic to

$$[T^2_{\mathcal{D}}(F \boxtimes F)]_!(X \boxtimes Y);$$

the RHS is isomorphic to

$$[FT_2^{\mathcal{C}}]_!(X \boxtimes Y)$$

The desired isomorphism (26) then comes from the isomorphism (25). It is straightforward to check that these isomorphisms satisfy all the properties of an SM-structure on a functor.

By the same token, one gets a SMS on the functors $F_! : \mathcal{C}^{\mathbf{free}} \to \mathcal{D}^{\mathbf{free}}$; $F_! : \mathcal{C}^{\mathbf{proj}} \to \mathcal{D}^{\mathbf{proj}}$.

9.2.4. In the same way, given SM functors $F : \mathcal{C} \to \mathcal{D}^{\wedge \mathrm{dg}}$ (resp. $F : \mathcal{C} \to \mathcal{D}^{\mathbf{free}}$; resp. $F : \mathcal{C} \to \mathcal{D}^{\mathbf{proj}}$), one gets a SM-structure on the induced functors

$$F_! : \mathcal{C}^{\mathbf{free}} \to \mathcal{D}^{\mathbf{free}}, \text{ resp. } \quad F_! : \mathcal{C}^{\mathbf{proj}} \to \mathcal{D}^{\mathbf{proj}}.$$

10. Appendix 2: Derived automorphisms of **ger**

According to [10] there exists a closed model structure on the category of dg-operads. Therefore, one can construct the derived category of dg-operads following [17]. Let us denote this derived category by D(op). In this section we will study the automorphism group $\mathrm{Aut}_{\mathrm{D(op)}}(\mathbf{ger})$. We will see that it is isomorphic to an extension of $\mathbf{k}^\times \times \mathbf{k}^\times$ by a pro-unipotent group, the latter group will be identified with the exponential group of a certain pro-nilpotent Lie algebra. We will conclude by showing that, roughly speaking, any homotopy non-trivial automorphism of **hoger** induces a homotopy non-trivial automorphism of the N-truncation $\mathbf{hoger}_{\leq N}$ for N large enough; see Sec. 10.1 for the definitions and a precise statement of the result.

10.0.5. Let $\mathbf{o}$ be a dg-operad. Quillen offers the following recipe for computing the set $\hom_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{o})$ in the derived category of dg-operads.

Let $\mathbf{hoger} \to \mathbf{ger}$ be the standard resolution. Let $k[t, dt]$ be the polynomial commutative algebra on two generators t of degree 0 and dt of degree 1 with the differential sending t to dt. Let $r_0, r_1 : \mathbf{k}[t, dt] \to \mathbf{k}$ be the maps defined by $r_0(dt) = r_1(dt) = 0$; $r_0(t) = 0$; $r_1(t) = 1$.

Let $\mathbf{o}[t, dt]$ be the dg-operad obtained from $\mathbf{o}$ by the extension of scalars $\mathbf{o}[t, dt](n) := \mathbf{o}(n) \otimes_{\mathbf{k}} \mathbf{k}[t, dt]$. The maps r_0, r_1 induce maps of operads

$$r_0, r_1 : \mathbf{o}[t, dt] \to \mathbf{o}$$

in the obvious way.

The set $\hom_{\mathrm{D(op)}}(\mathbf{ger}, \mathbf{o})$ is identified with the quotient of the set $\hom(\mathbf{hoger}, \mathbf{o})$ (in the usual category of dg-operads) by the following equivalence relation: two maps $f, g : \mathbf{hoger} \to \mathbf{o}$ are equivalent iff there exists a map $h : \mathbf{hoger} \to \mathbf{o}[t, dt]$ (a homotopy) such that $r_0 h = f; r_1 h = g$.

The set $\hom_{\mathrm{D(op)}}(\mathbf{hoger}, \mathbf{hoger})$ is then a monoid and we would like to describe its group of invertible elements $\mathfrak{G} := \mathrm{Aut}_{\mathrm{D(op)}}(\mathbf{hoger})$.

10.0.6. Given $\phi \in \mathfrak{G}$ it is represented by a map $f : \mathbf{hoger} \to \mathbf{hoger}$, and we can consider the induced map $\overline{f} : \mathbf{ger} = H^\bullet(\mathbf{hoger}) \to H^\bullet(\mathbf{hoger}) = \mathbf{ger}$. It follows that

1) $\overline{f}$ only depends on ϕ, not on the choice of a representative f, so that we can denote $\overline{\phi} := \overline{f}$;

2) the map $\phi \mapsto \overline{\phi}$ is a group homomorphism $\mathfrak{G} \to \mathrm{Aut}(\mathbf{ger})$.

On the other hand, the group $\mathrm{Aut}(\mathbf{ger})$ is isomorphic to $\mathbf{k}^\times \times \mathbf{k}^\times$. Given $(x, y) \in \mathbf{k}^\times \times \mathbf{k}^\times$, the corresponding automorphism of **ger** is given by dilating the commutative product by x and the Lie bracket by y. It is clear that this way we get all automorphisms of **ger**, whence an isomorphism $\mathbf{k}^\times \to \mathrm{Aut}(\mathbf{ger})$.

Thus, we have a map

$$\mathfrak{G} \to \mathrm{Aut}(\mathbf{ger}) = \mathbf{k}^\times \times \mathbf{k}^\times$$

Next, we have a natural map $\mathrm{Aut}(\mathbf{ger}) \to \mathfrak{G}$. Indeed, **hoger** is obtained by applying the co-bar construction to the shifted operad $\mathbf{ger}\{1\}$, which is the Koszul dual to **ger**. Therefore, the action of $\mathbf{k}^\times \times \mathbf{k}^\times$ on **ger** gives rise to an action of the same group on $\mathbf{ger}\{1\}$ and on **hoger**. One can check that the through map

$$\mathbf{k}^\times \times \mathbf{k}^\times \to \mathfrak{G} \to \mathbf{k}^\times \times \mathbf{k}^\times$$

is the identity, therefore, we have an extension

$$\mathfrak{G} = (\mathbf{k}^\times \times \mathbf{k}^\times) \ltimes \mathfrak{G}_0,$$

where $\mathfrak{G}_0 \subset \mathfrak{G}$ consists of all equivalence classes of automorphisms of **hoger** which induce the identity on $H^\bullet(\mathbf{hoger}) = \mathbf{ger}$.

Below, we will find a pro-nilpotent Lie algebra whose exponential is narurally isomorphic to $\mathfrak{G}_0$.

10.0.7. We have a natural grading on **hoger**: Let $\mathrm{Gr}^k\mathbf{hoger}(n) \subset \mathbf{hoger}(n)$ be the span of all elements which can be expressed as a $(k-1)$-fold composition of the generators.

It is clear that Gr is compatible with the operadic composition on **hoger** and that the differential increases the grading by 1.

Introduce a decreasing filtration on **hoger**:

$$F^k\mathbf{hoger}(n) := \bigoplus_{l \geq k} \mathrm{Gr}^l\mathbf{hoger}(n).$$

As $\mathrm{Gr}^{\geq n}\mathbf{hoger}(n) = 0$, the direct sum is finite.

Any map of operads $f : \mathbf{hoger} \to \mathbf{hoger}$ preserves this filtration. We say that $|f| \geq l$ if $(f - \mathrm{Id})F^k\mathbf{hoger}(n) \subset F^{k+l}\mathbf{hoger}(n)$ It follows that $|f| \geq 0$ for all maps of operads f.

LEMMA 10.1. *Suppose that $f : \mathbf{hoger} \to \mathbf{hoger}$ represents an element from $\mathfrak{G}_0$ (or, which is the same, induces the identity map $H^\bullet(\mathbf{hoger}) \to H^\bullet(\mathbf{hoger})$. Then $|f| \geq 1$.*

Proof. Suppose it is not true that $|f| \geq 1$. Let M be the minimal number such that the map

$$f - \mathrm{Id} : \mathbf{hoger}(M) \to \mathbf{hoger}(M)$$

does not increase filtration by 1.

Let us show that $M > 2$. Indeed the complex $\mathbf{hoger}(2)$ has zero differential, therefore, since f induces the identity on the cohomology, the map $f : \mathbf{hoger}(2) \to \mathbf{hoger}(2)$ must be the identity map.

Thus, $M > 2$. Let $G(k) \subset \mathbf{hoger}(k)$ be the spaces of generators.

Let $\mathbf{o}_k \subset \mathbf{hoger}$ be the sub-operad generated by $\mathbf{hoger}(l)$, $l \leq k$ (or, equivalently, by all $G(l)$, $l \leq k$.

It then follows that $\mathbf{o}_k$ is freely generated over $\mathbf{o}_{k-1}$ by $G(k)$ and that $d(G(k)) \subset \mathbf{o}_{k-1}(k)$. Let $D_k : G(k) \to \mathbf{o}_{k-1}(k)$ be the corresponding map.

Consider the map

$$\phi_M : G(M) \stackrel{i_M}{\to} \mathbf{hoger}(M) \stackrel{f}{\to} \mathbf{hoger}(M),$$

where i_M is the inclusion of the space of generators $G(M)$ into $\mathbf{hoger}(M)$.

The compatibility with the differential implies that

$$d\phi_M \pm \phi D = 0;$$

or

$$d(\phi_M - i_M) \pm (\phi - \mathrm{Id})D = 0.$$

The map D increases the filtration by 1, as well as $\phi - \mathrm{Id}$, because $\phi - \mathrm{Id} : \mathbf{hoger}(m) \to \mathbf{hoger}(m)$ increases filtration by 1 for all $m < M$ by assumption. Thus, $d(\phi_M - i_M) \in F^2\mathbf{hoger}(M)$. This implies that the following composition is zero:

$$G(M) \stackrel{\phi_M - i_M}{\longrightarrow} \mathbf{hoger}(M) \to \mathrm{Gr}^0\mathbf{hoger}(M) \stackrel{d}{\to} \mathrm{Gr}^1\mathbf{hoger}(M)$$

On the other hand, it is easy to see that the rightmost arrow in this sequence is injective: assuming the contrary, every non-zero element $x \in \mathrm{Gr}^0\mathbf{hoger}(M) = G(M)$ such that $dx = 0$ produces a non-trivial cycle in $\mathbf{hoger}(M)$, on the other hand the through map $G(M) \to \mathbf{hoger}(M) \to \mathbf{ger}(M)$ is zero for all $M > 2$. This contradicts to quasi-isomorphicity of the canonical map $\mathbf{hoger}(M) \to \mathbf{ger}(M)$.

Therefore, $(\phi_M - i_M)(G_M) \subset F^1\mathbf{hoger}(M)$. Since $\mathbf{hoger}(M) = \mathbf{o}_{M-1}(M) \oplus G(M)$, this readily implies that the map $f - \mathrm{Id} : \mathbf{hoger}(M) \to \mathbf{hoger}(M)$ increases the filtration by 1, which is a contradiction. $\square$

10.0.8. Extend the grading on **hoger** to that on $\mathbf{hoger}[t, dt]$ by setting the grading of t to be 0 and the grading of dt to be 1. We then see that the operadic composition preserves this grading and that the grading of the differential d is 1.

Introduce a filtration

$$F^n\mathbf{hoger}[t, dt](m) := \bigoplus_{N \geq n} \mathrm{Gr}^N\mathbf{hoger}[t, dt](m)$$

Note that the direct sum here is actually finite.

Every map of operads $f : \mathbf{hoger} \to \mathbf{hoger}[t, dt]$ preserves this filtration. We write $|f| \geq k$ if $(f - \mathrm{Id}) : \mathbf{hoger}(m) \to \mathbf{hoger}(m)$ increases the filtration by at least k.

LEMMA 10.2. *Let $f : \mathbf{hoger} \to \mathbf{hoger}[t, dt]$ be such that $|r_0 f| \geq 1$. Then $|f| \geq 1$.*

Proof. Let us choose an m and decompose $f : \mathbf{hoger}(m) \to \mathbf{hoger}[t, dt](m)$ as

$$\sum_k (u_k + v_k dt)t^k,$$

where $u_k, v_k : \mathbf{hoger}(m) \to \mathbf{hoger}(m)$. It follows that both u_k, v_k preserve the filtration on $\mathbf{hoger}(m)$.

The equality $df = 0$ implies that $ku_k + dv_{k-1} = 0$. Therefore, for all $k \geq 1$, u_k increases the filtration by 1, because so does d and v_{k-1} preserves the filtration.

Next, $u_0 = r_0 f$, therefore, $u_0 - \mathrm{Id}$ increases the filtration by 1.

Lastly, $v_k t^k dt$ increases the filtration by 1 as so does dt. This means that $|f| \geq 1$. $\square$

10.0.9. Let K be the group of all maps $f : \mathbf{hoger} \to \mathbf{hoger}[t, dt]$ for which $|f| \geq 1$. Let G be the group of all maps $f : \mathbf{hoger} \to \mathbf{hoger}$ with $|f| \geq 1$.

The maps $r_0, r_1 : \mathbf{hoger}[t, dt] \to \mathbf{hoger}$ induce group homomorphisms $r_0, r_1 : K \to G$. Next, we have a group homomorphism

$$p : G \to \mathfrak{G}_0$$

which is surjective in virtue of Lemma 10.1.

Lemma 10.2 implies that $p(x_0) = p(x_1)$ iff there exists a $y \in K$ such that $x_0 = r_0(y)$; $x_1 = r_1(y)$.

10.0.10. Let $\mathfrak{k}$ be the DGLA of $\mathbf{k}[t, dt]$-linear derivations of $\mathbf{hoger}[t, dt]$. The filtration on $\mathbf{hoger}[t, dt]$ induces that on $\mathfrak{k}$ and it follows that $\mathfrak{k} = F^0\mathfrak{k}$. It follows that $\mathfrak{k}$ is complete with respect to this filtration and that $F^1\mathfrak{k} \subset \mathfrak{k}$ is a nilpotent ideal. Let $Z^0F^1\mathfrak{k} \subset F^1\mathfrak{k}$ be the Lie sub-algebra of zero-cycles. This is a pro-nilpotent Lie algebra so that we can form a group

$$\mathrm{Exp}(Z^0F^1\mathfrak{k})$$

This group acts on **hoger** by automorphisms which are congruent to 1 modulo F^1, whence a map

$$\mathrm{Exp}(Z^0F^1\mathfrak{k}) \to K. \tag{27}$$

We have the inverse logarithm map showing that the map (27) is a group isomorphism.

Analogously, let $\mathfrak{g}$ be the DGLA of derivations of the operad **hoger**. It follows that $\mathfrak{k} = \mathfrak{g}[t, dt] := \mathfrak{g} \otimes_{\mathbf{k}} \mathbf{k}[t, dt]$. In a similar way, we get an isomorphism

$$\mathrm{Exp}(Z^0F^1\mathfrak{g}) \to G.$$

The maps $r_0, r_1 : K \to G$ are induced by the DGLA maps

$$r_0, r_1 : \mathfrak{k} = \mathfrak{g}[t, dt] \to \mathfrak{g}$$

induced by the maps $r_0, r_1 : \mathbf{k}[t, dt] \to \mathbf{k}[t]$.

We see that two elements

$$e^{X_0}, e^{X_1} \in \mathrm{Exp}(Z^0F^1\mathfrak{g}) = G$$

go to the same element in $\mathfrak{G}_0$ iff there is a $Y \in Z^0F^1\mathfrak{k}$ such that $r_0Y = X_0$, $r_1Y = X_1$. It is easy to see that such a Y exists iff $X_0 - X_1 = dZ$ for some $Z \in \mathfrak{g}$, i.e. if $X_0 - X_1$ is a boundary. Let $\mathfrak{b} \subset Z^0F^1\mathfrak{g}$ be the ideal formed by all elements dZ, $Z \in \mathfrak{g}^{-1}$ (note that since d increases filtration by 1, $dZ \in F^1\mathfrak{g}$). We then get that $\mathrm{Exp}(\mathfrak{b}) \subset \mathrm{Exp}(Z^0F^1\mathfrak{g})$ is the kernel of the projection $G \to \mathfrak{G}_0$, whence an isomorphism

$$\mathrm{Exp}(Z^0F^1\mathfrak{g}/\mathfrak{b}) \to \mathfrak{G}_0.$$

Lastly, we have an identification

$$Z^0F^1\mathfrak{g}/\mathfrak{b} = F^1H^0(\mathfrak{g}).$$

Thus we have proven:

THEOREM 10.3. *We have a natural isomorphism*

$$Exp(F^1H^0(\mathfrak{g})) \to \mathfrak{G}_0.$$

10.1. **Truncations of the operad hoger.**

10.1.1. *Truncated operads.* Define *an n-truncated operad* $\mathbf{o}$ *in a SMC* $\mathcal{C}$ as

— a functor from the groupoid of finite sets with at most $n - 1$ elements to $\mathcal{C}$;

— given a map of finite sets $f : S \to T$ with $|S|, |T| < n$, there should be given a composition map

$$\mathbf{o}(T) \otimes \bigotimes_{t \in T} \mathbf{o}(f^{-1}t) \to \mathbf{o}(S)$$

— the composition maps should be associative in the same way as in the setting of usual operads

Given a usual operad $\mathcal{A}$; its spaces $\mathcal{A}(S)$, $|S| \leq n$ form an n-truncated operad. Denote this truncated operad by $\mathcal{A}_{\leq n}$ The category of n-truncated dg-operads has a closed model structure. Hence, we can consider the derived category, to be denoted by $D(n\text{-op})$. We have an obvious functor $F_n : \mathrm{D}(\mathrm{op}) \to D(n\text{-op})$. The object $\mathbf{ger} \in \mathrm{D}(\mathrm{op})$ is of our particular interest, and we have natural homomorphisms

$$O_n : \mathrm{Aut}_{\mathrm{D}(\mathrm{op})}(\mathbf{ger}) \to \mathrm{Aut}_{D(n\text{-op})}\, \mathbf{ger}_{\leq n}$$

Let $\mathfrak{G}_0^{\leq n} \subset \mathrm{Aut}_{D(n\text{-op})}$ be the subgroup consisting of all elements inducing the identity of $H^\bullet(\mathbf{ger}_{\leq n})$. We then have homomorphisms

$$O_n : \mathfrak{G}_0 \to \mathfrak{G}_0^{\leq n}.$$

We want to prove:

PROPOSITION 10.4. *For every $X \in \mathfrak{G}_0$, $X \neq I$, there exists an n such that $O_n(X) \neq I$.*

In order to prove this theorem we need to rewrite $\mathfrak{G}_0^{\leq n}$ in terms of Lie algebras, in the same way as we did it with $\mathfrak{G}_0$.

Let $\mathfrak{g}^{\geq n}$ be the DGLA of derivations of the operad $\mathbf{hoger}_{\leq n}$. We have a grading and a filtration on $\mathbf{hoger}_{\leq n}$ in the same way as on $\mathbf{hoger}$, so that we have an induced grading and filtration on $\mathfrak{g}^{\geq n}$. We then have an identification $\mathfrak{G}_0^{\leq n} = \mathrm{Exp}(F^1 H^0(\mathfrak{g}^{\leq n}))$ in the same way as for $\mathfrak{G}_0$. The proof is similar and is omitted.

The map $O_n : \mathfrak{G}_0 \to \mathfrak{G}_0^{\leq n}$ is induced by the natural map $o_n : \mathfrak{g} \to \mathfrak{g}^{\leq n}$. So the proposition reduces to:

LEMMA 10.5. *For every $X \in F^1 H^0(\mathfrak{g})$, $X \neq 0$, there exists an n such that $o_n(X) \neq 0$.*

We have a grading on $\mathfrak{g}$; $\mathfrak{g} = \prod_n \mathrm{Gr}^n \mathfrak{g}$. The differential increases the grading by 1. Therefore, the complex $\mathfrak{g}$ splits as

$$\mathfrak{g} = \prod_n \mathfrak{g}^{(n)},$$

where $\mathfrak{g}^{(n)}$ is the following complex:

$$\cdots \to (\mathrm{Gr}^n \mathfrak{g})^0 \to (\mathrm{Gr}^{n+1} \mathfrak{g})^1 \to \cdots.$$

Same splitting takes place for $\mathfrak{g}_{\leq N}$ so that we have

$$\mathfrak{g}_{\leq N} = \prod_n \mathfrak{g}_{\leq N}^{(n)}$$

and the map $\mathfrak{g} \to \mathfrak{g}_{\leq N}$ is induced by maps

$$\mathfrak{g}^{(n)} \to \mathfrak{g}_{\leq N}^{(n)} \tag{28}$$

One sees that these maps are surjective. Let $\Phi^N(\mathfrak{g}^{(n)})$ be the kernel of (28) We see that Φ is a filtration on the complex $\mathfrak{g}^{(n)}$ and that $\Phi^N \mathfrak{g}^{(n)} \subset \mathfrak{g}^n$ is the subcomplex consisting of all derivations vanishing on $\mathbf{o}_N \subset \mathbf{hoger}$. From this one sees that Φ is a complete filtration. We have associated graded complexes $\mathrm{Gr}_\Phi^m \mathfrak{g}^{(n)}$. We have:

$$(\mathrm{Gr}_\Phi^m \mathfrak{g}^{(n)})^k \cong \hom^k(G(m); \mathrm{Gr}^{k+n} \mathbf{hoger}(m)).$$

The differential is induced by the differential on $\mathbf{hoger}$:

$$d : \mathrm{Gr}^{k+n} \mathbf{hoger}(m) \to \mathrm{Gr}^{k+1+n} \mathbf{hoger}(m)$$

The latter differential is acyclic unless $k + n \neq m - 1$.

Thus, $H^k(\mathrm{Gr}_\Phi^m \mathfrak{g}^{(n)}) = 0$ unless $k = m - 1 - n$.

Since the filtration Φ is complete we have a spectral sequence $E_2^{m,k-m} := H^k(\mathrm{Gr}^m_\Phi \mathfrak{g}^{(n)}) \Rightarrow H^k(\mathfrak{g}^{(n)})$ in which the differentials $d_r, r \geq 2$ are zero. This implies that the map

$$H^k(\mathfrak{g}^{(n)}) \to H^k(\mathfrak{g}^{(n)}/\Phi^N \mathfrak{g}^{(n)})$$

is isomorphisms for all $N \geq n+k$.

In particular, the maps

$$H^0(\mathfrak{g}^{(n)}) \to H^0(\mathfrak{g}^{(n)}_{\leq N})$$

are isomorphisms for all $N \geq n$.

Given an $X \in F^1 H^0(\mathfrak{g}) = \prod_{n\geq 1} H^0(\mathfrak{g}^{(n)})$, $X \neq 0$ there exists an $n \geq 1$, such that the component $X^{(n)} \in H^0(\mathfrak{g}^{(n)})$ is not zero, hence the image of X in $F^1 H^0(\mathfrak{g}_{\leq N})$ is non-zero for all $N \geq n$.

10.1.2. Let $\iota_x : \mathbf{hoger} \to \mathbf{hoger}$ be as in (19). This way we get maps

$$\iota_N : \mathbf{k}^\times \xrightarrow{\iota} \mathrm{Aut}_{\mathrm{D(op)}}(\mathbf{ger}) \to \mathrm{Aut}_{D(n\text{-op})}(\mathbf{ger}_{\leq n}).$$

PROPOSITION 10.6. *For every $X \in \mathfrak{G}$, $X \notin \iota(\mathbf{k}^\times)$, there exists an n such that $O_n(X) \notin \iota_n(\mathbf{k}^\times)$.*

Proof. Consider the image $(x_1, x_2) \in \mathbf{k}^\times \times \mathbf{k}^\times$ of X under the map

$$\mathfrak{G} \to \mathbf{k}^\times \times \mathbf{k}^\times.$$

Note that the latter map factors as:

$$\mathfrak{G} \to \mathfrak{G}_n \to \mathbf{k}^\times \times \mathbf{k}^\times$$

Therefore, if $x_1 \neq x_2$, the statement of the theorem is true for all n. Thus, $x_1 = x_2 = x$ and we have $Z := \iota_x^{-1} X \in \mathfrak{G}_0$. Since X is not in the image of $\iota(\mathbf{k}^\times$, we conclude that $Z \neq \mathrm{Id}$. Therefore, according to Proposition 10.4, the image of Z in $\mathfrak{G}_n$ is not identity.

On the other hand, the image of Z under the projection $\mathfrak{G}_n \to \mathbf{k}^\times \times \mathbf{k}^\times$ is the identity. This implies that the image of $X = \iota_x Z$ in $\mathfrak{G}_n$ is not equal to $\iota_n(x)$ for any $x \in \mathbf{k}^\times$. ☐

11. Appendix 3: Associators and GT

We only collect the information that is needed in this paper. The reader can find expositions of the theory of the associator and the GT group in the original paper [5], see also [3], [21].

For the purposes of the present paper, we only need to know the following facts:

11.0.3. Given an associator Φ over $\mathbf{k}$, one has a canonical element $A_\Phi \in \mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{braces}, \mathbf{ger})$. The construction is as follows.

1) proofs of Deligne's conjecture in [19] [13] [20] provide us with a zigzag quasi-isomoprhism of the operad **braces** with the operad of singular chains of the topological operad of little disks.

2) In [16], given an associator Φ, we construct a zigzag quasi-isomorphism between the chain operad of little disks and the operad of Gerstenhaber algebras.

Combining 1)-2) we get a zigzag quasi-isomorphism between the operads **braces** and **ger**. This zigzag defines element $A_\Phi \in \mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{braces}, \mathbf{ger})$.

Let us recall the construction. In [3] the associator is essentially defined as any isomorphism between two operads, **PaB** ans **PaCD** in the category of small categories, using the standard nerve and simplicial chain functors, one obtains an induced isomorphism of dg- operads

$$C_\bullet(N(\mathbf{PaB})) \xrightarrow{\Phi_*} C_\bullet(N(\mathbf{PaCD})).$$

Denote the operad on the LHS by $\mathbf{o}_1$ and the operad on the RHS by $\mathbf{o}_2$, so that we have an isomorphism

$$B(\Phi) : \mathbf{o}_1 \to \mathbf{o}_2$$

Lastly, one constructs:

a) a zigzag quasi-isomorphism between $\mathbf{o}_1$ and the singular chain operad of the little disks (hence, by 1) with the operad **braces**);

b) a zigzag quasi-isomorphism between $\mathbf{o}_2$ and **ger**

11.0.4. **GRT**. In [3] the group **GRT** defined as an automorphism group of the operad **PaCD**, hence a free and transitive **GRT**-action on the set of all associators and an action of **GRT**on $\mathbf{o}_2$, because the operad $\mathbf{o}_2$ is obtained from **PaCD** in a functorial way.

Given a $g \in \mathbf{GRT}$ and an associator Φ, let $g.\Phi$ be the result of the action of $\mathfrak{g}$ on Φ. Let us also denote by $g_* : \mathbf{o}_2 \to \mathbf{o}_2$ the automorphism induced by g. We then get $B_{g.\Phi} = g_* B_\Phi$.

A zig-zag quasi-isomorphism of $\mathbf{o}_2$ and **ger** gives rise to a canonical isomorphism $\mathbf{o}_2 \to \mathbf{ger}$ in the category D(op). Therefore, the **GRT**-action on $\mathbf{o}_2$ canonically defines a map

$$T : \mathbf{GRT} \to \mathrm{Aut}_{\mathrm{D(op)}}(\mathbf{ger}) = \mathfrak{G} \tag{29}$$

It easily follows that $A(g.\Phi) = T(g)A(\Phi)$.

11.0.5. The group **GRT** is known to be an extension of $\mathbf{k}^\times$ by a prounipotent group $\mathbf{GRT}_0$

From the previous subsection, we have a map $\mathbf{GRT} \to \mathfrak{G}$. The through map

$$\mathbf{GRT}_0 \to \mathbf{GRT} \to \mathfrak{G} \to \mathbf{k}^\times \times \mathbf{k}^\times$$

must be the identity, as easily follows from the theory of algebraic groups. Therefore, we get an induced map $\mathbf{GRT}_0 \to \mathfrak{G}_0$. This map produces a map of the Lie algebras

$$\mathfrak{g}rt_0 \to F^1 H^0(\mathfrak{g}).$$

In [16] we show that the latter map is injective, therefore

THEOREM 11.1. *The map of exponentials* $\mathbf{GRT}_0 \to \mathfrak{G}_0$ *is also injective.*

The map of the quotients

$$\mathbf{k}^\times = \mathbf{GRT}/\mathbf{GRT}_0 \to \mathfrak{G}/\mathfrak{G}_0 = \mathbf{k}^\times \times \mathbf{k}^\times \tag{30}$$

can be also proven to be injective. Furthermore, we have:

LEMMA 11.2. *The map (30) sends* $x \in \mathbf{k}^\times$ *to* $(1, x) \in \mathbf{k}^\times \times \mathbf{k}^\times$

Proof. The category **PaCD** is equivalent to the category with one object whose endomorphism space is the completed universal enveloping algebra of abelian one dimensional Lie algebra. Denote the generator of this Lie algebra by t. The action of **GRT** on $\mathbf{PaCD}(2)$ factors through the projection $p : \mathbf{GRT} \to \mathbf{k}^\times$ so that $g \in \mathbf{GRT}$ dilates t by $p(g)$: $g.t = \pi(g)t$.

The operad $C_\bullet(\mathbf{PaCD}(2))$ is canonically quasi-isomorphic to Chevalley-Eilenberg chain complex $C_{-\bullet}(\mathfrak{t})$, where $\mathfrak{t}$ is the one dimensional Lie algebra generated by t. This complex has one-dimensional zeroth and one dimensional negative first cohomology so that the cohomology of $C_{-\bullet}(\mathfrak{t})$ is canonically identified with $\mathbf{ger}(2)$.

The induced **GRT**-action on this cohomology (from the action on $\mathfrak{t}$ by dilations) can be easily found to be trivial on the zeroth cohomology; the action on the negative first cohomology is by dilations, whence the statement. □

We can now make the following statement.

Consider the diagonal embedding $\mathbf{k}^\times \to \mathbf{k}^\times \times \mathbf{k}^\times$. Let

$$\iota : \mathbf{k}^\times \to \mathbf{k}^\times \times \mathbf{k}^\times \to \mathfrak{G}$$

be the through map. Let $\mathfrak{G}' : \mathfrak{G}/\iota(\mathbf{k}^\times)$ be the quotient.

We then have the through map:

$$\mathbf{GRT} \stackrel{29}{\to} \mathfrak{G} \to \mathfrak{G}' \tag{31}$$

THEOREM 11.3. *The map (31) is injective*

Proof. Indeed, the the induced maps

$$\mathbf{GRT}_0 \to \mathfrak{G}_0 \to \mathfrak{G}'$$

and

$$\mathbf{k}^\times = \mathbf{GRT}/\mathbf{GRT}_0 \to \mathfrak{G}'/\mathfrak{G}_0 = \mathbf{k}^\times$$

are both injective: the arrow $\mathbf{GRT}_0 \to \mathfrak{G}_0$ by Theorem 11.1; the arrow $\mathfrak{G}_0 \to \mathfrak{G}'$ is injective by inspection; the arrow $\mathbf{GRT}/\mathbf{GRT}_0 \to \mathfrak{G}'/\mathfrak{G}_0$ is an isomorphism as follows from Lemma 11.2. $\square$

11.0.6. Let us now study elements $A_{\Phi_1}, A_{\Phi_2} \in \mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{braces}, \mathbf{ger})$.

We have a map $\iota : \mathbf{k}^\times \to \mathfrak{G} = \mathrm{Aut}_{\mathrm{D(op)}}(\mathbf{ger})$. Call two elements $U, V \in \mathrm{Iso}_{\mathrm{D(op)}}(\mathbf{braces}, \mathbf{ger})$ *equivalent* if they are conjugated by the action of an element $\iota(x)$ for some $x \in \mathbf{k}^\times$.

We can now prove

THEOREM 11.4. *Given two different associators Φ_1 and Φ_2, the elements*

$$A_{\Phi_1}, A_{\Phi_2} \in \mathrm{Iso}_{D(op)}(\mathbf{braces}, \mathbf{ger})$$

are not equivalent.

Proof. We have $\Phi_2 = g.\Phi_1$ for some $g \in \mathbf{GRT}$, $g \neq e$. Let $\overline{g} \in \mathfrak{G}$ be the image of g. We know that $\overline{g} \notin \iota(\mathbf{k}^\times)$ by Theorem 11.3.

Next, $A_{\Phi_2} = \overline{g}.A_{\Phi_1}$. As $\overline{g} \notin \iota(\mathbf{k}^\times)$, it follows that A_{Φ_2} and A_{Φ_1} are non-equivalent. $\square$

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