Microlocal Category

Dmitry Tamarkin

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PART 1: CATEGORIES

1 Closure of a dg-category

1.1 Generalities

1.1.1 dg-categories

By a *dg-category* we mean a category enriched over the category of complexes of $\mathbb{A}$-modules, where $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{Q}$.

An arrow, or a morphism $f : X \to Y$ is a cocycle in $Hom^0(X, Y)$. We say that two arrows f, g are *homotopy equivalent,* and write $f \sim g$, if the cocycle $f - g$ is exact.

We say that $f : X \to Y$ is *a homotopy equivalence* if there exists $g : Y \to X$ such that $fg \sim \mathrm{Id}_Y$ and $gf \sim \mathrm{Id}_X$.

We say that an object X is *acyclic* if $0 \sim \mathrm{Id}$ in $\mathrm{Hom}(X, X)$.

1.1.2 Enriched categories

Let $\mathcal{C}, \mathcal{D}$ be categories enriched over a SMC $\mathcal{M}$. Denote by $\mathcal{C} \otimes \mathcal{D}$ a category enriched over $\mathcal{M}$, where $\mathbf{Ob}\,\mathcal{C} \otimes \mathcal{D} = \mathbf{Ob}\,\mathcal{C} \times \mathbf{Ob}\,\mathcal{D}$ and $\mathrm{Hom}(X_1, Y_1); (X_2, Y_2)) = \mathrm{Hom}(X_1, X_2) \otimes \mathrm{Hom}(Y_1, Y_2)$.

1.2 Various completions

We will now introduce several operations, namely: twisting the differentual, adding a kernel of a projector, adding direct sums and direct products. We will end up with an operation **swell** so that **swell** $\mathcal{C}$ is closed under adding all the above listed objects.

1.3 Differential on an object

A differential on an object X of a dg-category $\mathcal{C}$ is an element $D \in \mathrm{Hom}^1(X, X)$ satisfying $dD + D^2 = 0$. Define a dg-category $D\mathcal{C}$ whose every object is a pair (X, D), where $X \in \mathcal{C}$ and D is a differential on X; we set

$$\mathrm{Hom}((X, D_X), (Y, D_Y)) := (\mathrm{Hom}(X, Y), D'),$$

where we introduce a new differential D' on $\mathrm{Hom}(X, Y)$ as follows. Let $f \in \mathrm{Hom}^n(X, Y)$; set:

$$D'f = df + D_Y f - (-1)^n f D_X$$

We have a natural functor $D\mathcal{C} \otimes D\mathcal{D} \to D(\mathcal{C} \otimes \mathcal{D})$. If $\mathcal{C}$ is a SMC, then $D\mathcal{C}$ inherits the structure. If $\mathcal{C}$ is enriched over an SMC $\mathcal{M}$, then $D\mathcal{C}$ is enriched over $D\mathcal{M}$. Call $\mathcal{C}$ D-closed if the obvious functor $\mathcal{C} \to D\mathcal{C}$ is an equivalence of DG categories. The category $D\mathcal{C}$ is always D-closed.

1.4 Kernels of projectors

Let X be an object of $\mathcal{C}$. *A projector* is an element $P \in \mathrm{Hom}^0(X,X)$ such that $dP = 0$ and $P^2 = P$. Define a dg-category $P\mathcal{C}$ whose every object is a pair (X, P_X), where P_X is a projector on X. Set $\mathrm{Hom}((X,P_X),(Y,P_Y))$ to be a sub-complex of $\mathrm{Hom}(X,Y)$ consisting of all elements f satisfying $P_Y f = f = f P_X$.

We have a natural map $P\mathcal{C} \otimes P\mathcal{D} \to P(\mathcal{C} \otimes \mathcal{D})$. If $\mathcal{C}$ is a SMC, then $P\mathcal{C}$ inherits the structure. If $\mathcal{C}$ is enriched over an SMC $\mathcal{M}$, then $P\mathcal{C}$ is enriched over $P\mathcal{M}$. We call a dg-category P-closed if the obvious inclusion $\mathcal{C} \to P\mathcal{C}$ is an equivalence of categories. If $\mathcal{C}$ is D-closed then so is $P\mathcal{C}$. Therefore, $PD\mathcal{C}$ is both P-and D-closed.

1.4.1

Call a category $\oplus\prod$-*closed* if all small direct products and direct sums exist in $\mathcal{C}$. It follows that $PD\mathcal{C}$ is $\oplus\prod$-closed if such is $\mathcal{C}$.

The goal of the subseqent subsection is to provide a tool for constructing $\oplus\prod$-closed dg-categories.

1.5 Precofilters

Let S be a set. By definition, *a pre-cofilter* $\mathcal{F}$ *on* S is a collection of subsets on S satisfying:

— if $X \in \mathcal{F}$ and $Y \subset X$, then $Y \in \mathcal{F}$;

— if $X_1, X_2 \in \mathcal{F}$, then so is $X_1 \cup X_2$.

1.5.1

Let P be any family of subsets of S. Let $\mathbf{precofilter}(P)$ be the smallest pre-cofilter containing P. We have: $U \in \mathbf{precofilter}(P)$ iff U is contained in a finite union of subsets from P. We call $\mathbf{precofilter}(P)$ the pre-cofilter generated by P.

1.5.2 Product of precofilters

Let S_1, S_2 be sets and $\mathcal{F}_1, \mathcal{F}_2$ precofilters. Let $\mathcal{F}_1 \times \mathcal{F}_2$ be the pre-cofilter generated by all subsets $U_1 \times U_2 \subset S_1 \times S_2$, where $U_1 \in \mathcal{F}_1$ and $U_2 \in \mathcal{F}_2$.

Let $p_i : S_1 \times S_2 \to S_i$ be the projections. We see that $U \in \mathcal{F}_1 \times \mathcal{F}_2$ iff $p_i(U) \in \mathcal{F}_i$, $i = 1, 2$.

1.5.3 Convolution of subsets

Finally, for $E \subset S_1 \times S_2$ and $F \subset S_2 \times S_3$ we define $E \circ F \subset S_1 \times S_3$ to consist of all $(s_1, s_3) \in S_1 \times S_3$, where there exists $s_2 \in S_2$ such that $(s_1, s_2) \in E$ and $(s_2, s_3) \in F$.

If $U \subset S_1$, $V \subset S_1 \times S_2$, and $W \subset S_3$, we define $U \circ V \subset S_2$ and $V \circ W \subset S_1$ in a similar way.

1.5.4 Properness

As above, let $E \subset S_1 \times S_2$ and $F \subset S_2 \times S_3$. We say that the convolution $E \circ F$ is *proper* if for all $(s_1, s_3) \in S_1 \times S_3$, the set

$$\{s_2 \in S_2 | (s_1, s_2) \in E;\ (s_2, s_3) \in F\}$$

is finite.

1.5.5 Precofilter hom

Let $\mathcal{F}_i$ be a cofilter on S_i, $i = 1, 2$. Define $\underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2)$ on $S_1 \times S_2$ to consist of all $U \subset S_1 \times S_2$, where
– for every $L \in \mathcal{F}_1$, $L \circ U \in \mathcal{F}_2$ and the convolution $L \circ U$ is proper.

1.5.6 Dual precofilter

Let $\mathcal{F}$ be a pre-cofilter on S. Define a cofilter $\mathcal{F}^\vee$ on S to consist of all subsets $U \subset S$, where $V \cap U$ is finite for every $V \in \mathcal{F}$.

We have $\mathcal{F}^\vee = \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{T})$, where $\mathcal{T}$ is a pre-cofilter on the one-element set consisting of all its subsets.

1.5.7 Cofilters

We have an inclusion $\mathcal{F} \subset (\mathcal{F}^\vee)^\vee$. Call $\mathcal{F}$ a *cofilter* if this inclusion is an equality. Observe that any pre-cofilter of the form $\mathcal{G}^\vee$ is a co-filter.

1.5.8 Formula for Hom

Let $\mathcal{F}_i$ be pre-cofilters on S_i, $\mathcal{F}_2$ being a co-filter, we then have

$$\underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2) = (\mathcal{F}_1 \times \mathcal{F}_2^\vee)^\vee.$$

In particular, $\underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2)$ is a co-filter.

1.5.9 Product of cofilters

Suppose that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are cofilters. Then so is $\mathcal{F}_1 \times \mathcal{F}_2$.

Sketch of the proof. Denote by Π_k a co-filter on S_k consisting of all its subsets. Let also $p_k : S_1 \times S_2 \to S_1$ be the projection.

We have $(\mathcal{F}_1 \times \mathcal{F}_2)^\vee \supset (\mathcal{F}_1 \times \Pi_2)^\vee$. The latter cofilter consists of all subsets $\Sigma \subset S_1 \times S_2$ satisfying $p_1(\Sigma) \in \mathcal{F}_1^\vee$ and every fiber of the projection $p_1 : \Sigma \to S_1$ must be finite. It follows that $(\mathcal{F}_1 \times \Pi_2)^{\vee\vee} = \mathcal{F}_1 \times \Pi_2$. Indeed, if $X \subset S_1 \times S_2$ and $p_1(X) \notin \mathcal{F}_1$, then there exists a $Y \in \mathcal{F}_1^\vee$ such that $p_1(X) \cap Y = Z$ is infinite. Therefore, there exists a subset $W \subset X$ which is mapped bijectively onto Z via p_1. It follows that $W \in (\mathcal{F}_1 \times \Pi_2)^\vee$ and $W \cap X$ is infinite.

Hence, $(\mathcal{F}_1 \times \mathcal{F}_2)^{\vee\vee} \subset \mathcal{F}_1 \times \Pi_2$. Similarly, $(\mathcal{F}_1 \times \mathcal{F}_2)^{\vee\vee} \subset \Pi_1 \times \mathcal{F}_2$, which implies

$$(\mathcal{F}_1 \times \mathcal{F}_2)^{\vee\vee} \subset \mathcal{F}_1 \times \Pi_2 \cap \Pi_1 \times \mathcal{F}_2 = \mathcal{F}_1 \times \mathcal{F}_2.$$

As $\mathcal{F}^{\vee\vee} \supset \mathcal{F}$ for any pre-cofilter $\mathcal{F}$, the statement follows.

1.5.10 Families of cofilters

Let $\pi : S \to T$ be a map of sets. Fix cofilters $\mathcal{F}$ on T and $\mathcal{F}_t$ on $S_t := \pi^{-1}t$, $t \in T$. Define a cofilter $\Phi := \prod_{t\in T}^{\mathcal{F}} \mathcal{F}_t$ to consist of all subsets $U \subset S$ such that $\pi(U) \in \mathcal{F}$ and $U \cap S_t \in \mathcal{F}_t$ for all $t \in T$.

Equivalently: given any $H \subset S$ such that $H \cap S_t \in \mathcal{F}_t^\vee$ and $\pi(H) \subset \mathcal{F}^\vee$, then $H \cap U$ is finite.

This implies:

$$\left(\prod_{t\in T}^{\mathcal{F}} \mathcal{F}_t\right)^\vee = \prod_{t\in T}^{\mathcal{F}^\vee} \mathcal{F}_t^\vee.$$

1.5.11 Product of $\mathbb{A}$-modules over a cofilter

Let S be a set and let X_s, $s \in S$ be a family of $\mathbb{A}$-modules. Let $\mathcal{F}$ be a cofilter on S. Set

$$\prod_{s\in S}^{\mathcal{F}} X_s \subset \prod_{s\in S} X_s$$

to consist of all families $\{x_s\}_{s\in S}$ where the set $\{s \in S | x_s \neq 0\}$ belongs to $\mathcal{F}$. We have natural maps

$$\left(\prod_{s\in S}^{\mathcal{F}} X_s\right) \bigotimes \prod_{t\in T}^{\mathcal{G}} Y_t \to \prod_{(s,t)\in S\times T}^{\mathcal{F}\times\mathcal{G}} X_s \otimes Y_t;$$

$$\prod_{(s,t)\in S\times T}^{\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})} \mathrm{Hom}(X_s, Y_t) \to \mathrm{Hom}\left(\prod_{s\in S}^{\mathcal{F}} X_s; \prod_{t\in T,\mathcal{G}} Y_t\right).$$

1.6 Aggrandizement

Let $\mathcal{C}$ be a category enriched over the category of $\mathbb{A}$-modules. Let us define a new category $\mathrm{agg}\,\mathcal{C}$ enriched over the same category as follows.

— Objects of $\mathrm{agg}\,\mathcal{C}$ are of the form $(S, \mathcal{F}, \{X_s\}_{s\in S})$, where S is a set, $\mathcal{F}$ is a cofilter on S, and $X_s \in \mathcal{C}$, $s \in S$.

— Let $\mathcal{X}_i := (S_i, \mathcal{F}_i, \{(X_i)_s\}_{s\in S})$, $i = 1, 2$. Set

$$\mathrm{Hom}_{\mathrm{agg}\,\mathcal{C}}(\mathcal{X}_1, \mathcal{X}_2) := \prod_{(s_1,s_2)\in S_1\times S_2}^{\underline{\mathrm{Hom}}(\mathcal{F}_1,\mathcal{F}_2)} \mathrm{Hom}(X_{s_1}, X_{s_2}).$$

We have a natural functor

$$\boxtimes : \mathrm{agg}(\mathcal{C}_1) \otimes \mathrm{agg}(\mathcal{C}_2) \to \mathrm{agg}(\mathcal{C}_1 \otimes \mathcal{C}_2),$$

where

$$(S, \mathcal{F}, \{X_s\}_{s\in S}) \boxtimes (T, \mathcal{G}.\{X_t\}_{t\in T}) := (S \times T, \mathcal{F} \times \mathcal{G}, \{X_s \otimes Y_t\}).$$

This implies that a (symmetric) monoidal structure on $\mathcal{C}$ caries over to $\mathrm{agg}\,\mathcal{C}$.

If $\mathcal{C}$ is enriched over a monoidal category $\mathcal{M}$, then $\operatorname{agg}\mathcal{C}$ is enriched over $\operatorname{agg}\mathcal{M}$.

It follows that $\operatorname{agg}\mathcal{C}$ is $\oplus\prod$-closed.

If $\mathcal{C}$ is a dg-catregory, then so is $\operatorname{agg}\mathcal{C}$.

1.6.1

We have natural functors: $\mathcal{K} : \operatorname{agg}\operatorname{agg}\mathcal{C} \to \operatorname{agg}\mathcal{C}$ and $\underline{\mathrm{Hom}} : (\operatorname{agg}\mathcal{C})^{\mathbf{op}} \otimes \operatorname{agg}\mathcal{D} \to \operatorname{agg}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D})$.

— $\mathcal{K}$. Let $\pi : S \to T$ be a map of sets, let $S_t := \pi^{-1}t$, $t \in T$. and $X : S \to \mathcal{C}$. Let F_T be a cofilter on T and F_{S_t} on S_t. Every object Y of $\operatorname{agg}\operatorname{agg}\mathcal{C}$ is of the form

$$Y = \prod_{t\in T}^{F_T} \prod_{s\in S_t} X_s.$$

Let Φ be a cofilter on S, where $U \in \Phi$ iff $U \cap S_t \in F_{S_t}$ for all t and $\pi(U) \in F_T$, that is

$$\Phi = \prod_{t\in T}^{F_T} F_{S_t}.$$

Set

$$\mathcal{K}(Y) := \prod_{s\in S}^{\Phi} X_s.$$

—$\underline{\mathrm{Hom}}$. Let $U = \prod_{s\in S}^{F} X_s \in \operatorname{agg}\mathcal{C}$ and $V = \prod_{t\in T}^{G} Y_s \in \operatorname{agg}\mathcal{D}$. Set

$$\underline{\mathrm{Hom}}(U, V) := \prod_{(s,t)\in S\times T}^{\mathrm{Hom}(F,G)} (X_s; Y_t).$$

1.7 Swell

For a dg-category $\mathcal{C}$ denote $\mathbf{swell}_0\,\mathcal{C} := PD\operatorname{agg}\mathcal{C}$ viewed as a dg-category. The resulting category is $PD \oplus \prod$-closed.

1.7.1 Graded free $\mathbb{A}$-modules

Let **grad** be a dg-category whose objects are of the form $[n]$, $n \in \mathbb{Z}$. Set $\mathrm{Hom}([n], [m]) = \mathbb{A}[m - n]$ Introduce a SMC on **grad** by setting $[n]\otimes[m] = [n+m]$; define the brading $B_{nm} : [n]\otimes[m] \to [m]\otimes[n]$ to be equal $(-1)^{nm}$.

1.7.2 Definition of swell

Set $\mathbf{swell}(\mathcal{C}) = \mathbf{swell}_0(\mathcal{C} \otimes \mathbf{grad})$. The advantage of $\mathbf{swell}(\mathcal{C})$ over $\mathbf{swell}_0(\mathcal{C})$ is the existence of cones and shifts.

1.7.3 Properties

There are natural functors

$$\boxtimes : \mathbf{swell}\,\mathcal{C} \otimes \mathbf{swell}\,\mathcal{D} \to \mathbf{swell}(\mathcal{C} \otimes \mathcal{D});$$

$$\underline{\mathrm{Hom}} : \mathbf{swell}\,\mathcal{C}^{\mathbf{op}} \otimes \mathbf{swell}\,\mathcal{D} \to \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D}).$$

These functors are obtained via extension from $\mathrm{agg}\,\mathcal{C}, \mathrm{agg}\,\mathcal{D}$.

Therefore, if $\mathcal{C}$ is an SMC, then so is $\mathbf{swell}\,\mathcal{C}$. If $\mathcal{D}$ is enriched over a SMC $\mathcal{C}$, then $\mathbf{swell}\,\mathcal{D}$ is enriched over $\mathbf{swell}\,\mathcal{C}$. Furthermore, if $\mathcal{C}$ is an SMC, then the tensor product is compatible with the direct products in the obvious way.

If $\mathcal{C}$ is an SMC with an inner hom then so is $\mathbf{swell}\,\mathcal{C}$ and that this inner hom is compatible with the direct sums and direct products in the obvious way.

We have a natural functor $\mathbf{swell}\,\mathbf{swell}\,\mathcal{C} \to \mathbf{swell}\,\mathcal{C}$.

1.7.4

Let $F : \mathcal{C} \to \mathbf{swell}\,\mathcal{D}$ be a dg-functor. It induces a functor

$$\mathbf{swell}\,F : \mathbf{swell}\,\mathcal{C} \to \mathbf{swell}\,\mathbf{swell}\,\mathcal{D} \to \mathbf{swell}\,\mathcal{D}.$$

1.8 Contraction and Co-contraction of Kernels

1.8.1 Preliminaries

Let $\mathbf{Com}$ be a dg-category of complexes of free $\mathbb{A}$-modules. We have an obvious functor $\mathbf{Com} \otimes \mathcal{C} \to \mathbf{swell}\,\mathcal{C}$.

1.8.2 Contraction

Let $h : \mathcal{D} \otimes \mathcal{D}^{\mathbf{op}} \to \mathbf{Com}$ be the hom functor. Define a contraction functor

$$\circ := \circ_D : \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D}) \otimes \mathbf{swell}(\mathcal{D}^{\mathbf{op}} \otimes \mathcal{E}) \xrightarrow{\boxtimes} \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D} \otimes \mathcal{D}^{\mathbf{op}} \otimes \mathcal{E}) \xrightarrow{h} \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathbf{Com} \otimes \mathcal{E}) \to \mathbf{swell}\,\mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{E}) \to \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{E}).$$

Define a co-contraction functor:

$$\underline{\mathrm{Hom}} := \underline{\mathrm{Hom}}_{\mathcal{C}} : \mathbf{swell}(\mathcal{C} \otimes \mathcal{D})^{\mathbf{op}} \otimes \mathbf{swell}(\mathcal{C} \otimes \mathcal{E}) \xrightarrow{\underline{\mathrm{Hom}}} \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D}^{\mathbf{op}} \otimes \mathcal{C} \otimes \mathcal{E}) \xrightarrow{h} \mathbf{swell}(\mathcal{D}^{\mathbf{op}} \otimes \mathbf{Com} \otimes \mathcal{E}) \to \mathbf{swell}\,\mathbf{swell}(\mathcal{D}^{\mathbf{op}} \otimes \mathcal{E}) \to \mathbf{swell}(\mathcal{D}^{\mathbf{op}} \otimes \mathcal{E}).$$

1.8.3 Associativity

The contraction functor has an obvious associativity property.

2 Category GZ

Let **pt** be the category with one object whose endomorphism group is $\mathbb{A}$. Set $\mathbf{GZ} := \mathbf{swell}(\mathbf{pt})$.

We have an internal Hom in **GZ** as well as a tensor functor $||$ from **GZ** to the category of complexes of $\mathbb{A}$-modules.

2.1 Explicit description of objects from GZ

Every object in **GZ** is the following collection of data:

$$(S, F, g, D, P),$$

where S is a set, $\mathcal{F}$ is a cofilter on S, $g : S \to \mathbb{Z}$ is an arbitrary map, and

$$D \in \mathrm{Hom}^1(\prod_{s\in S}^{\mathcal{F}}[g(s)]; \prod_{s\in S}^{F}[g(s)]); \quad D^2 = 0;$$

$$P \in \mathrm{Hom}^0(\prod_{s\in S}^{\mathcal{F}}[g(s)]; \prod_{s\in S}^{F}[g(s)]); \quad P^2 = P; \quad DP = PD.$$

2.2 Tensor product

Denote by $\otimes$ the functor

$$\otimes : \mathbf{GZ} \otimes \mathbf{swell}\,\mathcal{C} \stackrel{\boxtimes}{\to} \mathbf{swell}(\mathbf{pt} \otimes \mathcal{C}) = \mathbf{swell}(\mathcal{C})$$

and likewise for the isomorphic functor $\otimes : \mathbf{swell}\,\mathcal{C} \otimes \mathbf{GZ} \to \mathbf{swell}\,\mathcal{C}$.

2.3 Truncation

2.3.1 Categories $\mathbf{GZ}_{\leq k}$, $\mathbf{GZ}_{\geq k}$ etc.

Let $\mathbf{grad}_{\leq k}$ be the full subcategory of **grad** consisting of all objects $[n]$, $n \leq k$ and $\mathbf{grad}_{\geq k}$ be the full subcategory consisting of all $[n]$, $n \geq k$. Let $\mathbf{grad}_{=k}$ be the full sub-category consisting of one object $[k]$, etc. Let $\mathbf{GZ}_{\leq k} := \mathbf{swell}_0\,\mathbf{grad}_{\leq k}$; $\mathbf{GZ}_{\geq k} := \mathbf{swell}_0\,\mathbf{grad}_{\geq k}$ etc.

2.3.2 Stupid truncation

Let $X := (S, F, g, D, P) \in \mathbf{GZ}$, where (S, F, g, D, P) is as in Sec 2.1.

Let us define an object $X^{\leq k}$, where $k \in \mathbb{Z}$.

Set $S^{\leq k} := \{s \in S | g(s) \leq k\}$. Set

$$\mathcal{F}^{\leq k} := \{A | A \in \mathcal{F}; A \subset S^{\leq k}\}.$$

Set $g^{\leq k} := g|_{S^{\leq k}}$.

We have an obvious retraction in **GZ**:

$$\prod_{s\in S^{\leq k}}^{\mathcal{F}^{\leq k}} [g^{\leq k}(s)] \xrightarrow{I} \prod_{s\in S}^{\mathcal{F}}[g(s)] \xrightarrow{Q} \prod_{s\in S^{\leq k}}^{\mathcal{F}^{\leq k}} [g^{\leq k}(s)].$$

Set $D^{\leq k} := QDI$; $P^{\leq k} := QPI$.

Set $X^{\leq k} := (S^{\leq k}, F^{\leq k}, g^{\leq k}, D^{\leq k}, P^{\leq k})$.

We thus have constructed a functor of categories over **sets**:

$$-^{\leq k} : \mathbf{GZ} \to \mathbf{GZ}_{\leq k}.$$

It follows that this functor is the right adjoint to the embedding $\mathbf{GZ}_{\leq k} \to \mathbf{GZ}$.

Likewise one defined a functor $-^{\geq k}$ which is the left adjoint to the embedding $\mathbf{GZ}_{\geq k} \to \mathbf{GZ}$.

One has a natural map

$$\delta : X^{\leq k}[-1] \to X^{\geq k+1}$$

so that we have an isomorphism in **GZ**

$$X \cong \operatorname{Cone}\delta.$$

2.3.3 The object X^k

We set $X^k := (X^{\leq k})^{\geq k}$. The object X^k has zero differential.

2.3.4 Truncation

We say that an object $X \in \mathbf{GZ}$ *admits a truncation* if there exists a universal object $\tau_{\leq k}X \in \mathbf{GZ}_{\leq k}$ which maps into X. We say that $X \in \mathbf{GZ}$ *stably admits a truncation* if every object $Y \in \mathbf{GZ}$ which is homotopy equivalent to X, admits a truncation.

Likewise, for $X \in \mathbf{GZ}$, we denote by $\tau_{\geq k}(X)$ the universal object in $\mathbf{GZ}_{\geq}$ (if exists) endowed with a map $X \to \tau_{\geq k}X$.

2.3.5 Lemma

Lemma 2.1 *Let $X \in \mathbf{GZ}_{\geq 0}$ and suppose it admits a truncation. Then $\tau_{\leq X} \in \mathbf{GZ}_{=0}$.*

Sketch of the proof Let $Y := \tau_{\leq 0}X$. Let $\iota : Y \to X$ be the natural map. Let

$$C := \operatorname{Cone}(\mathrm{Id} : Y^{<0} \to Y^{<0})[-1],$$

in other words,

$$C = (Y^{<0} \oplus Y^{<0}[-1], D),$$

Where $D = \mathrm{Id} : Y^{<0} \to Y^{<0}[-1]$. It follows that $D \in \mathbf{GZ}_{\leq 0}$.

We have a natural map $c : C \to Y$, where $c|_{Y^{<0}} = I$; $c|_{Y^{<0}[-1]} = -dI$, where $I : Y^{<0} \to Y$ is the embedding.

It follows that $\iota c = 0$ which implies that $c = 0$, hence $I = 0$ and $Y^{<0} = 0$, which implies the statement.

2.3.6 Lemma

Lemma 2.2 *Let $X \in \mathbf{GZ}$ be homotopy equivalent to an object $Y \in \mathbf{GZ}_{\geq 0}$. Then there exists a direct sum decomposition $X = A \oplus B$, where $A \in \mathbf{GZ}_{\geq 0}$, $B \in \mathbf{GZ}_{\leq 0}$, and B is acyclic.*

Sketch of the proof

By definition we have maps $f : X \to Y$, $g : Y \to X$, $h \in \mathrm{Hom}^{-1}(X, X)$, where

$$\mathrm{Id}_X = gf + dh + hd. \tag{1}$$

Set $\pi := dh|_{X^0} : X^0 \to X^0$. We have

$$\pi^2 = dhdh|_{X^0} = d(hd)|_{X^{-1}} h|_{X^0}. \tag{2}$$

Let us restrict (1) onto X^{-1}. As $Y \in \mathbf{GZ}_{\geq 0}$, $gf = 0$. Therefore, $hd|_{X^{-1}} = \mathrm{Id}_{X^{-1}} - dh|_{X^{-1}}$. Substitute this equality into (2):

$$\pi^2 = d(hd)|_{X^{-1}} h|_{X^0} = dh_{X^0} + ddhh|_{X^0} = \pi.$$

Thus $\pi : X^0 \to X^0$ is a projector and we can decompose $X^0 = K \oplus L$ where $\pi|_K = 0$; $\pi|_L = \mathrm{Id}|_L$.

We have $d|_L = d\pi|_L = 0$. Denote $D_K : K \to X^0 \xrightarrow{d} X^{>0}$.

Consider $\pi d : X^{-1} \to X^0$. We have

$$\pi d = dhd|_{X^{-1}} = d(\mathrm{Id} - dh)|_{X^{-1}} = d|_{X^{-1}}.$$

This shows that $d|_{X^{-1}}$ factorizes through L: $d|_{X^{-1}} : X^{-1} \xrightarrow{d_L} L \to X^0$. Set

$$D_L := X^{<0} \xrightarrow{p} X^{-1} \xrightarrow{d_L} L,$$

where p is the obvious projection.

Set $A := (X^{<0} \oplus L, D_L)$; $B := K \oplus X^{>0}, D_K)$. The restriction of h onto B shows that B is acyclic. We see that thus chosen A and B satisfy all the conditions.

2.3.7 Lemma on stable truncation

Lemma 2.3 *Every object of $\mathbf{GZ}_{\geq 0}$ admitting a truncation admits it stably.*

Sketch of the proof Let $Y \in \mathbf{GZ}_{\geq 0}$ be an object admitting a truncation. Denote $H := \tau_{\leq 0} Y$. As follows from Lemma 2.1, $H \in \mathbf{GZ}_0$. Let $i : H \to Y$ be the structure map.

Let $X \in \mathbf{GZ}$ be an object homotopy equivalent to Y. Let us decompose $Y = A \oplus B$ according to Lemma **??**. It follows that A is homotopy equivalent to Y. It now sufices to show that A admits a truncation.

Fix a homotopy equivalence $f : A \to Y$; $g : Y \to A$; $gf = \mathrm{Id}_A + dh_A + h_A d$; $fg = \mathrm{Id}_Y + dh_Y + h_Y d$.

Let us prove that $gi : H \to A$ is the universal map from an object from $\mathbf{GZ}_{\le 0}$ to A. Let $f : U \to A$, where $U \in \mathbf{GZ}_{\le 0}$. It follows that f factors through U^0:

$$U \to U^0 \stackrel{\phi}{\to} A.$$

It follows that $d_A\phi = 0$. We therefore have $gf\phi = \phi + dh_A\phi + h_A d\phi = \phi$. On the other hand, the map $f\phi : U^0 \to Y$ factors uniquely through H: $f\phi = i\psi : U^0 \stackrel{\psi}{\to} H \to Y$ so that $\phi = gf\phi = gi\psi$. That is ϕ factors through gi. Let us check the uniqueness of this factorization which is equivalent to the following statement. Let $\chi : U^0 \to H$. Then $gi\chi = 0$ implies $\chi = 0$. Indeed, we have:

$$0 = fgi\chi = i\chi + h_Y di\chi + dh_Y i\chi = i\chi.$$

As i is a universal map, $i\chi = 0$ implies $\chi = 0$.

Corollary 2.4 *Let $X \in \mathbf{GZ}$ be an object homotopy equivalent to an object Y from $\mathbf{GZ}_{\ge 0}$. Then $\tau_{\le 0}X \cong \tau_{\le 0}Y \oplus B$, where $B \in \mathbf{GZ}_{\le 0}$ is an acyclic object.*

Proof. Follows directly from the proof of Lemma.

2.3.8 Complexes of free modules

Let $\mathbb{A}$**-freemod** be the category of complexes of finitely generated $\mathbb{A}$-modules concentrated in the non-negative degrees. One has an embedding of $\mathbb{A}$**-freemod** $\subset \mathbf{GZ}_{\ge 0}$ as a full sub-category.

Lemma 2.5 *Every object $X \in \mathbb{A}$**-freemod** admits a truncation.*

Sketch of the proof Let $H := H^0(X)$. We have a short exact sequence of $\mathbb{A}$-modules:

$$0 \to H \to X^0 \to \operatorname{Coker} d^0 \to 0. \tag{3}$$

The embedding $\operatorname{Coker} d^0 \hookrightarrow X^1$ implies that $\operatorname{Coker} d^0$ is a finitely generated free $\mathbb{A}$-module. Therefore, the exact sequence (3) splits and we can write $X = H \oplus Y$, where $Y \in \mathbb{A}$**-freemod**; $H^0(Y) = 0$.

For every $U \in \mathbb{A}_0$, the natural map $\operatorname{Hom}(U; Y^0) \to \operatorname{Hom}(U, Y^1)$ is an injection. Indeed, it suffices to check this statement for $U = \prod^{\mathcal{F}}_{s \in S}[0]$, in which case the statement can be checked directly. Therefore, $\tau_{\le 0}Y = 0$, whence $\tau_{\le 0}X = H$. This implies the statement.

2.4 The category GZtrunc

Let **GZtrunc** be the full subcategory of **GZ** consisting of all objects which are homotopy equivalent to an object from $\mathbb{A}$**-freemod**. It follows that **GZtrunc** is a full symmetric monoidal sub-category of **GZ**.

2.4.1 The category contract

Let **contract** $\subset \mathbf{GZ}_{\leq 0}$ be the full sub-category whose each object is isomorphic to a direct sum $M \oplus T$, where M is a finitely generated free $\mathbb{A}$-module (concentrated in degree 0) and $M \in \mathbf{GZ}_{\leq 0}$ is an acyclic object. The category **contract** is a full symmetric monoidal sub-category of $\mathbf{GZ}_{\leq 0}$.

It follows that every such an object $M \oplus T \in \mathbf{GZ}_{\geq 0}$ admits a truncation $\tau_{\geq 0}$, where $\tau_{\geq 0}(M \oplus T) = M$.

Therefore, we have a sequence of lax symmetric monoidal functors (enriched over **sets**):

$$\mathbf{GZtrunc} \stackrel{\tau_{\leq 0}}{\to} \mathbf{contract} \stackrel{\tau_{\geq 0}}{\to} \mathbb{A}\textbf{-freemod}_0,$$

where $\mathbb{A}$**-freemod**$_0$ is the category of finitely generated free $\mathbb{A}$-modules.

The lax structure on $\tau_{\leq 0}$ follows from the universal property of $\tau_{\leq 0}$. Indeed, $\tau_{\leq 0}A \otimes \tau_{\leq 0}B \in \mathbf{GZ}_{\leq 0}$, therefore, the natural map $\tau_{\leq 0}A \otimes \tau_{\leq 0}B \to A \otimes B$ factors through $\tau_{\leq 0}(A \otimes B)$.

Similarly, we have a natural map $\tau_{\geq 0}(A \otimes B) \to \tau_{\geq 0}A \otimes \tau_{\geq 0}B$ which is an isomorphism if $A, B \in$ **contract**, so that $\tau_{\geq 0}$ is a tensor functor.

We have embeddings as full sub-category $\mathbb{A}$**-freemod**$_0 \stackrel{I}{\to}$ **contract** $\stackrel{J}{\to}$ **GZtrunc**. Each of these embeddings is a tensor functor. By definition, I is left adjoint to $\tau_{\leq 0}$ and J is right adjoint to $\tau_{\geq 0}$ so that we have natural transformations of tensor functors

$$I\tau_{\leq 0} \to \mathrm{Id}_{\mathbf{GZtrunc}}; \quad \mathrm{Id}_{\mathbf{contract}} \to J\tau_{\geq 0}.$$

3 Filtered objects

Let $\mathcal{C}$ be a symmetric monoidal category enriched over $\mathbb{A}$-mod. Suppose $\mathcal{C}$ is $PD \bigoplus \prod$-closed. Finally, we assume that the tensor product in $\mathcal{C}$ commutes with direct sums.

3.1 Category filt$\mathcal{C}'$

Let **filt**$\mathcal{C}'$ be a dg category whose each object X is by definition a collection of objects $\mathbf{gr}^i X \in \mathcal{C}$, $i \in \mathbb{Z}$. We set

$$\mathrm{Hom}_{\mathbf{filt}\mathcal{C}'}(X, Y) = \prod_{n \leq m} \mathrm{Hom}_{\mathcal{C}}(X_n, X_m).$$

One has a SMC structure on **filt**$\mathcal{C}'$, where

$$\mathbf{gr}^n(X \otimes Y) = \bigoplus_{p=0}^{n} X^p \otimes Y^{n-p}.$$

Let us define a functor $|| : \mathbf{filt}\mathcal{C}' \to \mathcal{C}$, where

$$|X| = \bigoplus_{n<0} \mathbf{gr}^n X \oplus \prod_{n \geq 0} \mathbf{gr}^n X.$$

We call $|X|$ *the total of* X.

We have a lax tensor structure on $|X|$, that is we have a natural transformation

$$|X| \otimes |Y| \to |X \otimes Y|.$$

Indeed (we set $X_n = \mathbf{gr}^n X$; $Y_m = \mathbf{gr}^m Y$):

$$|X| \otimes |Y| = (\bigoplus_{m \le 0} X_m \otimes \bigoplus_{n \le 0} Y_m) \bigoplus (\bigoplus_{m \le 0} X_m \otimes \prod_{n \ge 0} Y_m) \bigoplus (\prod_{m \ge 0} X_m \otimes \bigoplus_{n \le 0} Y_m) \bigoplus (\prod_{m \ge 0} X_m \otimes \prod_{n \ge 0} Y_m)$$
$$\to (\bigoplus_{m,n \le 0} X_m \otimes Y_n) \oplus \bigoplus_{m<0} \prod_{n \ge 0} (X_m \otimes Y_n) \oplus \bigoplus_{n<0} \prod_{m \ge 0} X_n \otimes Y_m \oplus \prod_{n,m \ge 0} X_n \otimes Y_m$$
$$= \bigoplus_m \prod_{n \ge m} X_m \otimes Y_n \bigoplus_n$$

Next, for every $m \in \mathbb{Z}$ we have a map

$$\prod_{n \ge m} X_m \otimes Y_n = \prod_{n+m \ge 2m} X_m \otimes Y_n \to \prod_{n+k \ge 2m} \bigoplus_{k \le m} X_k \otimes Y_n \to |X \otimes Y|.$$

Likewise we have a map

$$\prod_{m > n} X_m \otimes Y_n \to |X| \otimes |Y|,$$

which finishes the construction.

Let $\mathbf{filt}\mathcal{C}'_- \subset \mathbf{filt}\mathcal{C}$ be the full sub-category of objects X satisfying: there exists an $M \in \mathbb{Z}$ such that $\mathbf{gr}^m X = 0$ for all $m > M$. The restriction of $||$ onto this sub-category is then a strict tensor functor.

For $X \in \mathbf{filt}\mathcal{C}'$ define an object $F^{\ge k} X$, where

$$\mathbf{gr}^l F^{\ge k} X = \mathbf{gr}^l X \text{ if } l \ge k;$$

$$\mathbf{gr}^l F^{\ge k} X = 0 \mathrm{if} l < k.$$

Define $F^{\le k}$ in a similar way. We have natural transformations

$$F^{\le kX} \to X \to F^{\ge k} X. \tag{4}$$

3.2 The category filt$\mathcal{C}$

Set $\mathbf{filt}\mathcal{C} := D\mathbf{filt}\mathcal{C}'$; $\mathbf{filt}\mathcal{C}_- := D\mathbf{filt}\mathcal{C}'_-$ etc. The functors $F^{\ge k}, F^{\le k}$ and the natural transformations (4) carry over to $\mathbf{filt}\mathcal{C}$. Let $(X, D) \in \mathbf{filt}\mathcal{C}$. The component of the differential D which maps X_{k-1} to X_k defines a natural transformation $\delta : F^{\le k-1} X \to F^k X[1]$ so that we have an isomorphism

$$(X, D) = \operatorname{Cone} \delta.$$

3.3 Filtered homotopy equivalences

For $(X,D) \in \mathbf{filt}\mathcal{C}$ set $\mathbf{Gr}^k X := |F^{\geq k} F^{\leq k} X| \in \mathcal{C}$. We have $\mathbf{Gr}^k X = (\mathbf{gr}^k, D_{kk})$, where $D_{kk} : X_k \to X_k$ is the component of D.

Proposition 3.1 *Suppose $\mathbf{Gr}^k X$ are acyclic. Then both X and $|X|$ are acyclic.*

Sketch of the proof Set $X_n = \mathbf{gr}^n X$; $X_m = \mathbf{gr}^m X$. By definition

$$D \in \prod_{n \geq m} \mathrm{Hom}^1(X_n, X_m) = \prod_{s \geq 0} H_s$$

where $H_s = \prod_n \mathrm{Hom}^1(X_n, X_{n+s})$. Thus we can write $D = \sum_{s \geq 0} D_s$, where $D_s \in H_s$. We are given that the object (X, D_0) is acyclic.

We are to solve an equation

$$Dh + hD = \mathrm{Id},$$

where $h \in \mathrm{Hom}^{-1}(X,X)$. Or, in the components,

$$dh_s + D_0 h_s + h_s D_0 = u_s$$

where $u_0 = \mathrm{Id}$ and for $s > 0$, $u_s = \sum_{0 < i \leq s} D_i h_{s-i} + h_{s-i} D_i$. One can resolve this system recursively by s, using the acyclicity of (X, D_0).

3.3.1 Corollary

Corollary 3.2 *Let $f : X \to Y$ be an arrow in $\mathbf{filt}\mathcal{C}$ such that all the induced maps $\mathbf{Gr}^k f : \mathbf{Gr}^k X \to \mathbf{Gr}^k Y$ are homotopy equivalences. Then f and $|f| : |X| \to |Y|$ are homotopy equivalences.*

Set $|X| = (\bigoplus_{n \geq 0} \mathbf{gr}^n X, D)$ We have thereby a strict symmetric monoidal functor $\mathbf{filt}\mathcal{C} \to \mathcal{C}$.

3.4 Derived Tensor product

Let $F : \mathcal{C} \to \mathbf{swell}\,\mathcal{U}$ and $G : \mathcal{C}^{\mathbf{op}} \to \mathbf{swell}\,\mathcal{V}$ be functors between **GZ-** categories (that is cateories enriched over **GZ**). $\mathcal{C}$ may be a non-unital category.

Define an object $F \otimes^L G \in \mathbf{swell}(\mathcal{U} \otimes \mathcal{V})$ as follows.

For $N \geq 0$, set

$$\mathbf{gr}^{-N} \otimes^L (F,G) := \bigoplus_{C_0, C_1, \ldots, C_N} F(C_0) \otimes \mathrm{Hom}(C_0, C_1) \otimes \cdots \otimes \mathrm{Hom}(C_{N-1}, C_N) \boxtimes G(C_N) \in \mathbf{swell}(\mathcal{U} \otimes \mathcal{V}).$$

We have the standard bar-differential on $\otimes^L(F,G)$ which gives rise to an object $(\otimes^L(F,G), D) \in \mathbf{filt}\,\mathbf{swell}(\mathcal{U} \otimes \mathcal{V})$. Set $F \otimes^L G := (\otimes^L(F,G), D)$.

3.4.1 Relative derived tensor product

Let $F : \mathcal{D}\otimes\mathcal{C} \to \mathbf{swell}\,\mathcal{U}$, $G : \mathcal{C}^{\mathbf{op}}\otimes\mathcal{E} \to \mathbf{swell}\,\mathcal{V}$. Let $d \in \mathcal{D}$; $e \in \mathcal{E}$. Let $F_d : \mathcal{C} \to \mathbf{swell}\,\mathcal{U}$; $G_e : \mathcal{C}^{\mathbf{op}} \to \mathbf{swell}\,\mathcal{V}$ be the restrictions. Let us define a functor $F\otimes^L_{\mathcal{C}} G : \mathcal{D}\otimes\mathcal{E} \to \mathbf{swell}(\mathcal{U}\otimes\mathcal{V})$, where

$$F\otimes^L_{\mathcal{C}} G(d,e) := F_d \otimes^L G_e.$$

3.4.2

Let I be a poset. Denote by the same symbol I a non-unital category, where $\mathrm{Hom}_I(i,j) = \mathbb{Z}$ if $i < j$, and $\mathrm{Hom}_I(i,j) = 0$ otherwise. Let I be a finite poset and let $F : I \to \mathbf{swell}\,\mathcal{U}$; $G : I^{\mathbf{op}} \to \mathbf{swell}\,\mathcal{V}$. Then we have $\mathbf{gr}^{-N}\otimes^L (F,G) = 0$ if N exceeds the number of elements in I.

3.5 Hocolim

Let $\mathcal{C}$ be a $\mathbf{GZ}$-category and I be a small category. Let J be the $\mathbb{A}$-span of I. Let $\mathbf{const} : J^{\mathbf{op}} \to \mathbf{GZ}$ be the constant functor, $\mathbf{const}^{\mathbf{op}}(j) = \mathbb{Z}$. Let $F : I \to \mathbf{swell}(\mathcal{C})$ be a functor. Still denote by F its extension $F : J \to \mathbf{swell}(\mathcal{C})$.

Set

$$\mathrm{hocolim}_I F := F\otimes^L \mathbf{const}.$$

3.6 Derived Hom

Let $F : \mathcal{C} \to \mathbf{swell}\,\mathcal{U}$ and $G : \mathcal{C} \to \mathbf{swell}\,\mathcal{V}$ be dg functors between $\mathbf{GZ}$- categories.

Define an object $\mathrm{RHom}_{\mathcal{C}}(F,G) \in \mathbf{swell}(\mathcal{U}^{\mathbf{op}}\otimes\mathcal{V})$ as follows. For $N \geq 0$, set

$$\mathbf{gr}^N\,\mathrm{RHom}(F,G) := \bigoplus_{C_0,C_1,\ldots,C_N} \underline{\mathrm{Hom}}(F(C_0)\otimes\mathrm{Hom}(C_0,C_1)\otimes\cdots\otimes\mathrm{Hom}(C_{N-1},C_N); G(C_N)) \in \mathbf{swell}(\mathcal{U}^{\mathbf{op}}\otimes\mathcal{V})$$

We have the standard bar-differential D on $\mathrm{RHom}(F,G)$. Still denote

$$R\mathrm{Hom}(F,G) := |\,\mathrm{RHom}(F,G)| \in \mathbf{swell}(\mathcal{U}^{\mathbf{op}};\mathcal{V}).$$

3.7 Holim

Let $\mathcal{C}$ be a $\mathbf{GZ}$-category and I be a small category. Let J be the $\mathbb{A}$-span of I. Let $C : J \to \mathbf{GZ}$ be the constant functor, $C(j) = \mathbb{Z}$. Let $F : I \to \mathbf{swell}(\mathcal{C})$ be a functor. Denote by the same letter the extension of F onto J. Set

$$\mathrm{holim}_I F := \mathrm{RHom}(C,F) \in \mathbf{swell}\,\mathcal{C}.$$

3.7.1 Homotopy stability

Let $F, H : \mathcal{C} \to \mathcal{U}$; $G : \mathcal{C}^{\mathbf{op}} \to \mathcal{V}$.

Proposition 3.3 *Supppose $F(c)$ is acyclic for all $c \in \mathcal{C}$. Then $F \otimes^L G$, $\mathrm{RHom}(F, H)$, and $\mathrm{RHom}(H, F)$ are acyclic.*

The Proposition follows from Prop. 3.1.

Corollary 3.4 *If $F(i)$ is acyclic for all $i \in I$, then so are $\operatorname{holim} F$ and $\operatorname{hocolim} F$.*

3.7.2 Functoriality

Let $f : I \to J$, $F : J \to C$ be functors. We have natural maps

$$f^* : \operatorname{holim}(F) \to \operatorname{holim}(Ff); \quad f_! : \operatorname{hocolim}(Ff) \to \operatorname{hocolim}(F).$$

Suppose $g : J \to I$ is a right (or left) adjoint to f. Then $f_!$ and $g_!$ are quasi-inverse to each other, same for f^* and g^*.

3.8 Filtered limits and colimits

Recall that a poset I is called *filtered* if for every finite subset $S \subset I$ there exists an $i \in I$ such that $i \geq s$ for all $s \in S$, such an i is called *an upper bound of S*. A subset $J \subset I$ is called *co-final* if every finite subset $S \subset I$ has an upper bound from J.

Let $\iota : J \to I$ be the embedding and let $F : I \to \mathcal{C}$ be a functor. We have a natural map

$$\iota_! : \operatorname{hocolim}_I F \circ i \to \operatorname{hocolim}_J F.$$

Proposition 3.5 *The map $\iota_!$ is a homotopy equivalence.*

Sketch of the proof. Still denote by J, I the $\mathbb{A}$-spans of J, I. 1. Let $h : J \otimes I^{\mathbf{op}} \to \mathbf{GZ}$; $h(j, i) = \mathrm{Hom}_I(i, j)$. We have a term-wise quasi-isomorphism of functors $J \to \mathcal{C}$

$$F \otimes_I^L h \to F \circ \iota.$$

2) We have natural map

$$h \otimes_J^L \mathrm{const}_J \to \mathrm{const}_I.$$

This map is a quasi-isomorphism of functors. Indeed, for each $i \in I$, we need to prove that the natural map

$$h(i, -) \otimes_J^L \mathrm{const}_J \to \mathbb{A} \tag{5}$$

is a homotopy equivalence.

2.1) We have an obvious embedding $I : \mathbf{Ab} \to \mathbf{GZ}$, where $\mathbf{Ab}$ is the category of complexes of free abelian groups bounded from above.

The map in (5) can be obtained from a similar map in **Ab** under I. The corresponding map in **Ab** is known to be a homotopy equivalence because it is isomorphic to the natural map

$$\mathrm{hocolim}_{j\in J; j\geq i}\, \mathbb{A} \to \mathbb{A}.$$

2.2) We have a commutative diagram

$$\begin{array}{ccc} F\circ i \otimes^L_J \mathrm{const}_J & \xrightarrow{i_!} & F\otimes^L_I \mathrm{const}_I \\ \sim\uparrow & \nearrow\sim & \\ F\otimes^L_I h \otimes^L_J \mathrm{const}_J & & \end{array}$$

which implies that the horizontal arrow is a homotopy equivalence.

3.8.1 Constant functor on a poset with the least element

Proposition 3.6 *Let I be a poset with the least element. Then the natural map* $\mathrm{hocolim}_I\, \mathbb{A} \to \mathbb{A}$ *is a homotopy equivalence.*

Sketch of the proof We have an isomorphism $\mathrm{const}_I(-) = \mathrm{Hom}_I(x, -)$. Therefore, we have a homotopy equivalence

$$\mathrm{Hom}_I(x, -) \otimes^L \mathrm{const}_{I^{\mathbf{op}}} \xrightarrow{\sim} \mathbb{A}(x) = \mathbb{A}.$$

3.8.2 Constant functor on a filtered poset

Proposition 3.7 *Let I be a filtered poset. Then the natural map*

$$\mathrm{hocolim}_{i\in I}\, \mathbb{A} \to \mathbb{A}$$

is a homopy equivalence.

Sketch of the proof Let $x \in I$ and let $I_{\geq x} \subset I$ consist of all $y \in I$, $y \geq x$. The subset $I_{\geq x}$ is cofinal. Consider the through map

$$\mathrm{hocolim}_{i\in I_x}\, \mathbb{A} \xrightarrow{\sim} \mathrm{hocolim}_{i\in I}\, \mathbb{A} \to \mathbb{A}.$$

It is a homotopy equivalence by the previous subsection. This implies the statement.

3.8.3 Reduction to the colimit over the set of all finite subsets

Let I be a poset. Let $P(I)$ be the poset of all non-empty finite subsets of I ordered with respect to the inclusion.

Let $F : I \to \mathcal{C}$ be a functor. Let $PF : P(I) \to \mathcal{C}$ be defined by

$$PF(S) := \mathrm{hocolim}_{s\in S}\, F(s)$$

Let $Q(I) \subset P(I)$ consist of all subsets S possessing the greatest element. Denote by $\mu(S)$ the greatest element of S. We then have a monotone map

$$\mu : Q(I) \to I.$$

We have a natural transformation $\varepsilon : PF|_{Q(I)} \to \mu^{-1}F$ of functors $Q(I) \to \mathcal{C}$. For every $S \in Q(I)$, μ induces a homotopy equivalence in $\mathcal{C}$

$$PF(\mu(S)) \xrightarrow{\sim} \mu^{-1}F(S) = F(\mu(S)). \tag{6}$$

The map μ induces maps

$$\operatorname{hocolim}_{Q(I)} PF \to \operatorname{hocolim}_{Q(I)} \mu^{-1}F \to \operatorname{hocolim}_I F. \tag{7}$$

The left arrow is a homotopy equivalence by (6). Let us show that the right arrow is a homotopy equivalence. It suffices to check it for $F(-) = \mathbb{A}[\operatorname{Hom}_I(i, -)]$, $i \in I$. Let $Z \subset Q(I)$ consist of all S with $\mu(S) \geq i$. The problem reduces to showing that the following map

$$\operatorname{hocolim}_Z \mathbb{A} \to \operatorname{hocolim}_{I_{\geq i}} \mathbb{A}$$

induced by μ is a homotopy equivalence which follows from the fact that Z is filtered and $I_{\geq i}$ has the least element so that both the natural map

$$\operatorname{hocolim}_{I_{\geq i}} \mathbb{A} \to \mathbb{A}$$

and the through map

$$\operatorname{hocolim}_Z \mathbb{A} \to \operatorname{hocolim}_{I_{\geq i}} \mathbb{A} \to \mathbb{A}$$

is a homotopy equivalence, whence the statement.

Thus, *the through map (7) is a homotopy equivalence.*

3.8.4 Nilpotent functors

Let I be a filtered poset and $F : I \to \mathcal{C}$ a functor. Call F *nilpotent* if for every $x \in I$ there exists a $y \in I$, $y \geq x$ such that the map $F(x) \to F(y)$ is homotopy equivalent to 0.

Theorem 3.8 *Let F be nilpotent. Then* $\operatorname{hocolim}_I F$ *is acyclic.*

Sketch of the proof

A. According to the previous subsection it suffices to show that $\operatorname{hocolim}_{Q(I)} PF$ is acyclic. It follows that $PF : Q(I) \to \mathcal{C}$ is nilpotent. Thus, replacing I with $Q(I)$ and F with PF allows us to assume without loss of generality that for every element $x \in I$ the set $I_{\leq x} := \{y | y \leq x\}$ is finite.

B. Using induction by $\# I_{\leq x}$, one can show that there exists a monotone map $\phi : I \to I$ such that $\phi(x) \geq x$ for all x and the natural map $F(x) \to F(\phi(x))$ is homotopy equivalent to 0 for all $x \in I$.

C. Let $G : I \to \mathcal{C}$ be an arbitrary functor. Show that the natural map

$$\operatorname{hocolim}_{x \in I} F(x) \to \operatorname{hocolim}_{x \in I} F(\phi(x))$$

is a homotopy equivalence.

It suffices to prove the statement for $F(x) = \mathrm{Hom}(i, x)$, $i \in I$. One can reprlace I with a cofinal subset $I_{\geq i}$, in which case all the maps $F(x) \to F(\phi(x))$ are isomorphisms, whence the statement.

D. Set $F^n : I \to \mathcal{C}$,

$$F^n(x) = F(\underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}(x)).$$

We have natural maps

$$i_n : F^n \to F^{n+1}, \quad n \geq 0.$$

It follows that the induced map

$$\mathrm{hocolim}_I\, F^n \to \mathrm{hocolim}\, F^{n+1}$$

is a homotopy equivalence. Therefore, $\mathrm{hocolim}_I\, F$ is homotopy equivalent to

$$\mathrm{hocolim}_n\, \mathrm{hocolim}_I\, F^n = \mathrm{hocolim}_I\, \mathrm{hocolim}_n\, F^n.$$

It also follows that the induced map $F^n(x) \to F^{n+1}(x)$ is homotopy equivalent to 0. This implies that $\mathrm{hocolim}_n\, F^n(x)$ is acyclic for every x. Therefore, the natural map

$$\mathrm{hocolim}_I\, \mathrm{hocolim}_n\, F^n \to \mathrm{hocolim}_I\, 0 = 0$$

is a homotopy equivalence, as we wanted.

3.9 Stability of a functor

Let $I_{\mathcal{C}} : \mathcal{C} \to \mathbf{swell}\,\mathcal{C}$ and $J_{\mathcal{C}} : \mathcal{C}^{\mathbf{op}} \to \mathbf{swell}(\mathcal{C}^{\mathbf{op}})$ be the embedding functors. Set

$$\Delta_{\mathcal{C}} := J_{\mathcal{C}} \otimes^L_{\mathcal{C}} I_{\mathcal{C}} \in \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{C}).$$

For every $S \in \mathbf{swell}(\mathcal{C})$ we have a narural map

$$S \circ \Delta_{\mathcal{C}} \to S.$$

Call S *stable* if this map is a homotopy equivalence.

3.9.1 Equivalent definition

The hom-functor $\mathrm{Hom} : \mathcal{C}^{\mathbf{op}} \times \mathcal{C} \to \mathbf{GZ}$ extends naturally to a functor

$$\mathrm{Hom} : \mathcal{C}^{\mathbf{op}} \otimes \mathbf{swell}\,\mathcal{C} \to \mathbf{GZ}.$$

For $S \in \mathbf{swell}\,\mathcal{C}$ we thus get a functor $h_S : \mathcal{C}^{\mathbf{op}} \to \mathbf{GZ}$. Let $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ be the identity functor. Set $R(S) := h_S \otimes^L \mathrm{Id}_{\mathcal{C}} \in \mathbf{swell}(\mathcal{C})$.

We have a natural map $R(S) \to S$. S is stable iff this map is a homotopy equivalence.

3.9.2

Let $F : \mathcal{C}^{\mathbf{op}} \to \mathbf{GZ}$ be a $\mathbf{GZ}$- functor. Set

$$\mathcal{R}(F) := F \otimes^L \mathrm{Id}_{\mathcal{C}}. \tag{8}$$

We have $S \circ \Delta_{\mathcal{C}} = \mathcal{R}(h_S)$ for any $S \in \mathbf{swell}(\mathcal{C})$.

Proposition 3.9 *Every object of the form $\mathcal{R}(F)$ is stable.*

Follows from the associativity of $\otimes$.

3.9.3

Let $X \in \mathbf{swell}(C^{\mathbf{op}} \otimes D)$ and $Y \in \mathbf{swell}(D^{\mathbf{op}} \otimes E)$ be stable, then so is $X \circ Y$.

3.9.4

Let $F : C \to D$ be a functor. Let $X \in \mathbf{swell}(C)$ be a stable functor. Then the functor $\mathbf{swell}(F)X$ is stable.

Indeed, we have $\mathbf{swell}(F)(X \circ \Delta_C) = (X \circ J_C) \otimes^L_C (\mathbf{swell}(F)I_C)$. The natural transformation

$$(\mathbf{swell}(F)I_C) \circ \Delta_D \to \mathbf{swell}(F)I_C$$

of functors $C \to \mathbf{swell}\, D$ is a term-wise weak equivalence. This implies the statement.

4 Classical categories

4.1 Categories Q_ε, Q_∞

Let ε be a positive real number or ∞. Let Q_ε be the following category enriched over the category $\mathbb{A}$-**freemod**. Set $\mathbf{Ob}\, Q_\varepsilon := \mathbb{R}$ Denote by $\mathbf{e}_a$ the object of Q_ε corresponding to a real number a. Set $\mathrm{Hom}(\mathbf{e}_a, \mathbf{e}_b) = \mathbb{Z}$ if $a \le b < a + \varepsilon$. Set $\mathrm{Hom}(\mathbf{e}_a, \mathbf{e}_b) = 0$ otherwise.

We have an SMC structure on Q_ε via $e_a \otimes e_b = e_{a+b}$. The categories Q_ε have internal hom. We have strict tensor functors $\mathbf{red} : Q_{\varepsilon_1} \to Q_{\varepsilon_2}$, $\varepsilon_1 \ge \varepsilon_2$.

4.1.1 The category Q_ω

Let Q_ω be the union of all Q_ε, $\varepsilon \in \{1, 1/2, 1/4, \ldots, 1/2^n, \ldots\} \cup \{\infty\}$. Let us define hom. Let $\mathbf{e}^{\varepsilon_1}_a \in Q_{\varepsilon_1}$ and $\mathbf{e}^{\varepsilon_2}_b \in Q_{\varepsilon_2}$. Set $\mathrm{Hom}(\mathbf{e}^{\varepsilon_1}_a, \mathbf{e}^{\varepsilon_2}_b) = 0$ if $\varepsilon_1 < \varepsilon_2$. Otherwise, set

$$\mathrm{Hom}(\mathbf{e}^{\varepsilon_1}_a, \mathbf{e}^{\varepsilon_2}_b) = \mathrm{Hom}_{Q_{\varepsilon_2}}(\mathbf{e}_a, \mathbf{e}_b).$$

We also have an SMC structure on Q_ω, where

$$e^{\varepsilon_1}_a \otimes e^{\varepsilon_2}_b := e^{\min(\varepsilon_1, \varepsilon_2)}_{a+b}.$$

We also have an internal hom
$$\underline{\mathrm{Hom}}(\mathbf{e}_a^{\varepsilon_1};\mathbf{e}_b^{\varepsilon_2})=0$$
if $\varepsilon_1\leq\varepsilon_2$. Otherwise,
$$\underline{\mathrm{Hom}}(\mathbf{e}_a^{\varepsilon_1};\mathbf{e}_b^{\varepsilon_2})=\mathbf{e}_{b-a}^{\varepsilon_2}.$$

4.1.2 The regularized categories $\mathbf{R}_{1/2^n}$, $\mathbf{R}_\omega$

Let $R_{1/2^n}\subset Q_{1/2^n}$ be the full sub-category consisting of all objects of the form $e_{m/2^n}$, $m\in\mathbb{Z}$. The sub-category $R_{1/2^n}$ is discrete and closed under the tensor product. The embedding $I_{1/2^n}:R_{1/2^n}\to Q_{1/2^n}$ has a right adjoint, to be denoted by $\mathbf{pr}_{1/2^n}$, where $\mathbf{pr}_{1/2^n}e_a=e_{m/2^n}$, where m is the largest integer satisfying $m/2^n\leq a$. Let $\mathbf{R}_{1/2^n}:=\mathbf{swell}\,R_{1/2^n}$, the functors $I,\mathbf{pr}$ extend to functors $I_{1/2^n}:\mathbf{R}_{1/2^n}\to\mathcal{Q}_{1/2^n}$, $\mathbf{pr}_{1/2^n}:\mathcal{Q}_{1/2^n}\to\mathbf{R}_{1/2^n}$.

Let us define a full sub-category of $R_\omega\subset Q_\omega$ consisting of all objects of the form $e_{m/2^n}^{1/2^n}$, $n=0,1,2,\ldots$, $m\in\mathbb{Z}$. R_ω is closed under the tensor product so that the embedding $I:R_\omega\to Q_\omega$ is a tensor functor. The functor I has a right adjoint, to be denoted by $\mathbf{pr}$, where $\mathbf{pr}(e_a^{1/2^n})=e_{m/2^n}^{1/2^n}$, where m is the largest integer satisfying $m/2^n\leq a$. We have a lax tensor structure on $\mathbf{pr}$ i.e. a natural transformation
$$\mathbf{pr}(X)\otimes\mathbf{pr}(Y)\to\mathbf{pr}(X\otimes Y)$$
satisfying the associativity condition.

Let $\mathbf{R}_\omega:=\mathbf{swell}\,R_\omega$. The functor $\mathbf{pr}$ extends to a lax tensor functor $\mathbf{pr}:\mathcal{Q}_\omega\to\mathbf{R}_\omega$. Via $\mathbf{pr}$, the SMC Q_ω^+ is enriched over the category $\mathcal{Q}_\omega$.

4.1.3 A Hopf algebra ℓ in $\mathbf{R}_\omega$

Let P be the set of all numbers of the form $m/2^n$, $m>0$, $n\geq 0$. For $a\in P$ let $\mathbf{den}(a):=1/2^n$, where n is the smallest non-negative integer such that $2^na\in\mathbb{Z}$. Let $\mathcal{V}\in\mathcal{Q}_\omega$ be defined by $\mathcal{V}:=\prod_{a\in P}\lambda_a$, where $\lambda_a:=e_a^{\mathbf{den}(a)}$. Let $p_a:\mathcal{V}\to f_a$ be the projection. Let
$$D_{b,a-b}^a:f_a\to f_b\otimes f_{a-b},\quad 0<a<b$$
be the natural map.

Let $D:\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}$ be defined as $D=\sum_{0<a<b}D_a^{b,a-b}\mathbf{pr}_a$.

Let
$$\ell:=\bigoplus_{k=0}^{\infty}(\mathcal{V}[-1])^{\otimes k}.$$
We have an obvious Hopf algebra structure on ℓ, where the product is the concatenation and the co-product is given by requiring that $\mathcal{V}[-1]$ is primitive.

4.1.4 ℓ-modules in $\mathbf{R}_\omega$: the category $\mathbf{R}_q$

Let R_q be the following category: its objects are of the form $f^{1/2^n}_{m/2^n}$, $m,n \in \mathbb{Z}, n \geq 0$. and $\mathrm{Hom}(f^{1/2^n}_{m/2^n}, f^{1/2^N}_{M/2^N}) = \mathbb{A}$ whenever $n \leq N$ and $m/2^n \leq M/2^N$; $\mathrm{Hom}(f^{1/2^n}_{m/2^n}, f^{1/2^N}_{M/2^N}) = 0$ otherwise.

It is clear that every ℓ-module X in $\mathbf{R}_\omega$ gives rise to an object of $\mathbf{swell}\, R_q$, to be denoted by $[X]$.

Let $\mathbf{R}_q$ be the category, enriched over $\mathbf{GZ}$, whose every object is a ℓ-module in $\mathbf{R}_\omega$ and we set

$$\mathrm{Hom}_{\mathbf{R}_q}(X,Y) := \mathrm{Hom}_{\mathbf{swell}\, R_q}([X],[Y]).$$

We have a tensor structure on $\mathbf{R}_q$, where we let $X \otimes Y$ to be the same as in the category $\mathbf{R}_\omega$ with the induced ℓ-module structure (coming from the co-product on ℓ). This tensor structure admits an inner hom, again borrowed from $\mathbf{R}_\omega$.

4.1.5 Tensor functor $\mathcal{Q}_\infty \to \mathbf{R}_q$

Let $Q^{1/2^n}_\infty \subset Q_\infty$ be the full sub-category formed by all objects of the form $e_{m/2^n}$, $m \in \mathbb{Z}$. We have a right adjoint functor to the embedding $p_n : Q_\infty \to Q^{1/2^n}_\infty$, where $p_n(e_a) = e_{m/2^n}$ and m is the largest integer such that $m/2^n \leq a$. We have an embedding $i_n : Q^{1/2^n}_\infty \to R_q$, $i_n(e_{m/2^n}) = f^{1/2^n}_{m/2^n}$. Let $\pi_n : \mathcal{Q}_\infty \to \mathbf{R}_q$ be induced by $i_n p_n$. We have a tensor structure on π_n.

We have a natural transformation of tensor functors $\pi_n \to \pi_{n+1}$. Set $\pi(X) = \mathrm{hocolim}_n\, \pi_n(X)$. We have an induced tensor structure on π. Via π, every category enriched over $\mathcal{Q}_\infty$ is enriched over $\mathbf{R}_q$.

PART 2. SHEAVES

5 The category of sheaves

We fix a ground SMC C enriched over the category of finite complexes of finitely generated free $\mathbb{A}$-modules.

Let X be a locally compact topological space. Let Open_X be the category whose objects are open sub-sets of X and we have a unique arrow $U \to V$ iff $U \subset V$. We denote by the same symbol the $\mathbb{A}$-span of Open_X.

Similarly, denote by $\mathrm{precompact}_X$ the poset of all open precompact sets in X.

5.1 Pre-sheaves

Denote $\mathrm{psh}(X,C) := \mathbf{swell}(\mathrm{Open}_X^{\mathbf{op}} \otimes C)$; $\mathrm{psh}(X) := \mathbf{swell}(\mathrm{Open}_X^{\mathbf{op}})$.

5.2 Coverings

Let $U \in \mathrm{Open}_X$. *A covering of* U is a subset $\mathcal{U} \subset \mathrm{Open}_U$ satisfying:

— $\mathcal{U}$ is closed under finite intersections;

— the union of all elements in $\mathcal{U}$ is U.

5.3 Various gluing conditions

5.3.1 Meyer-Vietoris Condition

Let $F : \mathrm{Open}_X \to \mathbf{swell}\, C$ be a functor. Say that F satisfies the Meyer-Vietoris conditon if, given a pair of open subsets U, V of X, the total of the complex

$$0 \to F(U \cap V) \to F(U) \oplus F(V) \to F(U \cap V) \to 0 \tag{9}$$

is homotopy equivalent to 0.

5.3.2 Coverings

Let U be an open subset of X. Let $\mathcal{U}$ be a family of open subsets of U whose union is U and which is closed under finite intersections.

We have an induced poset structure on $\mathcal{U}$ as well as an embedding $I_{\mathcal{U}} : \mathcal{U} \to \mathrm{Open}_X$. Call $\mathcal{U}$ a finite covering if $\mathcal{U}$ is a finite set.

5.3.3 Finite covering condition

Let $F : \mathrm{Open}_X \to \mathbf{swell}\, C$ be a functor. Let U be an open subset and $\mathcal{U}$ be its covering. We say that F satisfies the gluing condition with respect to $\mathcal{U}$ if the natural map

$$\mathrm{hocolim}_{\mathcal{U}}\, F \to F(U)$$

is a homotopy equivalence. The Myer-Vietoris condition (9) is equivalent to the gluing condition with respect to the covering $\{U, V, U \cap V\}$ of the set $U \cup V$ (where some of the sets $U, V, U \cap V$ may concide).

Proposition 5.1 *Suppose F satisfies the Meyer-Vietoris condition and $F(\emptyset) \sim 0$. Then F satisfies the gluing condition for any finite covering $\mathcal{U}$.*

Sketch of the proof Let $\mathcal{U}$ be a covering of U. Say that a subset $M \subset \mathcal{U}$ generates $\mathcal{U}$ if every element of $\mathcal{U}$ is a finite intersection of a finite number of elements from M.

Let us use induction by the number of elements in M. If M consists of one element, the statement is obvious.

Let now $M = \{U_1, U_2, \ldots, U_{N-1}\}$. Let $V := U_1 \cup U_2 \cup \cdots \cup U_{N-1}$. Let $\mathcal{V}$ be the covering of V generated by $U_1, U_2, \ldots, U_{N-1}$. Let $\mathcal{W}$ be the covering of $V \cap U_N$ generated by $U_1 \cap U_N, U_2 \cap U_N, \ldots, U_{N-1} \cap U_N$.

We have a complex:

$$0 \to \mathrm{hocolim}_{\mathcal{W}}\, F \to \mathrm{hocolim}_{\mathcal{V}}\, F \oplus F(U_N) \to \mathrm{hocolim}_{\mathcal{U}}\, F \to 0$$

whose totalization is acyclic for any functor $F : \mathrm{Open}_X \to \mathbf{swell}\, C$. We also have a map of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{hocolim}_{\mathcal{W}}\, F & \longrightarrow & \mathrm{hocolim}_{\mathcal{V}}\, F \oplus F(U_N) & \longrightarrow & \mathrm{hocolim}_{\mathcal{U}}\, F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(V \cap U_N) & \longrightarrow & F(V) \oplus F(U_N) & \longrightarrow & F(U) & \longrightarrow & 0 \end{array}$$

By the induction assumption all the vertical arrows except the rightmost one are homotopy equivalences. The bottom line is an acyclic complex by Meyer-Vietoris. Hence, the rightmost vertical arrow is a homotopy equivalence, which prove the induction transition.

5.3.4 Direct limit condition

We say that a functor $F : \mathrm{Open}_X \to \mathbf{swell}\, C$ satisfies the direct limit condition if given any filtered poset I and any monotone map $U : I \to \mathrm{Open}_X$, the natural map

$$\mathrm{hocolim}_{i\in I} F(U_i) \to F(\bigcup_{i\in I} U_i)$$

is a homotopy equivalence.

5.3.5

$F : \mathrm{Open}_X \to \mathbf{swell}\, C$ satisfies the gluing condition for any covering $\mathcal{U}$ iff $F(\emptyset) \sim 0$, F satisfies the Meyer-Vietoris condition and the direct limit condition.

5.4 Definition of a sheaf

Let $\mathrm{sh}(X, C) \subset \mathrm{psh}(X, C)$ be the full sub-category consisting of all objects F satisfying:

— F is stable;

— h_F satisfies the gluing condition for all coverings of all open subsets of X.

5.5 sections supported on a compact set

Let $K \in \mathrm{compact}_X$. Denote

$$\Gamma_K(F) := \mathrm{holim}_{U\in \mathrm{Open}_X; K\subset U}\, h_F(U).$$

We have

$$\Gamma_K(F) = \mathrm{Hom}_{\mathrm{psh}(X)}(\mathrm{hocolim}_{U\in \mathrm{Open}_X; K\subset U}\, U; F).$$

5.6 Representability

Let us define an object $\mathbb{A}_K \in \mathrm{sh}(X)$, for every $K \in \mathrm{compact}_X$, with the property that we have a natural transformation of functors $\mathrm{psh}(X, C) \to \mathbf{swell}\, C$:

$$\mathrm{Hom}(\mathbb{A}_; -) \to \Gamma_K(-).$$

which induces a homotopy equivalence

$$\mathrm{Hom}(\mathbb{A}_K; F) \to \Gamma_K(F)$$

whenever $F \in \mathrm{sh}(X, C)$.

5.6.1 Finite coverings of K

A finite covering of K is a finite subset $\mathcal{U} \subset \mathrm{Open}_X$ satisfying:

— every element of $\mathcal{U}$ is a precompact subset of X;

— the union of all elements in $\mathcal{U}$ contains K;

— $\mathcal{U}$ is closed under intersection.

Denote by $\mathbf{Cov}_K$ the set of all finite coverings of K.

5.6.2 A pre-sheaf $\mathbb{A}_{\mathcal{U}}$

For $\mathcal{U} \in \mathbf{Cov}_K$ set

$$\mathbb{A}_{\mathcal{U}} := \mathrm{holim}_{U \in \mathcal{U}} \mathbb{A}_U \in \mathrm{psh}(X).$$

denote by $\iota_X : X \to \mathbb{A}_{\mathcal{U}}$ the natural map.

5.6.3 Cap-product

Denote by $\cap : \mathrm{Open}_X \times \mathrm{Open}_X \to \mathrm{Open}_X$ the following functor: $\cap(U, V) = U \cap V$. This functor extends naturally to a functor

$$\cap : \mathrm{psh}(X) \otimes \mathrm{psh}(X) \to \mathrm{psh}(X).$$

This functor gives a tensor structure on $\mathrm{psh}(X)$. The unit of this structure is X.

5.6.4 Definition of $\mathbb{A}'_K$.

Let $S(K)$ be the poset of finite subsets of $\mathbf{Cov}_X$. For $I \in S(X)$, set

$$\mathbb{A}_I := \bigcap_{\mathcal{U} \in I} \mathbb{A}_{\mathcal{U}}$$

Let $I \subset J$. We then have an induced map $k_{IJ} : \mathbb{A}_I \to \mathbb{A}_J$ given by

$$\mathbb{A}_I = \bigcap_{\mathcal{U} \in I} \mathbb{A}_{\mathcal{U}} \cap \bigcap_{\mathcal{U} \in J \setminus I} X \to \bigcap_{\mathcal{U} \in J} \mathbb{A}_{\mathcal{U}},$$

which is induced by the maps $\iota_{\mathcal{U}} : X \to \mathbb{A}_{\mathcal{U}}, \mathcal{U} \in J \setminus I$.

It is clear that $k_{JK} k_{IJ} = k_{IK}$, $I \subset J \subset K$. Therefore, $\mathbb{A}_- : S(K) \to \mathrm{psh}(X)$ is a functor.

Set

$$\mathbb{A}'_K := \mathrm{hocolim}_{I \in S(K)} \mathbb{A}_I. \tag{10}$$

5.6.5 Lemma

Let $\mathcal{U}, \mathcal{V} \in \mathbf{Cov}_K$. Write $\mathcal{U} \leq \mathcal{V}$ if for every $U \in \mathcal{U}$ there exists a $V \in \mathcal{V}$ such that $U \subset V$.

Lemma 5.2 *Suppose $\mathcal{U} \leq \mathcal{V}$. Then the natural map*

$$\mathbb{A}_{\mathcal{U}} \to \mathbb{A}_{\mathcal{U}} \cap \mathbb{A}_{\mathcal{V}}$$

is a homotopy equivalence in $\mathrm{psh}(X)$.

Sketch of the proof The above map reads as

$$\mathrm{holim}_{U \in \mathcal{U}} U \to \mathrm{holim}_{U \in \mathcal{U}} \mathrm{holim}_{V \in \mathcal{V}} U \to \mathrm{holim}_{U \in \mathcal{U}} \mathrm{holim}_{V \in \mathcal{V}} U \cap V$$

Therefore, it suffices to show that for every $U \in \mathcal{U}$ the natural map

$$U \to \mathrm{hocolim}_{V \in \mathcal{V}} U \cap V$$

is a homotopy equivalence.

Denote by $\mathcal{W} \subset \mathrm{Open}_U$ the sub-set consisting of all subsets of the form $U \cap V$. We have a functor $\psi : \mathcal{V} \to \mathcal{W}$, $\psi(V) = U \cap V$. Let $I : \mathcal{W}^{\mathbf{op}} \to \mathrm{Open}_X^{\mathbf{op}}$ be the embedding.

Let $\Psi : \mathcal{W}^{\mathbf{op}} \times \mathcal{V} \to \mathbf{GZ}$ be given by $\Psi(W, V) = \mathrm{Hom}_{\mathcal{W}}(W; \psi(V))$. We have

$$I \circ \psi = \mathrm{Hom}_W(\Psi; I).$$

Whence a homotopy equivalence

$$\mathrm{holim}_{V \in \mathcal{V}} U \cap V = \mathrm{RHom}_{\mathcal{V}^{\mathbf{op}}}(\mathbb{A}; \mathrm{Hom}_W(\Psi; I)) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{W}^{\mathbf{op}}}(\mathbb{A}_{\mathcal{V}} \otimes^L_{\mathcal{V}} \Psi; I).$$

The natural map $\mathbb{A}_{\mathcal{V}} \otimes^L_{\mathcal{V}} \Psi \to \mathbb{A}_{\mathcal{W}}$ is a homotopy equivalence, because for every $W \in \mathcal{W}$ we have

$$\mathbb{A}_{\mathcal{V}} \otimes^L_{\mathcal{V}} \Psi(W) = \mathrm{hocolim}_{V \in V; V \cap U \supset W} \xrightarrow{\sim} \mathbb{A},$$

and there exists the least element in $\mathcal{V}$ containing W. Thus, we have a homotopy equivalence

$$I(U) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{W}^{\mathbf{op}}}(\mathbb{A}_{\mathcal{W}}; I) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{W}^{\mathbf{op}}}(\mathbb{A}_{\mathcal{V}} \otimes^L_{\mathcal{V}} \Psi; I)$$

because $U \in \mathcal{W}$ is the greatest element.

Corollary 5.3 *Let $S \in S(K)$ and let $\mathcal{V} \in S$ satisfy $\mathcal{V} \leq \mathcal{U}$ for all $\mathcal{U} \in S$. Then the natural map*

$$\mathbb{A}_{\mathcal{V}} \to \mathbb{A}_S$$

is a homotopy equivalence.

Lemma 5.4 *Let $I \in \mathbf{Cov}_X$. The two maps $i_1 : \mathbb{A}_I = \mathbb{A}_I \cap X \to \mathbb{A}_I \cap \mathbb{A}_I$ and $i_2 : \mathbb{A}_I = X \cap \mathbb{A}_I \to \mathbb{A}_I \cap \mathbb{A}_I$ are homotopy equivalent.*

Sketch of the proof We have a map

$$m : \mathbb{A}_I \cap \mathbb{A}_I = \mathrm{holim}_{(U_1, U_1) \in I \times I} U_1 \cap U_2 \to \mathrm{holim}_{(U,U) \in I \times I} U \cap U = \mathbb{A}_I.$$

We have $mi_1 = mi_2 = \mathrm{Id}$. As i_1, i_2 are homotopy equivalences, so is m. As $mi_1 = mi_2$, the statement follows.

5.6.6 Proof that $\mathbb{A}'_K$ belongs to $\mathrm{sh}(X)$.

Let us check the conditions from Sec 5.4.

A. Stability is stable under direct limits, so we are to check the stability of $\mathbb{A}_{\mathcal{U}}$, which is a finite complex of objects of the form U, $U \in \mathcal{U}$, which implies the statement.

B. Direct limit condition. Let $A \in \mathrm{Open}_X$ and let $\mathcal{A}$ be a family of open subsets of A which forms a filtered poset. We will show that the natural map

$$\mathrm{hocolim}_{B\in\mathcal{A}}\, \mathbb{A}'_K(B) \to \mathbb{A}'_K(A)$$

is a homotopy equivalence. Equivalently, we are to prove:

$$\mathrm{hocolim}_{I\in S(K)}\, \mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\, \mathbb{A}_I(B)) \to \mathbb{A}_I(A))$$

is acyclic. To this end we will show that for every $I \in S(K)$ there exists a $J \in S(K)$, $J \geq I$, such that the map

$$\mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\, \mathbb{A}_I(B)) \to \mathbb{A}_I(A)) \to \mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\, \mathbb{A}_J(B)) \to \mathbb{A}_J(A)) \tag{11}$$

is homotopy equivalent to 0.

B1. Let $I = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\}$. Let us construct a covering $\mathcal{V} \in \mathbf{Cov}_K$ with the following properties:

— there exist poset maps $\phi_k : \mathcal{V} \to \mathcal{U}_k$ such that every $V \in \mathcal{V}$ satisfies $\bar{}(V) \subset \phi_k(V)$ for all k.

Let $\mathcal{U} = \bigcup_k \mathcal{U}_k$. One can choose an open subset $U' \subset U$ for every $U \in \mathcal{U}$ such that $\overline{U'} \subset U$. Let $\mathcal{V}$ consist of all finite intersections of the sets U'. For every $V \in \mathcal{V}$, let $S_k(V) = \{U \in \mathcal{U}_k | V \subset U'\}$. Set $\phi_k(V) := \bigcap_{U\in S_k(V)} U$.

B2. Set $\mathcal{I} := \prod_k \mathcal{U}_k$. Set $\phi := \prod_k \phi_k : \mathcal{V} \to \mathcal{I}$. For $i = (U_1, U_2, \ldots, U_n) \in \mathcal{I}$, set $\mathcal{U}_i := U_1 \cap U_2 \cap \cdots \cap U_n$.

B3. Set $J = I \cup \{\mathcal{V}\}$.

It follows that $\mathcal{V} \leq \mathcal{U}_k$, $k = 1, 2, \ldots, n$.

Therefore, the natural map

$$\mathbb{A}_{\mathcal{V}} \to \mathbb{A}_J = \mathbb{A}_V \cap \mathbb{A}_I$$

is a homotopy equivalence.

The maps ϕ_k induce a map

$$\pi : \mathbb{A}_{\mathcal{I}} \to \mathbb{A}_{\mathcal{V}} \cap \mathbb{A}_{\mathcal{V}} \cap \cdots \cap \mathbb{A}_{\mathcal{V}} \xrightarrow{m} \mathbb{A}_{\mathcal{V}}.$$

We have a diagram

$$\begin{array}{ccc}
\mathbb{A}_{\mathcal{V}} & \xrightarrow[j]{\sim} & \mathbb{A}_{\mathcal{V}} \cap \mathbb{A}_I \\
\pi\uparrow & & \sim\uparrow\sigma \\
\mathbb{A}_I & \overset{i_1}{\underset{i_2}{\rightrightarrows}} & \mathbb{A}_I \cap \mathbb{A}_I
\end{array}$$

Here i_1, i_2 are as in Lemma 5.4 so that $i_1 \sim i_2$ and both i_1 and i_2 are homotopy equivalences. We have $j\pi = \sigma i_1$. Therefore $j\pi \sim \sigma i_2$, where σi_2 is the natural map $\mathbb{A}_I \to \mathbb{A}_J$. Therefore, one can replace in

(11) the natural map $\mathbb{A}_I \to \mathbb{A}_J$ by the map $j\pi$. As j is a homotopy equivalence, we can replace $j\pi$ with π, so that the problem now reduces to showing that the map

$$\mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathbb{A}_I(B)) \to \mathbb{A}_I(A)) \to \mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathbb{A}_{\mathcal{V}}(B)) \to \mathbb{A}_{\mathcal{V}}(A))$$

is homotopy equivalent to 0. This map factorizes as

$$\begin{aligned}
&\mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{i\in\mathcal{I}}\,\mathrm{Hom}(\mathcal{U}_i, B)) \to \mathrm{holim}_{i\in\mathcal{I}}\,\mathrm{Hom}(\mathcal{U}_i, A))\\
&\quad\to \mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(\mathcal{U}_{\phi(V)}, B)) \to \mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(\mathcal{U}_{\phi(V)}, A))\\
&\qquad\xrightarrow{G} \mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(V, B)) \to \mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(V, A))
\end{aligned}$$

Let us show that the arrow G is homotopy equivalent to 0.

We have a homotopy equivalence

$$\begin{aligned}
&\mathrm{Cone}((\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(\mathcal{U}_{\phi(V)}, B)) \to \mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(\mathcal{U}_{\phi(V)}, A))\\
&\quad\to \mathrm{Cone}(\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(\mathcal{U}_{\phi(V)}, B) \to \mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(\mathcal{U}_{\phi(V)}, A))\\
&\qquad= \mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}(\mathrm{Hom}(\mathcal{U}_{\phi(V)}, B) \to \mathrm{Hom}(\mathcal{U}_{\phi(V)}, A)).
\end{aligned}$$

Similarly, we have a homotopy equivalence

$$\begin{aligned}
&\mathrm{Cone}(\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(V, B) \to \mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Hom}(V, A))\\
&\qquad\xrightarrow{\sim} \mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Cone}(\mathrm{Hom}(V, B) \to \mathrm{Hom}(V, A)).
\end{aligned}$$

The arrow G is then homotopy equivalent to the arrow

$$\begin{aligned}
G_1 :\ &\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Cone}(\mathrm{Hom}(\mathcal{U}_{\phi(V)}, B) \to \mathrm{Hom}(\mathcal{U}_{\phi(V)}, A))\\
&\qquad\to \mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Cone}(\mathrm{Hom}(V, B) \to \mathrm{Hom}(V, A))
\end{aligned}$$

induced by the embedding $V \subset \mathcal{U}_{\phi(V)}$.

Let $\mathcal{V}_A \subset \mathcal{V}$ consist of all those $V \in \mathcal{V}$ satisfying $\mathcal{U}_{\phi(V)} \subset A$. It follows that $\overline{V} \subset A$ for all $V \in \mathcal{V}_A$. Hence, there exists $B_0 \in \mathcal{A}$ such that $\overline{V} \subset B_0$ for all $V \in \mathcal{V}_A$ because all $\overline{V}$ are compact.

Let $\delta_A : \mathcal{V}^{\mathbf{op}} \to \mathbf{GZ}$ be defined by $\delta_A(U) = \mathbb{A}$ if $U \in \mathcal{V}_A$ and $\delta_A(U) = 0$ otherwise. We have a natural transformation $\delta_A \to \mathbb{A}_{\mathcal{V}^{\mathbf{op}}}$.

The map G_1 factorizes as follows:

$$\begin{aligned}
&\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Cone}(\mathrm{Hom}(\mathcal{U}_{\phi(V)}, B) \to \mathrm{Hom}(\mathcal{U}_{\phi(V)}, A))\\
&\quad= \mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\delta_A(V) \otimes \mathrm{Cone}(\mathrm{Hom}(\mathcal{U}_{\phi(V)}, B) \to \mathrm{Hom}(\mathcal{U}_{\phi(V)}, A))\\
&\qquad\to \mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\delta_A(V) \otimes \mathrm{Cone}(\mathrm{Hom}(V, B) \to \mathrm{Hom}(V, A))\\
&\qquad\quad\to \mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\mathrm{Cone}(\mathrm{Hom}(V, B) \to \mathrm{Hom}(V, A)).
\end{aligned}$$

It therefore suffices to show that the object

$$\mathrm{hocolim}_{B\in\mathcal{A}}\,\mathrm{holim}_{V\in\mathcal{V}}\,\delta_A(V) \otimes \mathrm{Cone}(\mathrm{Hom}(V, B) \to \mathrm{Hom}(V, A))$$

is acyclic.

The set of all $B \in \mathcal{A}$, where $B \supset B_0$, is cofinal in $\mathcal{A}$. Therefore, the above written object is homotopy equivalent to

$$\mathrm{hocolim}_{B\in\mathcal{A},B\supset B_0}\, \mathrm{holim}_{V\in\mathcal{V}}\, \delta_A(V) \otimes \mathrm{Cone}(\mathrm{Hom}(V,B) \to \mathrm{Hom}(V,A))$$

But the map $\mathrm{Hom}(V,B) \to \mathrm{Hom}(V,A)$ is an isomorphism whenever $B \in \mathcal{A}$, $B \supset B_0$, $V \in \mathcal{V}_A$. This implies the statement.

C. Finite covering condition. Let $A \in \mathrm{Open}_X$ and let $\mathcal{T}$ be a finite covering of A. Show that the map

$$\mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B) \to \mathbb{A}'_K(A) \tag{12}$$

is a homotopy equivalence.

C1) Choose a finite subset $S \subset A$ such that $X, Y \in \mathcal{T}$, $X \cap S = Y \cap S$ implies $X = Y$. Consider the set $\mathcal{X}$ consisting of all open sets $U \in \mathrm{Open}_X$ such that $\overline{U} \subset A$ and $S \subset U$. The poset $\mathcal{X}$ is closed under union, hence, it is filtered.

C2) For each $U \in \mathcal{X}$, we have a natural map

$$\mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B\cap U) \to \mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B).$$

As follows from B), the natural map

$$\mathrm{hocolim}_{U\in\mathcal{X}}\, \mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B\cap U) \to \mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B)$$

is a homotopy equivalemce.

We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{hocolim}_{U\in\mathcal{X}}\, \mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B\cap U) & \xrightarrow{\sim} & \mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B) \\
\downarrow & & \downarrow \\
\mathrm{hocolim}_{U\in\mathcal{X}}\, \mathbb{A}'_K(U) & \xrightarrow{\sim} & \mathbb{A}'_K(A)
\end{array}$$

It therefore suffices to show that the left vertical arrow is a homotopy equivalence, which follows from

$$\mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B\cap U) \to \mathbb{A}'_K(U)$$

being a homotopy equivalence.

Observe that the open sets $B \cap U$ form an open covering of U, to be denoted by $\mathcal{T}_U$. It also follows that if $B_1, B_2 \in \mathcal{T}$ and $B_1 \cap U = B_2 \cap U$ implies $B_1 = B_2$. Therefore, the rule $B \mapsto B \cap U$ is an isomorphism of posets $\mathcal{T} \to \mathcal{T}_U$ and we have an isomorphism

$$\mathrm{hocolim}_{B\in\mathcal{T}}\, \mathbb{A}'_K(B\cap U) = \mathrm{hocolim}_{B'\in\mathcal{T}_U}\, \mathbb{A}'_K(B').$$

C3) Call a subset $V \in \mathrm{Open}_X$ *small* if $V \cap U$ is contained in some element of $\mathcal{T}_U$. Every point $x \in X$ has a small neighborhood U_x. Indeed, if $x \notin A$, then choose U_x so that it does not intersect U; if $x \in A$, then there exists a $B \in \mathcal{T}$ such that $x \in B$ and we can choose U_x so that $U_x \subset B$.

Call a covering $\mathcal{U} \in \mathbf{Cov}_K$ *small* if so is every element of $\mathcal{U}$. As the intersection of small sets is small, such coverings exist.

Let $\Sigma \subset S(K)$ be a subset, where $\{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\} \in \Sigma$ iff at least one of $\mathcal{U}_i$ is small. The subset Σ is cofinal, therefore, the map

$$\text{hocolim}_{I \in \Sigma} \mathbb{A}_I \to \mathbb{A}'_K$$

is a homotopy equivalence. The proplem now reduces to showing that the natural map

$$\text{hocolim}_{B \in \mathcal{T}} \mathbb{A}_I(B \cap U) \to \mathbb{A}_I(U)$$

is a homotopy equivalence for every $I \in \Sigma$.

It follows that every $\mathbb{A}_I$ is a finite complex whose every term is of the form

$$Z := W \cap A_1 \cap A_2 \cap A_n$$

where W is a small open set. Therefore, $\mathbb{A}_I$ is a finite comlex whose every term is of the form Z, where Z is small.

It therefore suffices to show that the map

$$\text{hocolim}_{B \in \mathcal{T}_U} \text{Hom}(Z, B) \to \text{Hom}(Z, U) \tag{13}$$

is a homotopy equivalence.

If Z is not contained in U, both sides are 0. If $Z \subset U$, then let $R \subset \mathcal{T}_U$ consist of all those B containing Z. R is non-empty because Z is small. R has the least element (the intersection of all its elements).

The map (13) is isomorphic to the natural map

$$\text{hocolim}_{B \in R} \mathbb{A} \to \mathbb{A}$$

which is a homotopy equivalence as R has the least element.

5.6.7 Lemma

Lemma 5.5 *Let $U \in \text{Open}_X$ be a neighborhood of K. Then the natural map $\mathbb{A}'_K = X \cap \mathbb{A}'_K \to U \cap \mathbb{A}'_K$ is a homotopy equivalence.*

Sketch of the proof Let $\delta := \text{Cone}\, X \to U$. We are to show that $\delta \cap \mathbb{A}'_K \sim 0$.

Choose $V \in \text{Open}_X$, $K \subset V$; $\overline{V} \subset U$.

Let $[V] \in \mathbf{Cov}_K$ be the covering consisting of a unique element V. It follows that $\delta \cap \mathbb{A}_V \sim 0$. Let $S_V \subset S(K)$ consist of all subsets containing $[V]$. Then it follows that

$$\delta \circ \text{hocolim}_{I \in S_V} \mathbb{A}_I \sim 0.$$

As $S_V \subset S(K)$ is cofinal, the natural map

$$\text{hocolim}_{I \in S_V} \mathbb{A}_I \to \text{hocolim}_{I \in S(K)} \mathbb{A}_I = \mathbb{A}'_K$$

is a homotopy equivalence, hence $\delta \cap \mathbb{A}'_K \sim 0$.

5.6.8 Fundamental system of coverings

A subset $\mathbf{T} \subset \mathbf{Cov}_K$ is called the fundamental system of coverings of K if for every $\mathcal{V} \in \mathbf{Cov}_K$ there exists $\mathcal{U} \in \mathbf{T}$ such that $\mathcal{U} \leq \mathcal{V}$. Let $S(\mathbf{T}) \subset S(K)$ consist of all finite subsets of $\mathbf{T}$. We have a natural map

$$\mathrm{hocolim}_{I \in S(\mathbf{T})} \mathbb{A}_I \to \mathrm{hocolim}_{I \in S(K)} \mathbb{A}_I = \mathbb{A}'_K.$$

Proposition 5.6 *This map is a homotopy equivalence.*

Sketch of the proof Define a subset $\Sigma \subset S(K)$ to consist of all $I \in S(K)$ such that for every $\mathcal{U} \in I$ there exists a $\mathcal{V} \in I \cap \mathbf{T}$ such that $\mathcal{V} \leq \mathcal{U}$. Observe that Σ is a cofinal subset of $S(K)$ so that we have a homotopy equivalence

$$\mathrm{hocolim}_{I \in \Sigma} \mathbb{A}_I \stackrel{\sim}{\to} \mathrm{hocolim}_{I \in S(K)} \mathbb{A}_I = \mathbb{A}'_K.$$

The problem reduces to showing that the natural map

$$\mathrm{hocolim}_{I \in S(\mathbf{T})} \mathbb{A}_I \to \mathrm{hocolim}_{I \in \Sigma} \mathbb{A}_I \tag{14}$$

is a homotopy equivalence.

For $I \in S(K)$ denote $r(I) := I \cap \mathbf{T} \in S(\mathbf{T})$. We have a natural map

$$\mathbb{A}_{r(I)} \to \mathbb{A}_I$$

which is a homotopy equivalence for all $I \in \Sigma$.

Let $i : S(\mathbf{T}) \subset \Sigma$ be the embedding of posets. Let $h : S(\mathbf{T})^{\mathbf{op}} \times \Sigma \to \mathbf{GZ}$ be defined by $h(x, y) = \mathrm{Hom}_\Sigma(i(x); y)$ We have $\mathrm{Hom}_\Sigma(i(x); y) = \mathrm{Hom}_{S(\mathbf{T})}(x; r(y))$.

We therefore have a commutative diagram

$$\begin{array}{ccc} \mathbb{A}_- \otimes^L_{S(\mathbf{T})} \mathrm{Hom}_{S(\mathbf{T})}(-; x) & \xrightarrow{\sim} & \mathbb{A}_- \otimes^L_\Sigma \mathrm{Hom}_\Sigma(i(-); y) \\ \downarrow \sim & & \downarrow \\ \mathbb{A}_{r(y)} & \xrightarrow{\sim} & \mathbb{A}_y \end{array}$$

This diagram proves that the natural map

$$\mathbb{A}_- \otimes^L_{S(\mathbf{T})} \mathrm{Hom}_\Sigma(i(-); y) \to \mathbb{A}_y.$$

is a homotopy equivalence

In order to prove that (14) is a homotopy equivalence, it now remains to show that the natural map

$$\mathrm{hocolim}_{y \in S(\mathbf{T})} \mathrm{Hom}_\Sigma(i(x); i(y)) \to \mathrm{hocolim}_{z \in \Sigma} \mathrm{Hom}_\Sigma(i(x); z)$$

is a homotopy equivalence for every $x \in S(\mathbf{T})$, which is obvious because we have an isomorphism $\mathrm{Hom}_{S(\mathbf{T})}(x, y) \to \mathrm{Hom}_\Sigma(i(x), i(y))$.

5.6.9 Definition of $\mathbb{A}_K$

Let $\mathcal{K}$ be the poset of all neighborhoods of K. Set

$$\mathbb{A}_K := \operatorname{hocolim}_{U\in\mathcal{K}} U \cap \mathbb{A}'_K.$$

We have a natural map $\mathbb{A}'_K \to \mathbb{A}_K$ which is a homotopy equivalence by Sec 5.6.7.

5.6.10 Represenatability

The map $X \to \mathbb{A}'_K$ induces a map

$$\operatorname{hocolim}_{U\in\mathcal{K}} U \to \mathbb{A}_K$$

Let $F \in \operatorname{sh}(X)$. We have an induced map

$$\operatorname{Hom}(\mathbb{A}_K; F) \to \operatorname{Hom}(\operatorname{hocolim}_{U\in\mathcal{K}} U; F) = \Gamma_K(F).$$

Theorem 5.7 *The above map is a homotopy equivalence.*

Sketch of the proof. Let us rewrite the map:

$$\operatorname{Hom}(\operatorname{hocolim}_{(U,I)\in\mathcal{K}\times S(K)} U \cap \mathbb{A}_I; F) \to \operatorname{Hom}(\operatorname{hocolim}_{U\in\mathcal{K}} U; F).$$

For $\mathcal{U} \in \mathbf{Cov}_K$, let $|\mathcal{U}|$ be the union of all elements in $\mathcal{U}$. For $I = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\} \in S(K)$, set

$$|I| = |\mathcal{U}_1| \cap |\mathcal{U}_2| \cap \cdots \cap |\mathcal{U}_n|.$$

The above map factors as:

$$\operatorname{Hom}(\operatorname{hocolim}_{(U,I)\in\mathcal{K}\times S(K)} U \cap \mathbb{A}_I; F)$$
$$\xrightarrow{\sim} \operatorname{Hom}(\operatorname{hocolim}_{I\in S(K);U\in\mathcal{K},U\subset|I|} U \cap \mathbb{A}_I; F) \xrightarrow{u} \operatorname{Hom}(\operatorname{hocolim}_{U\in\mathcal{K}} U; F)$$

The first arrow in this sequence is a homotopy equivalence because the subset $\{(U,I) \in \mathcal{K} \times S(K) | U \subset |I|\} \subset \mathcal{K} \times S(K)$ is cofinal. Therefore, the problem reduces to showing that the second arrow u is a homotopy equivalence. Let us rewrite u as

$$\operatorname{holim}_{U\in\mathcal{K}} \operatorname{holim}_{I\in S(K)_U} \operatorname{Hom}(U \cap \mathbb{A}_I; F) \to \operatorname{holim}_{U\in\mathcal{K}} \operatorname{Hom}(U, F).$$

It suffices to show that for every $U \in \mathcal{K}$, the map

$$\operatorname{hocolim}_{I\in S(K)_U} \operatorname{Hom}(U \cap \mathbb{A}_I; F) \to \operatorname{hocolim}_{I\in S(K)_U} \operatorname{Hom}(U, F) \to \operatorname{Hom}(U, F)$$

is a homotopy equivalence. The right arrow is a homotopy equivalence because the poset $S(K)_U$ is filtered. Let us show that the left arrow is a homotopy equivalence, which reduces to showing that for every $I \in S(K)_U$, the map

$$\operatorname{Hom}(U \cap \mathbb{A}_I; F) \to \operatorname{Hom}(U, F)$$

is a homotopy equivalence.

Let $I = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n\}$. Let $\mathcal{I} := \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$. For $i = (U_1, U_2, \ldots, U_n) \in \mathcal{I}$, denote $V_i := U_1 \cap U_2 \cap \cdots \cap U_n$. We have

$$\operatorname{Hom}(U \cap \mathbb{A}_I; F) = \operatorname{hocolim}_{i\in\mathcal{I}} F(V_i \cap U).$$

It now follows that the natural map

$$\operatorname{hocolim}_{i\in\mathcal{I}} F(V_i \cap U) \to F(|I| \cap U) = F(U)$$

is a homotopy equivalence by the finite covering gluing property of F. This finishes the proof.

5.6.11 The objects $\mathbb{A}_K$ generate $\operatorname{sh}(X)$

For $U \in \operatorname{precompact}_X$ denote $R_U := \mathbb{A}_{\overline{U}}$, $R : \operatorname{precompact}_X^{\mathbf{op}} \to \operatorname{sh}(X)$.

We have natural transformations

$$\mathbb{A}_{\overline{U}} \xleftarrow{\iota} \operatorname{hocolim}_{V\supset\overline{U}} V \xrightarrow{r} U. \tag{15}$$

For $U \in \operatorname{precompact}_X$ set

$$\mathbf{C}_U := \operatorname{hocolim}_{V\supset\overline{U}} V \in \operatorname{psh}(X),$$

$\mathbf{C} : \operatorname{precompact}_X^{\mathbf{op}} \to \operatorname{psh}(X)$. Let also $I : \operatorname{precompact}_X^{\mathbf{op}} \to \operatorname{psh}(X)$ be given by $I(U) = U$. so that $\iota : R \to I$. We can now rewrite (15) as a diagram of natural transformations of functors $\operatorname{precompact}_X \to \operatorname{psh}(X)$:

$$R \leftarrow \mathbf{C} \to I.$$

Let $F \in \operatorname{psh}(X, C)$. Denote

$$\mathcal{R}(F) := F \otimes^L_{\operatorname{precompact}_X} R \in \operatorname{sh}(X, C).$$

We then have an induced diagram

$$F \otimes^L_{\operatorname{precompact}_X} R \leftarrow F \otimes^L_{\operatorname{precompact}_X} \mathbf{C} \to F \otimes_{\operatorname{precompact}_X} I \to F. \tag{16}$$

Theorem 5.8 *Let $F \in \operatorname{sh}(X, C)$. Then every arrow in (16) is a homotopy equivalence.*

Sketch of the proof

Let us show that the arrow

$$F \otimes^L_{\operatorname{precompact}_X} \mathbf{C} \to F \otimes^L_{\operatorname{precompact}_X} R \tag{17}$$

is a homotopy equivalence.

Indeed, Let $G \in \operatorname{sh}(X)$, and consider the induced map

$$\operatorname{Hom}(F \otimes^L_{\operatorname{precompact}_X} R; G) \to \operatorname{Hom}(F \otimes^L_{\operatorname{precompact}_X} \mathbf{C}; G). \tag{18}$$

Denote $G', G'' : \operatorname{precompact}_X^{\mathbf{op}} \to \mathbf{swell}\, C$, where $G'(U) = \operatorname{Hom}(\mathbf{C}_U; G)$; $G''(U) = \operatorname{Hom}(R_U; G)$. We have a natural transformation $G'' \to G'$ induced by the natural transformation $\mathbf{C} \to R$. Then the map (18) is homotopy equivalent to

$$R\operatorname{Hom}(F, G') \to R\operatorname{Hom}(F, G''). \tag{19}$$

The Representability theorem implies that $G'(U) \to G''(U)$ is a homotopy equivalence for all U, therefore, (19) is a homotopy equivalence. Hence so is (18) and (17).

Let us switch to the remaining arrows in (16). Let $U \in \mathrm{Open}_X$ and consider the induced sequence:

$$F \otimes^L_{\mathrm{precompact}_X} \mathbf{C}(U) \to F \otimes^L_{\mathrm{precompact}_X} I(U) \to F(U). \tag{20}$$

Rewrite:

$$\mathrm{hocolim}_{V \in \mathrm{precompact}_X | \overline{V} \subset U} F(V) \to \mathrm{hocolim}_{V \in \mathrm{precompact}_X ; V \subset U} F(V) \to F(U)$$

both arrows are homotopy equivalences by the covering axiom for F.

As F is a stable object, it follows that both arrows in (20) are homotopy equivalences. This proves the theorem.

5.6.12 Meyer-Vietoris property of $\mathbb{A}_K$

Let $K, L \subset X$ be compact subsets. We then have a complex

$$MV(K, L) := [0 \to \mathbb{A}_{K \cup L} \to \mathbb{A}_K \oplus \mathbb{A}_L \to \mathbb{A}_{K \cap L} \to 0]$$

Proposition 5.9 *This complex is acyclic*

Sketch of the proof A. It suffices to prove that $\mathrm{Hom}(MV(K, L), G) \sim 0$ for any $G \in \mathrm{sh}(X)$. As follows from the Representability theorem, the complex $\mathrm{Hom}(MV(K, L), G)$ is homotopy equivalent to the complex

$$0 \to \Gamma_{K \cap L} G \to \Gamma_K(G) \oplus \Gamma_L(G) \to \Gamma(K \cup L) \to 0.$$

B. Let us show that the natural map

$$f : \mathrm{hocolim}_{U \supset K; V \supset L} U \cap V \to \mathrm{hocolim}_{W \supset K \cap L} W$$

is a homotopy equivalence in $\mathrm{psh}(X)$.

Let $A \in \mathrm{Open}^{\mathbf{op}}_X$. Consider

$$\mathrm{Hom}(A, \mathrm{Cone}\, f) = \mathrm{Cone}(\mathrm{hocolim}_{U \supset K; V \supset L; U \cap V \subset A} \mathbb{A} \to \mathrm{hocolim}_{W \supset K \cap L; W \subset A} \mathbb{A}.$$

Both colimits are filtered, therefore, $\mathrm{Hom}(A, \mathrm{Cone}\, f) \sim 0$, whence $\mathrm{Hom}(\mathrm{Cone}\, f; \mathrm{Cone}\, f) \sim 0$ as we wanted.

C. Similarly, one checks that the natural map

$$\mathrm{hocolim}_{U \supset K; V \supset L} U \cup V \to \mathrm{hocolim}_{W \supset K \cup L} W$$

is a homotopy equivalence in $\mathrm{psh}(X, C)$.

D. The natural map

$$\mathrm{hocolim}_{U \supset K; V \supset L} U \to \mathrm{hocolim}_{U \supset K} Ui$$

is a homotopy equivalence because the set $\{V \in \mathrm{Open}_X | V \supset L\}$ is filtered.

E. B,C,D imply that the natural maps

$$\Gamma_{K\cap L}G \to \mathrm{holim}_{U\supset K;V\supset L}\, G(U\cap V);$$

$$\Gamma_{K\cup L}G \to \mathrm{holim}_{U\supset K;V\supset L}\, G(U\cup V);$$

$$\Gamma_K G \to \mathrm{holim}_{U\supset K;V\supset L}\, G(U);$$

$$\Gamma_L G \to \mathrm{holim}_{U\supset K;V\supset L}\, G(V)$$

are homotopy equivalences.

Hence, $\mathrm{Hom}(MV(K,L);G)$ is homotopy equivalent to

$$\mathrm{holim}_{U\supset K;V\supset L}[0 \to G(U\cap V) \to G(U)\oplus G(V) \to G(U\cup V) \to 0]$$

which is acyclic because G satisfies Meyer-Vietoris.

5.7 Triangulations

We assume that X is a manifold with corners.

Fix a triangulation $\mathcal{T}$ of X. Denote by the same symbol $\mathcal{T}$ the poset of simplices of $\mathcal{T}$. Let $\mathcal{T}_n$ be the n-th baricentric subdivision of $\mathcal{T}$.

Let us identify each $x \in \mathcal{T}_n$ with the corresponding compact subset of X. Denote by $\mathbf{Star}_n(x) \in \mathrm{precompact}_X$ the star of x, which is by definition the interior of the union of all closed simplices of $\mathcal{T}_n$ containing x.

5.7.1 Theorem on $\mathbf{Hom}(\mathbb{A}_x;\mathbb{A}_y)$

Theorem 5.10 *Let $x, y \in \mathcal{T}$. If $x \subset y$, then the natural map $\mathbb{A} \to \mathrm{Hom}(\mathbb{A}_y;\mathbb{A}_x)$ is a homotopy equivalence. Otherwise, $\mathrm{Hom}(\mathbb{A}_y;\mathbb{A}_x) \sim 0$.*

Sketch of the proof

Denote by $U_n(y)$ the union of all $\mathbf{Star}_n(z)$ where $z \subset y$, $z \in \mathcal{T}_n$. The open sets $U_n(y)$ form a fundamental system of neighborhoods of y. Therefore we have a homotopy equivalence

$$\mathrm{Hom}(\mathbb{A}_y;\mathbb{A}_x) \xrightarrow{\sim} \mathrm{holim}_n\, \mathbb{A}_x(U_n(y)) \xrightarrow{\sim} \mathrm{holim}_n\, \mathbb{A}'_x(U_n(y)).$$

In the case $x \subset y$, the map $\mathbb{A} \to \mathrm{Hom}(\mathbb{A}_y;\mathbb{A}_x)$ gives rise to a map $\mathbb{A} \to \mathrm{holim}_n\, \mathbb{A}'_x(U_n(y))$. This map coincides with the map determined by the natural maps $\iota_n : \mathbb{A} \to \mathbb{A}'_x(U_n(y))$ coming from the inclusion $x \subset U_n(y)$.

Let $\mathcal{U}_n \in \mathbf{cov}_x$ be the covering formed by the stars of all simplices of $\mathcal{T}_n$ contained in x. Let $\mathcal{E} \subset \mathbf{cov}_x$ consist of all $\mathcal{U}_n$. $\mathcal{E}$ is a fundamental system of coverings of x.

Consider $\mathbb{A}_{\mathcal{U}_N}(U_n(y))$, $N > n$. In the case $x \subset y$, we have $\mathcal{U}_N \subset U_n(y)$, whence an isomorphism

$$\mathbb{A}_{\mathcal{U}_N}(U_n(y)) = \mathrm{holim}_{\mathcal{U}_N}\, \mathbb{A},$$

in which case we have a homotopy equivalence $\mathbb{A} \to \mathrm{holim}_{\mathcal{U}_N} \mathbb{A} = \mathbb{A}_{\mathcal{U}_N}(U_n(y))$. Likewise all the maps

$$\mathbb{A} \overset{\sim}{\to} \mathbb{A}_{\mathcal{U}_{N_k}}(U_n(y)) \overset{\sim}{\to} \mathbb{A}_{\{\mathcal{U}_{N_1},\mathcal{U}_{N_2},\ldots\mathcal{U}_{N_k}\}}(U_n(y)).$$

are homotopy equivalences, this proves that $\mathbb{A} \to \mathrm{Hom}(\mathbb{A}_y; \mathbb{A}_x)$ is a homotopy equivalence.

Suppose that x is not contained in y.

In this case we have

$$\mathrm{Cone}(\mathrm{holim}_{u\in\mathcal{U}_N|u\cap(x\backslash U_n(y))\neq\emptyset} \mathbb{A} \to \mathrm{holim}_{u\in\mathcal{U}_N} \mathbb{A}) \overset{\sim}{\to} \mathbb{A}_{\mathcal{U}_N}(U_n(y)).$$

Let us show that the LHS is acyclic, which would imply the statement.

Indeed, $\mathrm{holim}_{u\in\mathcal{U}_N|u\cap(x\backslash U_n(y))\neq\emptyset} \mathbb{A}$ computes Chech cohomology of

$$\bigcup_{u\in\mathcal{U}_N|u\cap(x\backslash U_n(y))\neq\emptyset} u$$

with respect to the covering by the elements of $\mathcal{U}_N$, which is contractible.

Likewise, $\mathrm{holim}_{u\in\mathcal{U}_N} \mathbb{A}$ computes Chech cohomology of $U_n(x)$, which is contracible as well.

5.8 Constructible subsets

Let $\mathcal{T}$ be a triangulation of X, call a closed subset $K \subset X$ *$\mathcal{T}$-constructible* if it is a finite union of closed simplices from $\mathcal{T}$. A locally closed subset $Z \subset X$ is called $\mathcal{T}$-constructible if it can be represented as a difference of two $\mathcal{T}$-constructible subsets of X.

Let Z_1, Z_2 be $\mathcal{T}$-constructible locally closed subsets of X. Denote $dZ_1 := \overline{Z_1}\backslash Z_1$.

Theorem 5.11 *1) $Hom(\mathbb{A}_{Z_1}, \mathbb{A}_{Z_2})$ is homotopy equivalence to a finite complex of finitely generated free $\mathbb{A}$-modules concentrared in the positive degrees, in particular it admits a truncation.*

2) We have a homotopy equivalence $\tau_{\leq 0} Hom(\mathbb{A}_{Z_1}, \mathbb{A}_{Z_2}) \to H$, where H is a finitely generated free $\mathbb{A}$-module of locally constant $\mathbb{A}$-valued functions on $\overline{Z_2}\backslash dZ_1$ supported on $Z_2 \cap Z_1$.

5.8.1 Generalization

Let $X \subset X'$ be an open embedding and $\mathcal{T}$ a triangulation of X'. A locally closed subset $Z_1 \subset X$ is called $\mathcal{T}$-constructible if it is such as a subset of X'. The above theorem still holds true in $\mathrm{sh}(X)$.

5.9 Base of topology

Let $\mathcal{B} \subset \mathrm{Open}_X$ be a poset which is a base of the topology on X. Let us define a full sub-category $\mathrm{sh}(\mathcal{B}, C) \subset \mathbf{swell}(\mathcal{B}^{\mathbf{op}} \otimes C)$ satisfying the same axioms as in Sec. 5.4 when all the open sets involved are in $\mathcal{B}$.

We have a functor

$$I_B : \mathrm{sh}(\mathcal{B}, C) \to \mathbf{swell}(\mathcal{B}^{\mathbf{op}} \otimes C) \to \mathrm{psh}(X, C).$$

Theorem 5.12 *The functor I_B establishes a quasi-equivalence* $\mathrm{sh}(\mathcal{B},C)\to\mathrm{sh}(X,C)$.

Sketch of the proof

1) Let us show that the functor I_B takes values in $\mathrm{sh}(X,C)$. Let $F\in\mathrm{sh}(\mathcal{B},C)$. The stability of $I_B(F)$ follows from Sec. 3.9.4. Let us check the covering axiom. Let U be an open subset of X and let $\mathcal{U}$ be an open covering of U. Let us inscribe a $\mathcal{B}$-covering $\mathcal{V}$ into $\mathcal{U}$ so that $\mathcal{V}\leq\mathcal{U}$.

We have

$$\mathrm{hocolim}_{A\in\mathcal{U}}F(A)\overset{\sim}{\leftarrow}\mathrm{hocolim}_{(A,B)\in\mathcal{U}\times\mathcal{V},B\subset A}F(B)\overset{\sim}{\to}\mathrm{hocolim}_{B\in\mathcal{V}}F(B)\overset{\sim}{\to}F(U),$$

which implies the covering axiom.

2) It follows readily that I_B is a fully faithful functor. Therefore, it now remains to show that I_B is essentially surjective. Indeed, for every compact $K\subset X$ let $\mathbf{cov}_{\mathcal{B}}(K)\subset\mathbf{cov}_K$ consist of all coverings $\mathcal{U}$ whose every element in in $\mathcal{B}$. It follows that $\mathbf{cov}_{\mathcal{B}}(K)\subset\mathbf{cov}_K$. is a fundamental system of coverings. Let $S_{\mathcal{B}}(K)\subset S(K)$ consist of all subsets of $\mathbf{cov}_{\mathcal{B}}(K)$.

Let

$$\mathbb{A}^{\mathcal{B}}_K:=\mathrm{hocolim}_{I\in S_{\mathcal{B}}(K)}\mathbb{A}_I.$$

We have a homotopy equivalence $\mathbb{A}^{\mathcal{B}}_K\to\mathbb{A}'_K$.

Therefore, we have a homotopy equivalence

$$F(U)\otimes_{\mathrm{precompact}_X}\mathbb{A}^{\mathcal{B}}_{\overline{U}}\overset{\sim}{\to}F(U)\otimes_{\mathrm{precompact}_X}\mathbb{A}_{\overline{U}}\overset{\sim}{\to}F$$

in $\mathrm{sh}(X,C)$. Finally,

$$F(U)\otimes_{\mathrm{precompact}_X}\mathbb{A}^{\mathcal{B}}_{\overline{U}}\in\mathrm{sh}(\mathcal{B},C).$$

5.9.1 Product

In particular, let $Z=X\times Y$ Let $\mathcal{B}$ be the base consisting of all open sets of the form $U\times V$, where $U\in\mathrm{Open}_X$, $V\in\mathrm{Open}_Y$. Denote $\mathrm{sh}(X|Y,C):=\mathrm{sh}(\mathcal{B},C)$.

5.9.2 Lemma

Lemma 5.13 *Let $K\subset X$, $L\subset Y$ be compact subsets. We have a zig-zag homotopy equivalence between $\mathbb{A}_K\boxtimes\mathbb{A}_L$ and $\mathbb{A}_{K\times L}$.*

Sketch of the proof Both objects homotopically represent the same functor.

5.10 Convolution of kernels

Let $\Delta:\mathrm{Open}^{\mathbf{op}}_X\times\mathrm{Open}^{\mathbf{op}}_X\to\mathbf{GZ}$ be given by $\Delta(U,V)=\mathbb{Z}$ if $U\cap V\neq\emptyset$ and $\Delta(U,V)=0$ otherwise.

Let us define the convolution functor as follows:

$$\circ_Y:\mathrm{psh}(X|Y,C)\otimes\mathrm{psh}(Y|Z,C)\to\mathrm{psh}(X|Y|Y|Z,C)\overset{\Delta}{\to}\mathrm{psh}(X|Z,C).$$

One checks that this functor induces a functor

$$\circ_Y : \mathrm{sh}(X|Y,C) \otimes \mathrm{sh}(Y|Z,C) \to \mathrm{sh}(X,Z).$$

This way we get a non-unital 2-category **kernels** whose 0-objects are locally compact spaces and $\mathbf{kernels}(X,Y) = \mathrm{sh}(X,Y)$.

5.11 Definition of $\mathbb{A}_C$, where C is a locally closed subset

5.11.1 One point compactification

Let $\overline{X} = X \cup \infty$ be the one point compactification of X. The topology on $\overline{X}$ is defined as follows: a subset $U \subset \overline{X}$ not containing ∞ is open iff it is an open subset of X. A subset $U \subset \overline{X}$ containing ∞ is open iff $X \backslash U$ is compact. The space $\overline{X}$ is compact and Hausdorff as long as X is locally compact.

5.11.2 Restriction of a sheaf onto an open subset

Let $U \subset X$ be an open subset Let $|_U : \mathrm{Open}_X^{\mathbf{op}} \to \mathrm{Open}_U^{\mathbf{op}}$, where $V|_U = V$ if $V \subset U$ and $V|_U = 0$ otherwise. This functor extends to a functor $|_U : \mathrm{psh}(X,C) \to \mathrm{psh}(U,C)$. It follows easily that this functor transforms sheaves into sheaves so that we have a functor

$$|_U : \mathrm{sh}(X,C) \to \mathrm{sh}(U,C).$$

5.11.3 Definition of $\mathbb{A}_C$, C is closed

Let $C \subset X$ be a closed subset. Let $\overline{C} \subset \overline{X}$ be the closure of C in $\overline{X}$. $\overline{C} = C$ if C is compact and $\overline{C} = C \cup \infty$ otherwise. The set $\overline{C}$ is compact.

Set

$$\mathbb{A}''_C := \mathbb{A}_{\overline{C}}|_X.$$

If C is compact, we have an isomorphism $\mathbb{A}''_C = \mathbb{A}_C$, therefore, we denote $\mathbb{A}''_C$ by $\mathbb{A}_C$.

5.11.4 $\mathbb{A}_C$, general case.

If $C \subset X$ is a locally closed subset, then let $dC := \overline{C}_{\backslash C} \subset \overline{X}$ and set

$$\mathbb{A}_C := \mathrm{Cone}\, \mathbb{A}_{\overline{X}} \to \mathbb{A}_{dC}.$$

Let $L \subset K$ be closed subsets of X. Let $C = K \backslash L$ We have $\overline{C} \subset K$; $dC \subset L$, $dC = K \cap L$. Whence an induced map

$$\mathrm{Cone}(\mathbb{A}_K \to \mathbb{A}_L) \to \mathrm{Cone}(\mathbb{A}_{\overline{C}} \to \mathbb{A}_{dC})$$

which is a homotopy equivalence. Indeed, let K', L', C' be the closures of K, L, C in $\overline{X}$. Let $dC' = C' \backslash C$. By definition, we have

$$\mathbb{A}_K = \mathbb{A}_{K'}|_X;\ \mathbb{A}_L = \mathbb{A}_{L'}|_X;\ \mathbb{A}_C = \mathbb{A}_{C'}|_X;\ \mathbb{A}_{dC'} = \mathbb{A}_{dC}|_X.$$

Therefore, it suffices to show that the natural map

$$\mathrm{Cone}(\mathbb{A}_{K'} \to \mathbb{A}_{L'}) \to \mathrm{Cone}(\mathbb{A}_{C'} \to \mathbb{A}_{dC'})$$

is a homotopy equivalence.

We have $dC' = C' \cap L'$ and $K' = L' \cup C'$, whence the statement.

5.12 Convolution with $\mathbb{A}_C$

5.12.1 Convolution with $U \in \mathrm{psh}(X, Z)$

For $H : \mathrm{Open}_X \to \mathbf{swell}\, C$ and $K \in \mathrm{compact}_X$, set

$$H(K) := \mathrm{Cone}(H(X \backslash K) \to H(X)).$$

The rule $K \mapsto H(K)$ determines a functor $\mathrm{compact}_X^{\mathbf{op}} \to \mathbf{swell}\, C$.

Consider the following complex of functors $\mathrm{Open}_X^{\mathbf{op}} \times \mathrm{Open}_X^{\mathbf{op}} \to \mathbf{GZ}$:

$$0 \to h \to \mathbb{A}_{\mathrm{Open}_X^{\mathbf{op}} \times \mathrm{Open}_X^{\mathbf{op}}} \to \delta \to 0,$$

where $h(U, V) = \mathbb{A}$ if $V \subset X \backslash \overline{U}$, $h(U, V) = 0$ otherwise. This complex is termwise acyclic. Let $F \in \mathrm{psh}(X, C)$ and $U \in \mathrm{precompact}_X$. We have the following acyclic complex in $\mathbf{swell}\, C$:

$$0 \to h([U], F) \to \mathbb{A}_{\mathrm{Open}_X^{\mathbf{op}} \times \mathrm{Open}_X^{\mathbf{op}}}([U], F) \to 0.$$

This complex is isomorphic to

$$0 \to F(X \backslash \overline{U}) \to F(X) \to U \circ F \to 0.$$

This can be reinterpreted as a term-wise homotopy equivalence of functors $\mathrm{precompact}_X^{\mathbf{op}} \to \mathbf{swell}\, C$:

$$F(\overline{U}) \xrightarrow{\sim} U \circ F.$$

5.12.2 Convolution with $\mathbb{A}_K$

Theorem 5.14 *We have a term-wise zig-zag homotopy equivalence of functors* $\mathrm{sh}(X, C) \times$ *compact*$_X^{\mathbf{op}} \to \mathbf{swell}\, C$*:* $(F, K) \mapsto F(K)$ *and* $(F, K) \mapsto \mathbb{A}_K \circ F$*.*

Sketch of the proof A. According to Sec. 5.6.10, we have a map $\mathrm{hocolim}_{U \in \mathrm{precompact}_X | K \subset U}\, U \to \mathbb{A}_K$ in $\mathrm{psh}(X)$. Consider the induced map

$$\mathrm{hocolim}_{U \in \mathrm{precompact}_X | K \subset U}\, U \circ F \to \mathbb{A}_K \circ F.$$

Using the argument similar to those from Sec. 5.6.10, one can show that this map is a homotopy equivalence whenever $F \in \mathrm{sh}(X, C)$.

Next, we have homotopy equivalences

$$F(K) \xleftarrow{\sim} \mathrm{hocolim}_{U \in \mathrm{precompact}_X | K \subset U}\, U \circ F(\overline{U}) \xrightarrow{\sim} \mathrm{hocolim}_{U \in \mathrm{precompact}_X | K \subset U}\, U \circ F.$$

This finishes the proof.

Corollary 5.15 *Let* $F \in \mathrm{sh}(X, C)$*. We have a zig-zag homotopy equivalence of functors* $\mathrm{sh}(X, C) :$ $\mathrm{Open}_X \to \mathbf{swell}\, C$ *between* $(F, U) \mapsto F(U)$ *and* $(F, U) \mapsto \mathbb{A}_U \circ F$*.*

5.13 Direct image

Let $f : X \to Y$ be a continuous map of locally compact topological spaces. We then have a functor $f^{-1} : \mathrm{Open}_Y \to \mathrm{Open}_X$. Let $F :\in \mathrm{sh}(X, C)$. Set $f_! F \in \mathrm{sh}(X, C)$ to be defined by

$$f_! F = F(f^{-1}U) \otimes^L_{U \in \mathrm{Open}_X} U \in \mathrm{psh}(Y, C).$$

It follows that we have a term-wise homotopy equivalence

$$h_{f_! F}(U) \xrightarrow{\sim} h_F(f^{-1}U).$$

It now follows easily that $h_{f_! F}$ satisfies all the sheaf axioms so that $f_! F \in \mathrm{sh}(Y, C)$.

Theorem 5.16 *1) There exists a kernel* $K_f \in \mathrm{sh}(Y|X)$ *and a zig-zag term-wise homotopy equivalence of functots* $\mathrm{sh}(X) \to \mathrm{sh}(Y)$ *between* $f_!$ *and* $F \mapsto K_f \circ_X F$*.*

2) One can choose $K_f = \mathbb{A}_{\Gamma_f}$, where $\Gamma_f \subset Y \times X$ is the graph of f.

Sketch of the proof 1) The functor $f_!$ is homotopy equivalent to $\mathcal{R}f_!$. We have to

$$\mathcal{R}f_! F = h_{f_! F}(U) \otimes^L_{U \in \mathrm{precompact}_Y} \mathbb{A}_{\overline{U}} \xrightarrow{\sim} F(f^{-1}U) \otimes^L_{U \in \mathrm{precompact}_Y} \mathbb{A}_{\overline{U}}.$$

According to Corollary 5.15 the latter functor is term-wise homotopy equivalent to

$$\begin{aligned} F \mapsto \mathbb{A}_{\overline{U}} \otimes^L_{U \in \mathrm{precompact} X} \mathbb{A}_{f^{-1}U} \circ F \\ \xleftarrow{\sim} \mathrm{hocolim}_{(T,U) \in \mathrm{compact}^{\mathbf{op}}_X \times \mathrm{precompact}_X | T \supset U} \mathbb{A}_T \otimes (\mathbb{A}_{f^{-1}U} \circ F) \\ \cong \left(\mathrm{hocolim}_{\{(T,U) \in \mathrm{compact}^{\mathbf{op}}_X \times \mathrm{precompact}_X | T \supset U\}} \mathbb{A}_T \boxtimes \mathbb{A}_{f^{-1}U} \right) \circ F \end{aligned}$$

Thus, we can set

$$K_f := \mathrm{hocolim}_{\{(T,U) \in \mathrm{compact}^{\mathbf{op}}_X \times \mathrm{precompact}_Y | T \supset U\}} \mathbb{A}_T \boxtimes \mathbb{A}_{f^{-1}U}. \tag{21}$$

2) If X is compact, the statement follows from the fact that K_f represents the functor Γ_{Γ_f}. The general case reduces to this one via passage to the compatifications: let $\overline{Y}, \overline{X}$ be the one point compactification and let X' be the closure of Γ_f in $\overline{Y} \times \overline{X}$. The projection onto $\overline{Y}$ determines a map $f' : X' \to \overline{Y}$. We also have an open embedding $i : X = \Gamma_f \hookrightarrow X'$ such that $f'i = f$.

5.13.1 Convolution with the constant sheaf on the diagonal

Corollary 5.17 *We have a zig-zag homotopy equivalence of the endofunctors on* $\mathrm{sh}(X, C)$*: Id and* $F \mapsto \mathbb{A}_{\Delta_X} \circ F$*, where* $\Delta_X \subset X \times X$ *is the diagonal.*

Set $f = \mathrm{Id}_X$ in the above theorem.

5.14 The inverse image functor

Let $f : X \to Y$ be a continuous map of locally compact topological spaces. Let $F \in \mathrm{sh}(Y)$. Set $f^{-1} : \mathrm{sh}(Y, C) \to \mathrm{sh}(X, C)$: $f^{-1}F = F \circ \mathbb{A}_{\Gamma_f}$, where $G_F \subset Y \times X$ is the graph of f.

We have a zig-zag homotopy equivalence of bifunctors $\mathrm{sh}(Y, C) \times \mathrm{sh}(X, C) \to \mathbf{swell}\, C$,

$$(F, G) \mapsto f^{-1}F \circ G \text{ and } F \circ f_! G.$$

5.14.1

Theorem 5.18 *We have a zig-zag term-wise homotopy equivalence of functors* $\mathrm{Open}_Y \to \mathrm{sh}(X)$*:* $U \mapsto f^{-1}\mathbb{A}_U$ *and* $U \mapsto \mathbb{A}_{f^{-1}U}$*.*

Sketch of the proof We have a zig-zag homotopy equivalence of functors $\mathrm{Open}_Y \to \mathbf{GZ}$: $V \mapsto (f^{-1}\mathbb{A}_U)(V)$ and $V \mapsto (f^{-1}\mathbb{A}_U) \circ \mathbb{A}_V$; which is zig-zag homotopy equivalent to

$$V \mapsto \mathbb{A}_U \circ (f_!\mathbb{A}_V); \quad V \mapsto f_!\mathbb{A}_V(U); \quad V \mapsto \mathbb{A}_V(f^{-1}U);$$

$$V \mapsto \mathbb{A}_V \circ \mathbb{A}_{f^{-1}U}; \quad V \mapsto \mathbb{A}_{f^{-1}U}(V).$$

5.14.2 Inverse image under closed embedding

Let $i : X \to Y$ be a closed embedding.

Proposition 5.19 *We have a homotopy equivalence of functors* $compact_X^{\mathbf{op}} \times \mathrm{sh}(Y) \to \mathbf{swell}\, C$*:* $(K,F) \mapsto (i^{-1}F)(K)$ *and* $(K,F) \mapsto F(i(K))$*.*

Sketch of the proof Assume for simplicity $X \subset Y$. Use the notation $\approx$ for 'zig-zag pointwise homotopy equivalent'. The functor $(K,F) \mapsto i^{-1}F(K)$ is zig-zag pointwise homotopy equivalent to

$$\begin{aligned}(F,K) \mapsto i^{-1}F \circ \mathbb{A}_K \approx F \circ i_!\mathbb{A}_K \approx F(\overline{U}) \otimes^L_{U \in \mathrm{Open}_Y} i_!\mathbb{A}_K(U) \\ \approx (F,K) \mapsto F(T) \otimes^L_{T \in \mathrm{compact}_Y} \mathrm{Hom}_{\mathrm{compact}_Y}(\overline{U};T) \otimes^L_{U \in \mathrm{precompact}_X} i_!\mathbb{A}_K(U) \\ \xrightarrow{\sim} (F,K) \mapsto F(T) \otimes^L_{T \in \mathrm{compact}_Y} \mathbb{A}_K(i^{-1}\mathbf{int}T) \\ \approx (F,K) \mapsto F(T) \otimes^L_{T \in \mathrm{compact}_X} \mathrm{Hom}(V, i^{-1}\mathbf{int}T) \otimes^L_{V \in \mathrm{precompact}_Y} \mathbb{A}_K(V) \\ \xrightarrow{\sim} (F,K) \mapsto F(\overline{V}) \otimes^L_{\mathrm{Open}_X} \mathbb{A}_K(V)\end{aligned}$$

Set $R(U) := \mathrm{Cone}(F(X \cap U) \to F(X))$, $R : \mathrm{Open}_X \to \mathbf{swell}\, C$. It is easy to see that R satisfies the gluing properties for all coverings. Hence, we have

$$F(\overline{V}) \otimes^L_{\mathrm{Open}_X} \mathbb{A}_K(V) \approx R(\overline{V}) \otimes^L_{\mathrm{Open}_X} \mathbb{A}_K(V) \approx R(K) \approx F(K).$$

This proves the statement.

5.14.3 Direct image under closed embedding of $\mathbb{A}_K$

As above, let $i : X \to Y$ be a closed embedding.

Corollary 5.20 *We have a zig-zag pointwise homotopy equivalence of functors* $compact_X^{\mathbf{op}} \to \mathrm{sh}(Y)$*:* $K \mapsto \mathbb{A}_{i(K)}$ *and* $K \mapsto i_!\mathbb{A}_K$*.*

Sketch of the proof We have

$$i_!\mathbb{A}_K(L) \approx \mathbb{A}_K \circ i^{-1}\mathbb{A}_L \approx i^{-1}\mathbb{A}_L(K) \approx \mathbb{A}_L(i(K)) \approx \mathbb{A}_L \circ \mathbb{A}_{i(K)} \approx \mathbb{A}_{i(K)}(L).$$

5.15 Convolutions of constant sheaves on simplices

Fix a triangulation $\mathcal{T}$ of $\mathbb{R}^n$. Let A be a star of a simplex from $\mathcal{T}$.

5.15.1 Lemma

Lemma 5.21 *1) We have a homotopy equivalence*

$$\mathbb{A}_{\mathbb{R}^n}[n](A) \xrightarrow{\sim} \mathbb{A}.$$

Sketch of the proof Follows from a standard computation.

5.15.2 Corollary

Corollary 5.22 *We have a zig-zag homotopy equivalence of functors* $\mathcal{T}^{\mathbf{op}} \to \mathbf{GZ}$*:*

$$u \mapsto \mathbb{A}_{\mathbb{R}^n}[n](\mathbf{Star}(u)) \text{ and } u \mapsto \mathbb{A}.$$

Sketch of the proof As follows from the previous Lemma, $\mathbb{A}_{\mathbb{R}^n}(\mathbf{Star}(u))$ admits a trunctation and the natural transformation of functors $\mathcal{T}^{\mathbf{op}} \to \mathbf{GZ}$:

$$\tau_{\leq 0}\mathbb{A}_{\mathbb{R}^n}[n](\mathbf{Star}(u)) \to \mathbb{A}_{\mathbb{R}^n}[n](\mathbf{Star}(u))$$

is a termwise homotopy equivalence.

Finally, we have a natural transformation of functors $\mathcal{T}^{\mathbf{op}} \to \mathbf{GZ}$:

$$\tau_{\leq 0}\mathbb{A}_{\mathbb{R}^n}[n](\mathbf{Star}(-)) \to \mathbb{A}_{\mathcal{T}^{\mathbf{op}}}$$

which is a homotopy equivalence as well.

5.16 Dualization of convolution

In this section we assume that C has internal hom. Consider a functor

$$\mathrm{sh}(X,C)^{\mathbf{op}} \otimes \mathrm{sh}(X|Y,C)^{\mathbf{op}} \otimes \mathrm{sh}(Y,C) \to \mathbf{swell}\, C;$$

$$(F,K,G) \mapsto \mathrm{Hom}_C(F \circ_X K; G).$$

Theorem 5.23 *There exists a functor* $\mathrm{sh}(X|Y,C)^{\mathbf{op}} \otimes \mathrm{sh}(Y,C) \to \mathrm{sh}(X,C)$*,* $(K,G) \mapsto K^!G$*, and a zig-zag pointwise homotopy equivalence of functors*

$$(F,K,G) \mapsto Hom_C(F \circ_X K; G) \text{ and } (F,K,G) \mapsto Hom(F; K^!G).$$

Sketch of the proof. A. We have a homotopy equivalence

$$h_F(U) \otimes^L_{U \in \text{precompact}_X} U \xrightarrow{\sim} F.$$

We have an induced homotopy equivalence

$$\text{Hom}(F \circ K, G) \xrightarrow{\sim} \text{Hom}(h_F(U) \otimes^L_{U \in \text{precompact}_X} U \circ K; G).$$

Denote $\Lambda : \text{Open}_X \to \mathbf{swell}\, C$, $\Lambda(U) := \text{Hom}(U \circ K; G)$. We can now continue

$$\text{Hom}(h_F(U) \otimes^L_{U \in \text{precompact}_X} U \circ K; G) \cong \text{RHom}_{\text{precompact}_X}(h_F; \Lambda).$$

B. Let us also introduce a functor $Z : \text{compact}_X \to \mathbf{swell}\, C$. For $P \in \text{compact}_X$, let $\delta_P : \text{Open}_X^{\mathbf{op}} \to \mathbf{GZ}$, $\delta_P(U) = \mathbb{A}$ if $P \cap U \neq \emptyset$ and $\delta_P(U) = 0$ otherwise. Set $\delta_P^Y : \text{Open}_X^{\mathbf{op}} \times \text{Open}_Y^{\mathbf{op}} \to \text{psh}(Y)$, $\delta_P^Y(U, V) = \delta_P(U) \otimes V$. Set

$$Z(P) := \text{Hom}_{\text{psh}(Y)}(\delta_P^Y(K); G).$$

We have a natural isomorphism $\Lambda(U) = Z(\overline{U})$ for every $U \in \text{precompact}_X$.

C. It follows that Z satisfies Meyer-Vietoris. For every $P, Q \in \text{compact}_X$, we have

$$[0 \to Z(P \cap Q) \to Z(P) \oplus Z(Q) \to Z(P \cup Q) \to 0] \sim 0,$$

where [] denote the totalization of a complex. Indeed, denote $\delta_{P,Q} : \text{Open}_X^{\mathbf{op}} \to \mathbf{GZ}$,

$$\delta_{P,Q} := [0 \to \delta_{P \cup Q} \to \delta_P \oplus \delta_Q \to \delta_{P \cap Q} \to 0].$$

Observe that $\delta_{P,Q}(U) = 0$ whenever $U \cap (P \cup Q) \subset P$ or $U \cap (P \cup Q) \subset Q$. Denote by $\mathcal{B}_X$ the set of all pre-compact subsets of X with this property. They form a base of topology on X. Hence, there is an object in $\text{psh}(\mathcal{B}_X \times \text{Open}_Y, C)$ which is homotopy equivalent to K. It follows that $\delta_{P,Q} \circ K \sim 0$ which proves the statement.

D. Define a functor $M : \text{Open}_X \to \mathbf{swell}\, C$, where Set $M(U) := \text{hocolim}_{P \in \text{compact}_X | K \subset U} Z(K)$. We have a natural transformation $M \to \Lambda$ because $\Lambda(U) = Z(\overline{U})$.

Let us show that the induced map $\text{RHom}(h_F; M) \to \text{RHom}(h_F; \Lambda)$ is a homotopy equivalence. Equivalently $\text{RHom}(h_F; \text{Cone}(M \to \Lambda))$ is acyclic. Let $r : \text{precompact}_X^{\mathbf{op}} \times \text{precompact}_X \to \mathbf{GZ}$ be given by $r(U, V) = \mathbb{A}$ if $\bar{(}U) \subset V$; $r(U, V) = 0$ otherwise. as $F \in \text{sh}(X, C)$, the natural map $h_F \otimes^L_{\text{precompact}_X} r \to h_F$ is a termwise homotopy equivalence. Therefore, it suffices to show that

$$R\text{Hom}(h_F \otimes^L_{\text{precompact}_X} r; \text{Cone}(M \to \Lambda)$$

is acyclic. Equivalently:

$$\text{holim}_{V \in \text{precompact}_X | \overline{U} \subset V} \text{Cone}(M(V) \to \Lambda(V)) \sim 0.$$

As the holim is filtered, it suffices to show that for every $V \in \text{precompact}_X$, $V \supset \overline{U}$ there exists a $W \in \text{precompact}_X$, $W \supset \overline{U}$, $W \subset V$, such that the induced map

$$\text{Cone}(M(W) \to \Lambda(W)) \to \text{Cone}(M(V) \to \Lambda(V))$$

is homotopy equivalent to 0. To this end, it suffices to choose W so that $\overline{W} \subset V$.

F. Set $K^!G := M(U) \otimes^L_{\mathrm{precompact}_X} U$ and show that $K^!G \in \mathrm{sh}(X, C)$. The stability axiom is obvious. Let us check the remaining properties. We have a term-wise homotopy equivalence of functors $\mathrm{precompact}_X \to \mathbf{swell}\, C$: $h_{K^!G} \to M$. Therefore, it suffices to show that M satisfies the direct limit gluing property, Meyer-Vietoris, and $M(\emptyset) \sim 0$. The direct limit gluing property and $M(\emptyset) \sim 0$ is obvious. Let us check Meyer-Vietoris.

F1. Let $U, V \in \mathrm{precompact}_X$. Let $\mathrm{compact}_U$ be the poset of compact subsets of U and similar for $\mathrm{compact}_V$. Let us prove that the natural map

$$\mathrm{hocolim}_{K \in \mathrm{compact}_U; L \in \mathrm{compact}_V} F(K \cup L) \to \mathrm{hocolim}_{M \in \mathrm{compact}_{U \cup V}} F(M)$$

is a homotopy equivalence for every $F : \mathrm{compact}_X \to \mathbf{swell}\, C$. Indeed, it suffices to check this statement for $F(M) = \mathrm{Hom}(N, M)$, $N \in \mathrm{compact}_{U \cup V}$, in which case the statement reduces to

$$\mathrm{hocolim}_{K \in \mathrm{compact}_U; L \in \mathrm{compact}_V; N \subset K \cup L} \mathbb{A} \to \mathrm{hocolim}_{M \in \mathrm{compact}_{U \cup V}; N \subset M} \mathbb{A}.$$

As both colimits are filtered, the statement follows.

F2. Similarly, we can prove that the natural map

$$\mathrm{hocolim}_{K \in \mathrm{compact}_U; L \in \mathrm{compact}_V} F(K \cap L) \to \mathrm{hocolim}_{M \in \mathrm{compact}_{U \cap V}} F(M)$$

is a homotopy equivalence.

F3. The natural map

$$\mathrm{hocolim}_{K \in \mathrm{compact}_U; L \in \mathrm{compact}_V} F(K) \to \mathrm{hocolim}_{K \in \mathrm{compact}_U} F(K)$$

is a homotopy equivalence because the set $\mathrm{compact}_V$ is filtered.

F4 For $A \in \mathrm{precompact}_X$, set

$$F'(A) := \mathrm{hocolim}_{K \in \mathrm{compact}_X | K \subset U} F(K).$$

The natural map

$$\begin{aligned} \mathrm{hocolim}_{K \in \mathrm{compact}_U; L \in \mathrm{compact}_V} [0 \to F(K \cap L) \to F(K) \oplus F(L) \to F(K \cup L) \to 0] \\ \to [0 \to F'(U \cap V) \to F'(U) \oplus F'(V) \to F'(U \cup V) \to 0]. \end{aligned} \quad (22)$$

is a homotopy equivalence.

F5. It now remains to apply F4 to $F = Z$, where Z is as in C. Then $F' = M$ and the LHS of (22) is acyclic.

5.16.1 Projection along $\mathbb{R}^n$

Let $Z \in \mathrm{sh}(\mathbb{R}^n \times X | X)$, $Z = \mathbb{A}_{\mathbb{R}^n \times \Delta_X}$ so that Z is the graph of the projection $p : \mathbb{R}^n \times X \to X$.

Proposition 5.24 *We have a zig-zag homotopy equivalence of functors* $\mathrm{sh}(X) \to \mathrm{sh}(\mathbb{R}^n \times X)$ *between* $G \mapsto G \boxtimes \mathbb{A}_{\mathbb{R}^n}[n]$ *and* $G \mapsto Z^!G$.

Sketch of the proof.

Choose a triangulation $\mathcal{T}$ of $\mathbb{R}^n$. Let $\mathcal{B}$ be the base of topology on $\mathbb{R}^n$ formed by stars of all simplices of all baricentric sub-divisions of $\mathcal{T}$.

According to Sec 5.9, it suffices to construct a zig-zag homotopy equivalence between the following functors $\mathcal{B} \times \mathrm{Open}_X \times \mathrm{sh}(X, C) \to \mathbf{swell}\, C$:

$$(A, U, G) \mapsto Z^! G(A \times U) \text{ and } (A, U, G) \mapsto \mathbb{A}_{\mathbb{R}^n}[n](A) \otimes G(U).$$

According to Sec 5.9.2 and 5.15 we have homotopy equivalences

$$\mathbb{A}_{\mathbb{R}^n} \boxtimes \mathbb{A}_{\Delta_X} \xrightarrow{\sim} \mathbb{A}_Z;$$

$$\mathbb{A}_{\overline{U}} \leftarrow (U \circ \Delta_X) \to (A \circ \mathbb{A}_{\mathbb{R}^n}) \otimes (U \circ \mathbb{A}_{\Delta_X}) \xrightarrow{\sim} (A \times U) \circ Z;$$

These equivalences induce a zig-zag pointwise homotopy equivalence between $\Lambda(A \times U)$ and $\mathrm{Hom}(\mathbb{A}_{\overline{U}}; G)$, hence $\Gamma_{\overline{U}} G$. Here Λ is as in the previous subsection.

Therefore $Z^! G(A, U)$ is zig-zag homotopy equivalent to

$$\mathrm{hocolim}_{U' | \overline{U'} \subset U} \Gamma_{\overline{U'}} G.$$

We have a natural transformation $G(U') \to \Gamma_{\overline{U'}} G$, which induces a map

$$\mathrm{hocolim}_{U' | \overline{U'} \subset U} G(U') \, \mathrm{hocolim}_{U' | \overline{U'} \subset U} \Gamma_{\overline{U'}} G$$

Let us show that this transformation is a homotopy equivalence. Indeed, set

$$C(U') := \mathrm{Cone}\, G(U') \to \Gamma_{\overline{U'}} G.$$

The problem now reduces to showing that

$$\mathrm{hocolim}_{U' | \overline{U'} \subset U} C(U') \sim 0.$$

As the colimit is over a filtered poset, the statement follows from: let $\overline{U'} \subset U''$, then the induced map $C(U') \to C(U'')$ is homotopy equivalent to 0, which is immediate.

We also have a homotopy equivalence $\mathrm{hocolim}_{U' | \overline{U'} \subset U} G(U') \to G(U)$, which establishes a zig-zag homotopy equivalence between $Z^! G(A, U)$ and $G(U)$.

As follows from Sec 5.15, we have a zig-zag homotopy equivalence between

$$\mathbb{A}_{\mathbb{R}^n}[n](A) \text{ and } \mathbb{A},$$

which finishes the proof.

5.16.2 Inverse image under closed embedding

Let $i : X \to Y$ be a closed embedding. Let $W \in \text{sh}(Y|X)$; $W = \mathbb{A}_{\Gamma_i}$.

Theorem 5.25 *We have a zig-zag pointwise homotopy equivalence of functors* $\text{sh}(X) \to \text{sh}(Y)$ $G \mapsto W^! G$ *and* $G \mapsto i_! G$.

Sketch of the proof Let $F \in \text{sh}(Y)$. We have $F \circ_Y W \approx i^{-1}F$.
Therefore, we have

$$Z(L) \approx \text{RHom}_X(i^{-1}\mathbb{A}_L; G) \approx R\text{Hom}_X(\mathbb{A}_{i^{-1}L}; G) \approx \text{holim}_{U \in \text{precompact}_X | U \supset i^{-1}L}\, G(U)$$

Next,

$$M(U) = \text{hocolim}_{V \in \text{precompact}_Y | \overline{V} \subset U}\, \text{holim}_{W \in \text{precompact}_X | \overline{V} \cap X \subset W}\, G(U).$$

We have natural maps

$$G(U \cap X) \leftarrow \text{hocolim}_{V \in \text{precompact}_Y | \overline{V} \subset U}\, G(V \cap X) \to M(U)$$

both of which are homotopy equivalences, whence the statement.

5.16.3 Direct images under proper map

Let $p : X \to Y$ be a map. Let $\Gamma_p \subset X \times Y$ be the graph of p and $\Gamma_p^t \subset Y \times X$ be the transposed graph of p.
We then set $p_! : \text{sh}(X, C) \to \text{sh}(Y, C)$; $p_! F = F \circ_X \mathbb{A}_{\Gamma_p}$; $p^{-1} : \text{sh}(Y, C) \to \text{sh}(X, C)$: $p^{-1}G = G \circ_Y \mathbb{A}_{\Gamma_p^t}$.
We have $\mathbb{A}_{\Gamma_p} \circ_Y \mathbb{A}_{\Gamma_p^t} \approx \mathbb{A}_{X \times_Y X}$. Let $\Delta_X \subset X \times X$ be the diagonal. As $\Delta_X \subset X \times_Y X$, we have a map $\mathbb{A}_{X \times_Y X} \to \mathbb{A}_{\Delta_X}$, whence a zig-zag map

$$p^{-1} p_! F \to F,$$

we then have an induced zig-zag map

$$p_! F \to p_* F.$$

Theorem 5.26 *Assume p is proper on the support of F. Then the above map is a homotopy equivalence.*

Sketch of the proof The statement reduces to the case p is proper. Next, one reduces the statement to showing that the through map

$$\text{Hom}(\mathbb{A}_K; p_! F) \to \text{Hom}(p^{-1}\mathbb{A}_K; p^{-1}p_! F) \to \text{Hom}(p^{-1}\mathbb{A}_K; F)$$

is a homotopy equivalence for any compact set $K \subset Y$. As p is proper, $p^{-1}K$ is compact and the above map is homotopy equivalent to

$$\text{hocolim}_{U \in \text{precompact}_Y; K \subset U}\, p_! F(U) \to \text{hocolim}_{V \in \text{precompact}_X; p^{-1}K \subset V}\, F(V)$$

which can be rewritten as

$$\text{hocolim}_{U \in \text{precompact}_Y; K \subset U}\, F(p^{-1}U) \to \text{hocolim}_{V \in \text{precompact}_X; p^{-1}K \subset V}\, F(V).$$

As p is proper, the open subsets of the form $p^{-1}U$, $U \supset K$ form a base of neighborhoods of $f^{-1}K$. Therefore, the above map is a homotopy equivalence by the cofinality argument.

6 Quantum/Semi-classical sheaves

6.0.4 Definition of $\mathrm{sh}_\varepsilon(X,C)$

Let $\varepsilon\in\mathbb{R}_{>0}\cup\{\infty\}$. We will use the SMC Q_ε as in Sec. 4.

Let $\mathrm{sh}_\varepsilon(X,C)\subset\mathbf{swell}(X^{\mathbf{op}}\otimes Q_\varepsilon\otimes C)$ be the full sub-category satisfying the following conditions below.

A. Stability. Every object $F\in\mathrm{sh}_\varepsilon(X,C)$ must be stable. Recall the meaning of this condition. Let $h_F:\mathrm{Open}_X\otimes Q_\varepsilon^{\mathbf{op}}\to\mathbf{swell}\,C$ be defined by $h_F(U,a)=\mathrm{Hom}((U,a);F)$. Then the natural map

$$h_F(U,a)\otimes^L_{(U,a)\in\mathrm{Open}_X\times Q_\varepsilon^{\mathbf{op}}}(U,a)\to F$$

is a homotopy equivalence,

B. Sheaf condition 'along X'. F must belong to $\mathrm{sh}(X,C\otimes Q_\varepsilon)\subset\mathbf{swell}(\mathrm{Open}_X^{\mathbf{op}}\otimes C\otimes Q_\varepsilon^{\mathbf{op}})$.

C. Direct limit condition for Q_ε. For every $U\in\mathrm{Open}_X$ and every $a\in\mathbb{R}$, the natural map

$$\mathrm{hocolim}_{b|b>a}\,h_F((U,b))\to h_F(U,a) \tag{23}$$

must be a homotopy equivalence.

D. Completeness condition. For every $U\in\mathrm{Open}_X$ there must be:

$$\mathrm{hocolim}_{b\in\mathbb{R}^{\mathbf{op}}}\,F((U,b))\sim 0.$$

6.0.5 The category $\mathrm{sh}_\omega(X,C)$

We set

$$\mathrm{psh}_\omega(X,C):=\mathbf{swell}(\mathrm{Open}_X^{\mathbf{op}}\otimes C\otimes Q_\omega).$$

Let us define a full sub-category $\mathrm{sh}_\omega(X,C)$ of objects satisfying the conditions A,B from the previous subsection and the condition C for any $\varepsilon>0$: the natural

$$\mathrm{hocolim}_{b|b>a}\,h_F((U,f^\varepsilon_b))\to h_F(U,f^\varepsilon_a) \tag{24}$$

must be a homotopy equivalence.

The category $\mathrm{sh}_\omega(X,C)$ is enriched over $\mathcal{Q}_\omega$ hence $\mathbf{R}_\omega$.

6.0.6 A fully faithful embedding of $\mathrm{sh}_\infty(X,C)$ into $\mathrm{sh}(X\times\mathbb{R},C)$

Let $\mathbf{int}\subset\mathrm{Open}_\mathbb{R}$ be a subset consisting of all open intervals (both finite and infinite). The subsets from $\mathbf{int}$ form a base of topology on $\mathbb{R}$. Therefore, we have a quasi-equivalence of categories

$$\mathrm{sh}(X\times\mathbf{int},C)\to\mathrm{sh}(X\times\mathbb{R},C).$$

Let $\pi:\mathbf{int}\to Q_\infty^{\mathbf{op}}$, $\pi(a,b)=a$ if $a\neq-\infty$; $\pi(-\infty,b)=0$. We then have an induced functor

$$\pi:\mathrm{psh}(X\times\mathbf{int},C)\to\mathbf{swell}(\mathrm{Open}_X^{\mathbf{op}}\otimes Q_\infty\otimes C)$$

One checks that π induces a map

$$\pi : \mathrm{sh}(X \times \mathbf{int}, C) \to \mathrm{sh}_\infty(X, C).$$

We have a homotopy equivalence

$$p(F) := F((U,u)) \otimes^L_{(U,u)\in \mathrm{Open}_X \times \mathbf{int}} (U, \pi(u)) \to \pi(F).$$

Let us now define a functor $s : \mathrm{sh}_\infty(X, C) \to \mathrm{sh}(X \times \mathbf{int}, C)$.

$$s(F) = F(U, \pi(u)) \otimes_{(U,u)\in \mathrm{Open}_X \times \mathbf{int}} (U, u).$$

So that we have a homotopy equivalence

$$s(F)(V, v) \overset{\sim}{\to} F(V, \pi(v)), \quad (V, v) \in \mathrm{Open}_X \times \mathbf{int}. \tag{25}$$

We have natural transformations

$$ps(F) \overset{\sim}{\to} F(V, \pi(v)) \otimes_{(V,v)\in \mathrm{Open}_X \times \mathbf{int}} (V, \pi(V)) \to F; \tag{26}$$

$$\begin{aligned} sp(V) = F((U,u)) \otimes^L_{(U,u)\in \mathrm{Open}_X \times \mathbf{int}} \mathrm{Hom}((U, \pi(u); (V, \pi(v)) \otimes_{(V,v)\in \mathrm{Open}_X \times \mathbf{int}} \otimes (V, v) \\ \leftarrow F((U,u)) \otimes^L_{(U,u)\in \mathrm{Open}_X \times \mathbf{int}} \mathrm{Hom}((U, u); (V, v)) \otimes_{(V,v)\in \mathrm{Open}_X \times \mathbf{int}} \otimes (V, v) \\ \overset{\sim}{\to} F. \end{aligned} \tag{27}$$

Let $\mathbf{shq}(X, C) \subset \mathrm{sh}(X \times \mathbf{int}, C)$ be the full sub-category consisting of all objects F satisfying $F(-\infty, a) \sim 0$.

Theorem 6.1 *1) The functor s takes values in $\mathbf{shq}(X, C)$.*

2) The natural transformation (26) is a termwise homotopy equivalence.

3) The natural transformation (27) induces a homotopy equivalence for all $F \in \mathbf{shq}(X)$.

The functors p and s, therefore, establish a quasi-equivalence between $\mathrm{sh}_\infty(X, C)$ and $\mathbf{shq}(X, C)$.

Sketch of the proof. 1) Follows from (25).

2)

$$\begin{aligned} &ps(F)(U, a) \overset{\sim}{\to} F(V, \pi(v)) \otimes_{(V,v)\in \mathrm{Open}_X \times \mathbf{int}} \mathrm{Hom}_{\mathrm{Open}_X \times Q^{\mathbf{op}}_\infty}((V, \pi(v)); (U, a)) \\ &\overset{\sim}{\leftarrow} F(W, b) \otimes_{\mathrm{Open}_X \times Q^{\mathbf{op}}_\infty} \mathrm{Hom}_{\mathrm{Open}_X \times Q^{\mathbf{op}}_\infty}((W, b); (V, \pi(v))) \otimes^L_{(V,v)\in \mathrm{Open}_X \times \mathbf{int}} \mathrm{Hom}((V, \pi(v)); (U, a)). \end{aligned}$$

We have a homotopy equivalence

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Open}_X \times Q^{\mathbf{op}}_\infty}((W, b); (V, \pi(v))) \otimes^L_{(V,v)\in \mathrm{Open}_X \times \mathbf{int}} \mathrm{Hom}_{\mathrm{Open}_X \times Q^{\mathbf{op}}_\infty}((V, \pi(v)); (U, a)) \\ &= \mathrm{Hom}_{\mathrm{Open}_X \times Q^{\mathbf{op}}_\infty}((W, b); (V, \pi(v))) \otimes^L_{(V,v)\in \mathrm{Open}_X \times \mathbf{int}} \mathrm{Hom}_{\mathrm{Open}_X \times \mathbf{int}}((V, v); (U, (a, \infty)) \\ &\overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Open}_X \times Q^{\mathbf{op}}_\infty}((W, b); (U, a)). \end{aligned}$$

So that we have an induced homotopy equivalence

$$F(W,b)\otimes_{\mathrm{Open}_X\times Q_\infty^{\mathbf{op}}}\mathrm{Hom}_{\mathrm{Open}_X\times Q_\infty^{\mathbf{op}}}((W,b);(V,\pi(v)))\otimes^L_{(V,v)\in \mathrm{Open}_X\times\mathbf{int}}\mathrm{Hom}((V,\pi(v));(U,a))$$
$$\tilde{\to} F(W,b)\otimes_{\mathrm{Open}_X\times Q_\infty^{\mathbf{op}}}\mathrm{Hom}_{\mathrm{Open}_X\times Q_\infty^{\mathbf{op}}}((W,b),(U,a))\tilde{\to} F(U,a)$$

which proves the statement.

3) We have

$$sp(F)(U,u)\tilde{\to} F(V,v)\otimes^L_{\mathrm{Open}_X\times\mathbf{int}}\mathrm{Hom}_{\mathrm{Open}_X\times Q_\infty^{\mathbf{op}}}((V;\pi(v));(U,\pi(u)))$$
$$= F(V,v)\otimes^L_{\mathrm{Open}_X\times\mathbf{int}}\mathrm{Hom}_{\mathrm{Open}_X\times\mathbf{int}}((V;v);(U,(\pi(u),\infty)))$$
$$\tilde{\to} F((U,(\pi(u),\infty))).$$

The induced map

$$F(U,u)\to sp(U,u)\to F((u,(\pi(u),\infty)))$$

coincides with the natural map induced by the embedding $u\subset(\pi(u),\infty)$, whence the statement.

Below we will use the notation $\mathrm{sh}_q(X)$ instead of $\mathrm{sh}_\infty(X)$.

6.0.7 Objects in $\mathrm{sh}_q(X)$

Let $F: Q_\infty^{\mathbf{op}}\to \mathrm{sh}(X,C)$ be a functor. Say that F satisfies the direct limit condition if 1) The natural map $\mathrm{hocolim}_{d\in Q_\infty^{\mathbf{op}}|d>c}F(d)\to F(c)$ is a homotopy equivalence;

2) $\mathrm{hocolim}_{d\in Q_\infty^{\mathbf{op}}}F(d)\sim 0$.

Denote

$$\mathcal{R}(F):=F(u)\otimes^L_{u\in Q_\infty^{\mathbf{op}}}u\in \mathbf{swell}(\mathrm{Open}_X^{\mathbf{op}}\otimes Q_\infty\otimes C).$$

It follows that $\mathcal{R}(F)\in \mathrm{sh}_q(X,C)$. The stability follows from the fact that $\mathcal{R}(F)$ is a bounded from above complex consisting of objects of the form $F(u)\otimes v$, $(u,v)\in Q_\infty^{\mathbf{op}}\otimes Q_\infty$, which are stable. Next, we have

$$h_{\mathcal{R}(F)}(U,c)=F(d)(U)\otimes^L_{d\in Q_\infty^{\mathbf{op}}}\mathrm{Hom}_{Q_\infty}(c,d)\tilde{\to}F(c)(U),$$

which implies the statement.

6.0.8 Object $\mathbb{A}_{[K,f]}$

Let $K\subset X$ be a compact subset and let $f:K\to\mathbb{R}\cup\infty$ be a lower -continuous function. That is $f^{-1}(a,\infty)\in\mathrm{Open}_X$ for all $a\in\mathbb{R}$. Let

$$K_{f\le c}:=\{x\in K|f(x)\le c\}.$$

Set $F_{[K,f]}(c)=\mathbb{A}_{K_{f\le c}}$ so that $F_{[K,f]}:Q_\infty^{\mathbf{op}}\to\mathrm{sh}(X,\mathbb{A})$. One checks that $F_{[K,f]}$ satisfies the direct limit property so that $\mathcal{R}F_{[K,f]}\in\mathrm{sh}_q(X)$.

Proposition 6.2 *We have*

$$s(\mathcal{R}F_{[K,f]})\approx\mathbb{A}_{(x,t)|t\ge f(x)}[1]\in\mathbf{shq}(X).$$

Sketch of the proof

We have

$$s(\mathcal{R}F_{[K,f]})(U,u) \approx (\mathcal{R}F_{[K,f]})(U,\pi(u)) \approx F_{[K,f]}(\pi(u))(U).$$

We have a zig-zag homotopy equivalence

$$s\mathcal{R}(F_{[K,f]}) \approx F_{[K,f]}(\pi(u))(U) \otimes^L_{(U,u)\in \mathrm{Open}_X \times \mathbf{int}} \mathbb{A}_{\overline{U}} \boxtimes \mathbb{A}_{\overline{u}} \xrightarrow{\sim} F(\pi(u)) \otimes_{u\in\mathbf{int}} \mathbb{A}_{\overline{u}}.$$

Let $\pi : \mathbf{int} \to \mathbb{R}^{\mathbf{op}}$; $\sigma : \mathbf{int} \to \mathbb{R}$ be given by $\pi((a,b)) = a$; $\sigma((a,b)) = b$.

We have a homotopy equivalence

$$\mathrm{Cone}(\mathbb{A}_{(\sigma(u),\infty)} \oplus \mathbb{A}_{(-\infty,\pi(u))} \to \mathbb{A}_{\mathbb{R}}) \xrightarrow{\sim} \mathbb{A}_{\overline{u}}.$$

Let us consider

$$F(\pi(u)) \otimes^L_{u\in\mathbf{int}} \mathbb{A}_{\mathbb{R}} = F(\pi(u)) \otimes^L_{u\in\int} \mathrm{Hom}_{\mathbf{int}}(u,\mathbb{R}) \otimes \mathbb{A}_{\mathbb{R}} \xrightarrow{\sim} F(\pi(\mathbb{R})) \boxtimes \mathbb{A}_{\mathbb{R}} \sim 0.$$

$$\begin{aligned} F(\pi(u)) \otimes^L_{u\in\mathbf{int}} \mathbb{A}_{(\sigma(u),\infty)} &\xleftarrow{\sim} F(\pi(u)) \otimes_{u\in\mathbf{int}} \mathrm{Hom}_{Q_\infty}(\sigma(u),v) \otimes_{v\in Q_\infty} \mathbb{A}_{(v,\infty)} \\ &= F(\pi(u)) \otimes_{u\in\mathbf{int}} \mathrm{Hom}_{\mathbf{int}}(u,(-\infty,\sigma(v))) \otimes_{v\in Q_\infty} \mathbb{A}_{(v,\infty)} \xrightarrow{\sim} F(\pi(-\infty,\sigma(v))) \otimes_{v\in Q_\infty} \mathbb{A}_{(v,\infty)} \sim 0, \end{aligned}$$

because $F(\pi(-\infty,\sigma(v))) = F((-\infty,\infty)) \sim 0$.

We now have a homotopy equivalence

$$F(\pi(u)) \otimes^L_{u\in\mathbf{int}} (\mathbb{A}_{(\sigma(u),\infty)} \oplus \mathbb{A}_{(-\infty,\pi(u)} \to \mathbb{A}_{\mathbb{R}}) \xrightarrow{\sim} F(\pi)) \otimes^L_{u\in\mathbf{int}} \mathbb{A}_{(\sigma(u),\infty)}[1]$$

Finally, we have

$$\begin{aligned} F(\pi(u)) \otimes^L_{u\in\mathbf{int}} \mathbb{A}_{(-\infty,\pi(u))} &\xleftarrow{\sim} F(\pi(u)) \otimes_{u\in\mathbf{int}} \mathrm{Hom}_{Q_\infty}(v;\pi(u)) \otimes^L_{Q^{\mathbf{op}}_\infty} \mathbb{A}_{(-\infty,v)} \\ &= F(\pi(u)) \otimes_{u\in\mathbf{int}} \mathrm{Hom}_{\mathbf{int}}((u,\infty);(v\infty)) \otimes^L_{v\in Q_\infty} \mathbb{A}_{(-\infty,v)} \xrightarrow{\sim} F(\pi(v,\infty)) \otimes^L_{v\in Q_\infty} \mathbb{A}_{(-\infty,v)} \\ &= F(v) \otimes^L_{v\in Q^{\mathbf{op}}_\infty} \mathbb{A}_{(-\infty,v)} \xleftarrow{\sim} \mathrm{hocolim}_{(v,w)\in\mathbf{int}} F(v) \boxtimes \mathbb{A}_{(-\infty,w)} = \mathrm{hocolim}_{(v,w)\in\mathbf{int}} \mathbb{A}_{\{(x,t)|f(x)\le v;t<w\}} \\ &\xleftarrow{\sim} \mathrm{Cone}\,\mathrm{hocolim}_{(v,w)\in\mathbf{int}} \mathbb{A}_{(x,t)|f(x)>v;t<w} \to \mathrm{hocolim}_{(v,w)\in\mathbf{int}} \mathbb{A}_{K\times(-\infty,w)}) \end{aligned}$$

The open sets $\{(x,t)|f(x) > v; t < w\} \subset K\times\mathbb{R}$ form an open covering of the set $\{(x,t)|t < f(x)\}$. The open sets $K\times(-\infty,w)$ form an open covering of $K\times\mathbb{R}$. Therefore, we have a homotopy equivalence

$$\mathrm{Cone}(\mathrm{hocolim}_{(v,w)\in\mathbf{int}} \mathbb{A}_{(x,t)|f(x)>v;t<w} \to \mathrm{hocolim}_{(v,w)\in\mathbf{int}} \mathbb{A}_{K\times(-\infty,w)}) \xrightarrow{\sim} \mathrm{Cone}(\mathbb{A}_{(x,t)|t<f(x)} \to \mathbb{A}_{K\times\mathbb{R}}) \xrightarrow{\sim} \mathbb{A}_{(x,t)|t\ge f(x)}.$$

This proves the statement.

6.0.9 Definition of $\mathbb{A}_{[K,f]}$

We have

$$\mathcal{R}_{F_{[K,f]}} = \mathbb{A}_{x|f(x)\leq c} \otimes^L_{Q^{\mathbf{op}}_\infty} c \overset{\sim}{\leftarrow} \operatorname{hocolim}_{\{L\in\mathrm{compact}_X|\ f|_{K\setminus L}>c\}} \otimes^L_{c\in Q^{\mathbf{op}}_\infty} c \overset{\sim}{\to} \mathbb{A}_{K\setminus U} \otimes^L_{U\in\mathrm{precompact}_K} f(U)$$

where we set

$$f(U) := \inf_{x\in U\cap K} f(x).$$

In the case $U\cap K=\emptyset$ we let $f(U)$ to be the zero-object of $\mathbf{swell}\, Q_\infty$.

Let $C\subset X$ be a locally closed sub-set and let $f: C\to\mathbb{R}$ be a lower-continuous function. Set

$$\mathbb{A}_{[C,f]} := \mathbb{A}_{C\setminus U} \otimes^L_{U\in\mathrm{precompact}_X} f(U).$$

We have

$$s(\mathbb{A}_{[C,f]}) \approx s\mathcal{R}(F_{[C,f]}) \approx \mathbb{A}_{(x,t)|x\in C, t\geq f(x)}.$$

6.0.10 Functoriality of $\mathbb{A}_{[K,f]}$

Let C_1, C_2 be closed subsets of X, If $C_1\subset C_2$, f_1 is a lower continuous function on C_1, f_2 on C_2 and $f_2|_{C_1}\leq f_1$, we have a natural map $\mathbb{A}_{[C_2,f_2]}\to\mathbb{A}_{[C_1,f_1]}$ coming from the inequality $f_2(U)\leq f_1(U)$ for any $U\in\mathrm{precompact}_X$.

6.0.11 The functors $\mathbf{red}_{\varepsilon_1\varepsilon_2}$

Let $\varepsilon_1\geq\varepsilon_2$, $\varepsilon_1,\varepsilon_2\in\mathbb{R}\cup\{\infty\}$. The functors $\mathbf{red}_{\varepsilon_1\varepsilon_2}: Q_{\varepsilon_1}\to Q_{\varepsilon_2}$ induce functors $\mathbf{red}_{\varepsilon_1\varepsilon_2}: \mathrm{sh}_{\varepsilon_1}(X,C)\to \mathrm{sh}_{\varepsilon_2}(X)$

6.0.12 Reduction of $\mathbb{A}_{[K,f]}$

In the notation of Sec. 6.0.10 suppose $g|_K+\varepsilon\leq f$. Then the natural map

$$\mathbf{red}_{\infty\varepsilon}\mathbb{A}_{[M,g]}\to\mathbf{red}_{\infty\varepsilon}\mathbb{A}_{[K,f]}$$

equals 0 because such are all the maps $g(M\setminus L)\to f(K\setminus L)$ in Q_ε.

6.0.13 The functor $\boxtimes: \mathrm{sh}_\varepsilon(X,C)\otimes\mathrm{sh}_\varepsilon(Y,C)\to\mathrm{sh}_\varepsilon(X|Y,C)$

We have a natural functor

$$\otimes: \mathrm{psh}_\varepsilon(X,C)\otimes\mathrm{psh}_\varepsilon(Y,C)\to\mathrm{psh}_\varepsilon(X|Y,C).$$

which descends onto the corresponding categories of sheaves.

6.0.14 Convolution

Let $F \in \mathrm{psh}_\varepsilon(X|Y,C)$, $G \in \mathrm{psh}_\varepsilon(Y|Z,C)$. Let $g : \mathrm{Open}_Y \times \mathrm{Open}_Y \to \mathbf{GZ}$; $g(U,V) = \mathbb{A}$ if $U \cap V \neq \emptyset$ and $g(U,V) = \emptyset$ otherwise. We get an induced functor $g : \mathrm{psh}_\varepsilon(X|Y|Y|Z,C) \to \mathrm{psh}_\varepsilon(X|Z,C)$. We then get an object $F \boxtimes G \in \mathrm{psh}_\varepsilon(X|Y|Y|Z,C)$. Set

$$F *_Y G := g(F \boxtimes G) \in \mathrm{psh}_\varepsilon(X|Z,C).$$

Let $F \bullet_Y G \in \mathrm{psh}_\varepsilon(X|Z)$ be given by

$$F \bullet_Y G = (F *_Y G)|_0,$$

where $|_0 : \mathrm{psh}_\varepsilon(\mathcal{B},C) \to \mathrm{psh}(\mathcal{B})$ is given by $(U,a)|_0 = \mathrm{Hom}_{Q_\varepsilon}(0,a) \otimes U$. In particular, if $F,G \in \mathrm{sh}_\varepsilon(X)$, then $F \bullet G \in \mathbf{swell}\, C$.

All the above functors descend onto the corresponding categories of sheaves.

6.0.15 Convolution with the constant sheaf on a graph

Let $X \subset Y$. Let $F : X \to \mathbb{R}$ be an upper continuous functions. Let $C \subset X$ be a closed subset and $f : C \to \mathbb{R}$ a lower continuous function. Let $\Gamma \subset X \times Y$ be the graph of the embedding $X \subset Y$. Let $F' = F \circ \iota^{-1}$. Let $\iota : X \to \Gamma$ be the identification.

Proposition 6.3 *We have a natural zig-zag homotopy equivalence*

$$\mathbb{A}_{[C,f]} *_X \mathbb{A}_{\Gamma,F'} \approx \mathbb{A}_{[X,F'+f]} \in \mathrm{sh}_\infty(Y).$$

Sketch of the proof We have

$$\begin{aligned}
\mathbb{A}_{[C,f]} *_X \mathbb{A}_{\Gamma,F'} &\stackrel{\sim}{\leftarrow} \mathbb{A}_{C\backslash U} *_X \mathbb{A}_{\Gamma\backslash(V\times W)} \otimes^L_{(U,V,W)\in \mathrm{precompact}_{X\times X\times Y}} (f(U\cap C) + F(V\times W\cap\Gamma))\\
&= \mathbb{A}_{C\backslash U} *_X \mathbb{A}_{\Gamma\backslash\iota(V\cap W)} \otimes^L_{(U,V,W)\in \mathrm{precompact}_{X\times X\times Y}} (f(U\cap C) + F(\iota(V\cap W)))\\
&\approx \mathbb{A}_{C\backslash(U\cup(V\cap W))} \otimes^L_{(U,V,W)\in \mathrm{precompact}_{X\times X\times Y}} (f(U) + F(\iota(V\cap W)))\\
&\stackrel{\sim}{\leftarrow} \mathrm{hocolim}_{A\in\mathrm{precompact}_X} \mathbb{A}_{C\backslash(U\cup A)} \otimes^L_{(U,V,W)\in \mathrm{precompact}_{X\times X\times Y}} (f(U) + F(\iota(V\cap W)))\\
&\approx \mathbb{A}_{C\backslash(U\cup A)} \otimes^L_{A\in\mathrm{precompact}_X} \mathrm{Hom}(A, V\cap W) \otimes^L_{(U,V,W)\in \mathrm{precompact}_{X\times X\times Y}} (f(U\cap C) + F(\iota(V\cap W)))\\
&\approx \mathbb{A}_{C\backslash(U\cup A)} \otimes^L_{A\in\mathrm{precompact}_X, U\in\mathrm{precompact}_X} \mathrm{hocolim}_{\{V\times W\in \mathrm{precompact}^{\mathbf{op}}_{X\times Y}|A\subset V\cap W\}} (f(U\cap C) + F(\iota(V\cap W)))\\
&\stackrel{\sim}{\to} \mathbb{A}_{C\backslash(U\cup A)} \otimes^L_{A\in\mathrm{precompact}_X, U\in\mathrm{precompact} X} (f(U\cap C) + F(A)).
\end{aligned}$$

The last arrow is a homotopy equivalence because the poset

$$\{V\times W \in \mathrm{precompact}^{\mathbf{op}}_{X\times Y} | A \subset V\cap W\}$$

is filtered.

Let us continue:

$$\mathbb{A}_{C\backslash(U\cup A)}\otimes^L_{(A,U)\in\text{precompact}_{X\times X}}(f(U)+F(A))$$
$$\approx\mathbb{A}_{C\backslash B}\otimes^L_{B\in\text{precompact}_X}\text{Hom}(B,U\cup A)\otimes^L_{(A,U)\in\text{precompact}_{X\times X}}(f(U\cap C)+F(A)) \tag{28}$$

We have a homotopy equivalence (we assume $(A,U)\in\text{precompact}^{\mathbf{op}}_{X\times X}$):

$$\text{Cone}(\text{hocolim}_{(A,U)|B\backslash(A\cup U)\neq\emptyset}f(U\cap C)+F(A))\to\text{hocolim}_{(A,U)}f(U\cap C)+F(A))$$
$$\to\text{Hom}(B,U\cup A)\otimes^L_{(A,U)}(f(U\cap C)+F(A)).$$

As

$$\text{hocolim}_{(A,U)}f(U\cap C)+F(A)\sim f(\emptyset)+F(\emptyset)=0,$$

we have

$$\text{Hom}(B,U\cup A)\otimes^L_{(A,U)}(f(U\cap C)+F(A))\approx\text{hocolim}_{(A,U)|B\backslash(A\cup U)\neq\emptyset}f(U\cap C)+F(A))[1].$$

Next, we have an acycilic complex

$$\text{hocolim}_{(A,U)|B\backslash(A\cup U)\neq\emptyset}f(U\cap C)+F(A)$$
$$\to(\text{hocolim}_{(A,U)|B\backslash A\neq\emptyset}f(U\cap C)+F(A))\oplus\text{hocolim}_{(A,U)|B\backslash U\neq\emptyset}f(U\cap C)+F(A)$$
$$\to\text{hocolim}_{(A,U)|B\backslash(A\cap U)\neq\emptyset}f(U\cap C)+F(A).$$

As $f(\emptyset)=F(\emptyset)$ is the 0 object of Q_∞, the middle term in this complex is acyclic. Therefore,

$$\text{Hom}(B,U\cup A)\otimes^L_{(A,U)}(f(U\cap C)+F(A))\approx\text{hocolim}_{(A,U)|B\backslash(A\cap U)\neq\emptyset}f(U\cap C)+F(A)\xrightarrow{\sim}(f(B\cap C)+F(B)).$$

so that we can continue (28):

$$\mathbb{A}_{C\backslash B}\otimes^L_{B\in\text{precompact}_X}\text{Hom}(B,U\cup A)\otimes^L_{(A,U)\in\text{precompact}_{X\times X}}(f(U\cap C)+F(A))$$
$$\approx\mathbb{A}_{C\backslash B}\otimes^L_{B\in\text{precompact}_X}f(B\cap C)+F(B)$$
$$\xleftarrow{\sim}\mathbb{A}_{C\backslash B}\otimes^L_{B\in\text{precompact}_X}\text{Hom}(c,f(B\cap C)+F(B))\otimes^L_{c\in Q^{\mathbf{op}}_\infty}c$$
$$\approx(\text{hocolim}_{\{B\in\text{precompact}_X|f(B\cap C)+F(B)\geq c\}}\text{Cone}(\mathbb{A}_B\otimes\mathbb{A}_C\to\mathbb{A}_c))\otimes^L_{c\in Q^{\mathbf{op}}_\infty}c.$$

Let

$$S_c:=\{B\in\text{precompact}_X|f(B\cap C)+F(B)\geq c\}.$$

The set S_c is closed under finite intersections; the union of all elements of S_c equals

$$U_{f+F>c}=X\backslash C\cup\{x\in C|f(x)+F(x)>c\}.$$

We therefore have a homotopy equivalence

$$\text{hocolim}_{\{B\in\text{precompact}_X|f(B\cap C)+F(B)\geq c\}}\text{Cone}(\mathbb{A}_B\otimes\mathbb{A}_C\to\mathbb{A}_C))\otimes^L_{c\in Q^{\mathbf{op}}_\infty}c$$
$$\xrightarrow{\sim}\text{Cone}(\mathbb{A}_{U_{f+F>c}}\otimes\mathbb{A}_C\to\mathbb{A}_C)\otimes^L_{c\in Q^{\mathbf{op}}_\infty}c\xrightarrow{\sim}\mathbb{A}_{\{x\in C|f(x)+F(x)\leq c\}}\otimes_{c\in Q^{\mathbf{op}}_\infty}c\approx\mathbb{A}_{[C,f+F]}.$$

Corollary 6.4 *1) We have a zig-zag homotopy equivalences*

$$\mathbb{A}_{\Delta_X,f} *_X \mathbb{A}_{\Delta_X;g} \approx \mathbb{A}_{\Delta_X,f+g};$$

$$Id \approx \mathbb{A}_{\Delta_X,f} *_X \mathbb{A}_{\Delta_X,-f}.$$

6.0.16 Universal property of $\mathbb{A}_{[X,f]}$

Let X be compact.

Theorem 6.5 *We have a zig-zag homotopy equivalence of functors* $\mathrm{sh}_q(X) \to \mathbf{GZ}$*:* $F \mapsto$ *Hom*$(\mathbb{A}_{[X,f]}; F)$*,*

$$F \mapsto \mathrm{Cone}(\mathrm{holim}_{C\to\infty} \mathbb{A}_{X,-f-C} \to \mathrm{holim}_{\delta\downarrow 0} F \bullet_X \mathbb{A}_{[X,\delta-f]})[-1],$$

and

$$F \mapsto \mathrm{Cone}(\mathrm{holim}_{C\to\infty} \mathbb{A}_{X,-C} \to \mathrm{holim}_{\delta\downarrow 0} F \bullet_X \mathbb{A}_{[X,\delta-f]})[-1]$$

1) The third functor is zig-zag homotopy equivalent to the second one by the cofinality argument. Below we construct a zig-zag homotopy equivalence of the first two functors.

2) By virtue of Corollary 6.4 2, the endofunctor $F \mapsto F *_X \mathbb{A}_{\Delta_X;f}$ on $\mathrm{sh}_\varepsilon(X)$ is a homotopy equivalence, therefore, the statement reduces to the case $f = 0$.

3) We have

$$\mathbb{A}_{[X,0]}(c) \approx \mathrm{Cone}\,\mathrm{hocolim}_{\delta\downarrow 0} \mathbb{A}_X \otimes \mathrm{Hom}(c, -\delta) \to \mathrm{hocolim}_{C\to\infty} \mathbb{A}_X \otimes \mathrm{Hom}(c, C).$$

Which implies

$$\mathrm{Hom}(\mathbb{A}_{[X,0]}; F) \approx \mathrm{holim}_{\delta\downarrow 0, C\to\infty} \mathrm{Hom}(\mathbb{A}_X; \mathrm{Cone}(F(C) \to F(-\delta))[-1])$$
$$\mathrm{holim}_{\delta\downarrow 0, C\to\infty} \mathrm{Cone}(F(X,C) \to F(X,-\delta))[-1] \approx \mathrm{holim}_{\delta\downarrow 0, C\to\infty} F \bullet_X \mathrm{Cone}(\mathbb{A}_{[X,-C]} \to \mathbb{A}_{[X,\delta]})[-1].$$

7 Singular support

7.1 Lenses

Let X be a smooth manifold. Let $\Omega \subset T^*X \times \mathbb{R}$ be an open subset. Call Ω *fiberwise convex* if every fiber of Ω under the map $\Omega \to T^*X \times \mathbb{R} \to X$ is convex. Fix a fiberwise convex Ω.

7.1.1

Let $K \subset X$ be a compact set. *A lense ℓ supported on $K \subset X$* is a collection of the following data:

— a pair of lower continuous functions $f^{\mathbf{k}} := f^{\mathbf{k}}_\ell$, $\mathbf{k} = 0, 1$, defined on K such that $f^1 + \varepsilon \geq f^0 \geq f^1$ for all $x \in K$.

An Ω-lense with support K is a lense ℓ with support $K' \subset K$ additionally satisfying:

there exists a neighborhood U of K' such that the functions f^0, f^1 can be extended to smooth functions U satisfying:

a) for each $x \in K'$, the point $(x, -df^{\mathbf{k}}_x, f^{\mathbf{k}}(x))$ is in Ω, $\mathbf{k} = 0, 1$.

b) f^0 and f^1 coincide outside of K'.

7.1.2 The sheaf $\mathbb{A}_\ell$

Given a lense ℓ, let us define an object $\mathbb{A}_\ell \in \mathrm{sh}_\varepsilon(X)$ as follows.

A. Let $a, b \in \mathbb{R}$, $0 \leq b - a \leq \varepsilon$. Let $h_{ab} : Q^{\mathbf{op}}_\varepsilon \to \mathbb{A}$, $\chi_{ab}(x) = \mathbb{A}$ if $x \in (a, b]$ and $\chi_{ab}(x) = 0$ otherwise.

A1. Let $\delta_{ab} \in \mathcal{Q}_\varepsilon)$ be represented by the following complex

$$\cdots \to (a - 2\varepsilon) \to (b - 2\varepsilon) \to (a - \varepsilon) \to (b - \varepsilon) \to a \to b \to 0.$$

We have a termwise homotopy equivalence $h_{\delta_{ab}} \to \chi_{ab}$.

A2. Set

$$\mathbb{A}_\ell := \mathbb{A}_L \otimes_{L \in \mathrm{compact}_K} \delta_{f^0(K \backslash L); f^1(K \backslash L)}.$$

We then can represent $\mathbb{A}_\ell$ by a complex in $\mathrm{sh}_\varepsilon(X)$

$$\cdots \to \mathbb{A}_{[K, f^0 - 2\varepsilon]} \to \mathbb{A}_{[K, f^1 - 2\varepsilon]} \to \mathbb{A}_{[K, f^0 - \varepsilon]} \to \mathbb{A}[K, f^1 - \varepsilon] \to \mathbb{A}_{[K, f^0]} \to \mathbb{A}_{[K, f^1]} \to 0,$$

where we denote, by abuse of notation $\mathbf{red}_{\infty,\varepsilon}\mathbb{A}_{[K,f]}$ by $\mathbb{A}_{[K,f]}$. The composition of every two successive arrows in this complex is 0 via Sec (6.0.12).

7.1.3 Sections of $\mathbb{A}_\ell$

We have

$$\begin{aligned}
\mathbb{A}_\ell(U, a) = \mathbb{A}_L(U) \otimes_{L \in \mathrm{compact}_K} \delta_{f^0(K \backslash L); f^1(K \backslash L}(a) \\
\mathbb{A}_L(U) \otimes_{L \in \mathrm{compact}_K} \chi_{f^0(K \backslash L); f^1(K \backslash L}(a) \\
\xleftarrow{\sim} \mathbb{A}_L(U) \otimes_{L \in \mathrm{compact}_X} \mathrm{Cone}\, \mathrm{Hom}_{Q_\infty}(a, f^0(K \backslash L)) \to \mathrm{Hom}_{Q_\infty}(a, f^1(K \backslash L)) \\
\xrightarrow{\sim} \mathrm{Cone}\, \mathbb{A}_{K_{f^0 \leq a}}(U) \to \mathbb{A}_{K_{f^1 \leq a}}(U) \\
\approx \mathrm{Cone}(\mathbb{A}_{K_{f^0 > a}}(U) \to \mathbb{A}_{K_{f^1 > a}}(U))[1].
\end{aligned}$$

7.1.4 Fitlered colimits of $\mathbb{A}_\ell$

Let ℓ_1, ℓ_2 be lenses supported on K. Write $\ell_1 \leq \ell_2$ if $f^{\mathbf{k}}_{\ell_1} \leq f^{\mathbf{k}}_{\ell_2}$, $\mathbf{k} = 0, 1$. wherever the two functions are defined. This gives a partial order to the set of lenses supported on K.

Whenever $\ell_1 \leq \ell_2$ we have an induced map $\mathbb{A}_{\ell_1} \to \mathbb{A}_{\ell_2}$. Let I be a filtrant poset and let $\ell_i, i \in I$ be a monotone I-family of lenses supported on K. set $f^{\mathbf{k}}_{\ell_I}(x) := \sup_{i\in I} f^{\mathbf{k}}_{\ell_i}$; $\mathbf{k} = 0, 1$. We see that ℓ_I is a lense supported on K. We also have a homotopy equivalence

$$\text{hocolim}_{i\in I}\, \mathbb{A}_{\ell_i} \to \mathbb{A}_{\ell_I}.$$

Call a lense ℓ a generalized Ω-lense supported on K if $\ell = \ell_I$ and all ℓ_i are Ω-lenses supported on K.

7.1.5 Maximum of a pair of lenses

Let ℓ_1, ℓ_2 be Ω-lenses supported on K. Let $\lambda := \sup(\ell_1, \ell_2)$ be defined by $f^{\mathbf{k}}_\lambda = \sup(f^{\mathbf{k}}_{\ell_1}, f^{\mathbf{k}}_{\ell_2})$. Then λ is a generalized Ω-lense supported on K. Sketch of the proof:

1) we have a monotone sequence of smooth non-decreasing functions $\phi_n(x)$, where

— $\phi_n(x) = 0$ if $x \leq 0$;

— $\phi_n(x) = 1$ if $x \geq 1/n$.

Let $\Phi_n(x) = \int\limits_0^x \phi_n(x)$. In particular

$$0 \leq \Phi_n(x) \leq \max(0, x); \quad 0 \leq \phi_n(x) \leq 1. \tag{29}$$

2) Set $f^{\mathbf{k}}_n(x) := f^{\mathbf{k}}_{\ell_1}(x) + \Phi_n(f^{\mathbf{k}}_{\ell_2}(x) - f^{\mathbf{k}}_{\ell_1}(x))$.

We have $df^{\mathbf{k}}_n(X) = df^{\mathbf{k}}_{\ell_1}(x) + \phi_n(f^{\mathbf{k}}_{\ell_2}(x) - f^{\mathbf{k}}_{\ell_1}(x))(df^{\mathbf{k}}_{\ell_2}(x) - df^{\mathbf{k}}_{\ell_1}(x))$. As follows from (29) and from the fiberwise convexity of Ω, each $f^{\mathbf{k}}_n$ is an Ω-lense with support K. Since $f^{\mathbf{k}}_n(x) \uparrow \max(f^{\mathbf{k}}_{\ell_1}(x), f^{\mathbf{k}}_{\ell_2}(x))$, the statement follows.

7.1.6 Infinite suprema of lenses

Let $\ell_s = \{f^{\mathbf{k}}_s\}$, $s \in S$ be generalized Ω-lenses with support K. Let $f^{\mathbf{k}}(x) := \sup\limits_{s\in S} f^{\mathbf{k}}_s(x)$ Then $\ell = \{f^{\mathbf{k}}\}$ is also a generalized lense with support K. Indeed, we first consider the case of finite S. This reduces to a two-element set S, which follows from the previous subsection.

If S is infinite, pass to the filtrant poset P of finite subsets of S. To each finite $I \subset S$, associate $f^{\mathbf{k}}_I := \max_{i\in I} f^{\mathbf{k}}_i$. Let ℓ_I be the lense supported on K determined by the functions $f^{\mathbf{k}}_I$. The lenses ℓ_I satisfy all the conditions from Sec.7.1.4.

7.2 Localization of Ω

Let $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$ be a finite cover by fiberwise convex subsets. Let $\{f^{\mathbf{k}}\}$ be an Ω-lense supported on a compact K. One then has smooth functions $f^0 = g^0 \leq g^1 \leq \cdots \leq g^N = f^1$ such that (g^i, g^{i+1}) are Ω_{n_i}-lenses supported on K.

This follows from the following 2 particular cases.

Case 1. Let $K \subset \bigcup_{i=1}^{n} U_i$ be an open cover and let $\Omega_i = p^{-1}U_i \cap \Omega$, where $\pi : T^*X \times \mathbb{R} \to X$ is the projection.

Case 2. $\pi(\Omega_i) \supset K$ for all i.

Proof for Case 1. 1)Choose a partition of unit, i.e. smooth functions ρ_i supported on U_i such that $0 \leq \rho_i \leq 1$ and $\sum_i \rho_i(x) = 1$ for all $x \in K$. Let $E \subset \Omega$ be a compact fiberwise convex subset which conains all the points $(x, -df^{\mathbf{k}}(x), f^{\mathbf{k}}(x))$, $x \in K$.

2)There exist a positive integer $\mathbf{K} > 0$ such that

— for every i and for every function $\psi(x)$ such that $(x, -d\psi(x), \psi(x)) \in E)$ for all $x \in K$, we have: $(x, -dT_i\psi(x), T_i\psi(x)) \in \Omega$ for all $x \in K$, where

$$T_i\psi(x) = \psi(x) + \frac{(\rho_1(x) + \rho_2(x) + \cdots + \rho_i(x))(f^1(x) - f^0(x))}{\mathbf{K}}.$$

3) Consider the sequence of functions $f^0(x), T_1f^0(x), \ldots, T_nf^0(x)$. It follows that $(T_if^0(x), T_{i+1}f^0(x))$ is an Ω_i-variation. Next,

$$T_nf^0(x) = f^0 + \frac{(f^1 - f^0)}{\mathbf{K}} \in E.$$

We therefore can continue our sequence by adding

$$T_1(f^0 + \frac{f^1 - f^0}{\mathbf{K}}), T_2(f^0 + \frac{f^1 - f^0}{\mathbf{K}}, \ldots, T_n(f^0 + \frac{f^1 - f^0}{\mathbf{K}}) = f^0 + \frac{2(f^1 - f^0)}{\mathbf{K}}.$$

By repeating this process $\mathbf{K}$ times we prove the statement.

Case 2 It suffices to choose $\phi^k : (1 - k/\mathbf{K})f^0 + k/\mathbf{K}f^1$, where $\mathbf{K}$ is large enough and $k = 0, 1, 2, \ldots, \mathbf{K}$.

7.2.1 Convolution $\mathbb{A}_{[K,f]} \star \mathbb{A}_\ell$

Let X, Y be smooth manifolds. Let $\iota : X \to Y$ be a closed embedding. Let f be a smooth function on X. Let Γ be a graph of ι. Let f be a lower continuous function on X. Denote by $\kappa : \Gamma \to X$ the identification. Let $\ell = \{f^{\mathbf{k}}\}$ be a lense on Y. Let

$$T_f\ell := \{f^{\mathbf{k}} \circ \iota + f\},$$

so that $T_f\ell$ is a lense on X.

Proposition 7.1 *We have*

$$\mathbb{A}_{[X,f]} *_Y \mathbb{A}_\ell \approx \mathbb{A}_{T_f\ell}.$$

Sketch of the proof Follows from Sec. 6.0.15.

7.3 Definition of Singular Support

7.3.1 Ω-stable objects

Denote by $a : T^*X \times \mathbb{R} \to T^*X \times \mathbb{R}$ the following reflection map $a(x, \omega, t) = (x, -\omega, -t)$. Let $F \in \mathrm{psh}_\varepsilon(X)$. Call F Ω-*stable* if $F \bullet \mathcal{R}_\ell \sim 0$ for every Ω^a-lense ℓ.

7.3.2 Definition of Singular Support

Let $F \in \mathrm{sh}_\varepsilon(X)$. Define an open subset $U \subset T^*X$ as the union of all open fiberwise open subsets $\Omega \subset T^*X \times \mathbb{R}$ such that F is Ω-stable. Observe that F is Ω-stable iff $\Omega \subset U$. Indeed, if $\Omega \subset U$, then

$$\Omega \subset \bigcup_{a \in A} \Omega_a,$$

where F is Ω_a stable for all $a \in A$. Let ℓ be an Ω-lense, then there exists a pre-compact firerwise convex subset $\Omega' \subset \Omega$ such that ℓ is an Ω'-lense. One then can select a finite subset $B \subset A$ such that $\Omega \subset \bigcup_{b \in B} \Omega_b$. The statement now follows from Sec. 7.2.

Denote $\mathrm{SS}(F) := T^*X \times \mathbb{R} \backslash U$ so that F is Ω-stable iff $\Omega \cap \mathrm{SS}(F) = \emptyset$.

7.4 Properties of Singular support

7.4.1 Dual definition

Proposition 7.2 *Let $F \in \mathrm{sh}(X, C)$. Then F is non-singular on an open subset $\Omega \subset T^*X \times \mathbb{R}$ iff $Hom(\mathbb{A}_\ell, F) \sim 0$ for any Ω-lense ℓ supported on a compact $K \subset X$.*

Sketch of the proof Let $\ell = \{f^{\mathbf{k}}\}$ be a lense. Let $\ell_\delta^\vee := \{-f^{\mathbf{k}} - \delta\}$. As follows from Theorem 6.5, we have a zig-zag homotopy equivalence

$$\mathrm{Hom}(\mathbb{A}_\ell; F) \approx \mathrm{holim}_{\delta \downarrow 0} F \bullet_X \mathbb{A}_{\ell_\delta^\vee},$$

The statement now follows.

7.4.2 Convolution with a graph

Let $f : X \to \mathbb{R}$ be a smooth function. Let $T_f : T^*X \times \mathbb{R} \to T^*X \times \mathbb{R}$ be given by $T_f(x, \omega, t) = (x, \omega - df_x; t + f_x)$. If ℓ is an Ω-lense supported on K, then $T_f \ell$ is a $T_f \Omega$-lense supported on K.

Let $\Delta_X \subset X \times X$ be the diagonal. Let $f_\Delta : \Delta = X \xrightarrow{f} \mathbb{R}$.

Proposition 7.3 *Let $F \in \mathrm{sh}_\varepsilon(X)$ and let $SS(F) \subset C$. We have*

$$SS(F *_X \mathbb{A}_{[\Delta_X, f_\Delta]}) \subset T_f C.$$

Sketch of the proof. Let $\Omega \subset T^*X \times \mathbb{R}$ be an open fiberwise convex subset such that $\Omega \cap (T_f C)^a = \emptyset$. Let ℓ be an Ω-lense. We have

$$(F *_X \mathbb{A}_{[\Delta_X, f_\Delta]}) \bullet \mathbb{A}_\ell = F \bullet (\mathbb{A}_{[\Delta_X, f_\Delta]} *_X \mathbb{A}_\ell) \overset{(1)}{\approx} F \bullet \mathbb{A}_{T_f \ell} \overset{(2)}{\sim} 0,$$

where (1) follows from Sec 7.2.1 and (2) follows from $T_f \ell$ being a $T_f \Omega$ lense, where

$$T_f \Omega \cap C^a = T_f \Omega \cap T_f (T_f C)^a = T_f (\Omega \cap (T_f C)^a) = \emptyset.$$

7.4.3 Variation of lenses

Proposition 7.4 *Let M be a smooth manifold and let $F^{\mathbf{k}}$ be smooth functions on $X \times M$ such that for every $m \in M$, $\{F^{\mathbf{k}}(m, -)\}$ is an Ω-lense supported on a compact K. Let $\ell := \{F^{\mathbf{k}}\}$. Let $F \in \mathrm{sh}_\varepsilon(X)$ and $SS(F) \cap \Omega^a = \emptyset$. Then $\mathbb{A}_\ell \bullet_X F \sim 0$ as an object of $\mathrm{sh}(M)$.*

Sketch of the proof. 1) It suffices to show that $\mathbb{A}_\ell \bullet_X F(U) \sim 0$, where $U \subset M$ is an arbitrary pre-compact subset.

2) There exists a $\delta > 0$ such that for every $m \in M$ and every $\delta' \in [0, \delta)$, $\{F^{\mathbf{k}}(x, m) - \delta'\}$ is an Ω-lense supported on K.

3) Let $V \subset U$ be an open subset. Set

$$f^{\mathbf{k}}(x)_V := \sup_{v \in V} f^{\mathbf{k}}(x, v), \quad x \in X.$$

As follows from Sec. 7.1.5, $\{f_V^{\mathbf{k}}\}$ is an Ω-lense supported on K, and so is $\ell_{V,\delta'} := \{f_V^{\mathbf{k}} - \delta'\}$ for all $\delta' \in [0, \delta)$.

4) Call V δ'-small if $f(x) - f(y) > -\delta'$ for all $x, y \in V$. We then have $f_V^{\mathbf{k}} - \delta' \leq f$ on $X \times V$.

5) For an open δ'-small subset $V \Subset U$, set

$$\mathcal{F}_{V,\delta'} := \mathbb{A}_{\ell_{V,\delta'}} \boxtimes \mathbb{A}_V \in \mathrm{sh}_\varepsilon(X \times M).$$

Let P be the poset whose each element is a pair (V, δ'), where V is δ' small. The order is defined by $(V_1, \delta_1') \leq (V_2, \delta_2')$ if $V_1 \subset V_2$ and $\delta_1' \geq \delta_2'$. Then $\mathcal{F} : P \to \mathrm{sh}_\varepsilon(X \times M)$. We have a natural map

$$\mathrm{hocolim}_P \mathcal{F} \to \mathbb{A}_\ell.$$

7) Let us show that this map induces a homotopy equivalence

$$\mathrm{hocolim}_{(V,\delta') \in P} \mathcal{F}_{V,\delta'}(W \times U, a) \to \mathbb{A}_\ell(W \times U, a)$$

for all $a \in Q_\varepsilon^{\mathbf{op}}$ and all $W \in \mathrm{Open}_K$.

Using (7.1.3), the problem reduces to showing that the natural map

$$\mathrm{hocolim}_{(V,\delta') \in P} \mathbb{A}_{\{x \in K | \exists v \in V : f^{\mathbf{k}}(x,v) - \delta' > a\} \times V}(W \times U) \to \mathbb{A}_{\{(x,v) \in K \times U | f^{\mathbf{k}}(x,v) > a\}}(W \times U) \quad (30)$$

is a homotopy equivalence.

Fix a value of $\mathbf{k}$. For $p = (V, \delta') \in P$, denote

$$W_p^{\mathbf{k}} := \{x \in K | \exists v \in V : f^{\mathbf{k}}(x, v) - \delta' > a\} \times V \in \mathrm{Open}_{K \times U}$$

and

$$W^{\mathbf{k}} := \{(x, v) \in K \times U | f^{\mathbf{k}}(x, v) > a\}.$$

The natural zig-zag homotopy equivalences $\mathbb{A}_A(B) \approx \mathbb{A}_A \circ \mathbb{A}_B \approx \mathbb{A}_B(A)$ show that the arrow in (30) is zig-zag homotopy equivalent to

$$\mathrm{hocolim}_{p \in P} \mathbb{A}_{W \times U}(W_p^{\mathbf{k}}) \to \mathbb{A}_{W \times U}(W^{\mathbf{k}}). \tag{31}$$

Observe that the set $\{W_p^{\mathbf{k}}\}_{p \in P}$ is closed under finite intersection:

$$W^{\mathbf{k}}_{V_1, \delta_1} \cap W^{\mathbf{k}}_{V_2, \delta_2} = W^{\mathbf{k}}_{V_1 \cap V_2; \max(\delta_1, \delta_2)}.$$

Therefore, $\{W_p^{\mathbf{k}}\}_{p \in P}$ is an open covering of

$$\bigcup_{p \in P} W_p^{\mathbf{k}} = W^{\mathbf{k}}$$

so that the map in (31) is a homotopy equivalence by the gluing property for the sheaf $\mathbb{A}_{W \times U}$.

7.5 Singular support of $F \boxtimes G$

Let $F \in \mathrm{sh}_\varepsilon(X)$ and $G \in \mathrm{sh}_\varepsilon(Y)$. Suppose $\mathrm{SS}(F) \subset A$ and $\mathrm{SS}(G) \subset B$. Consider the following subset of $T^*(X \times Y) \times \mathbb{R}$

$$C_0(F, G) = \{(x, \omega, y, \eta, t_1 + t_2) | (x, \omega, t_1) \in A; (y, \eta, t_2) \in B\}.$$

Let $C(F, G)$ be the closure of $C_0(F, G)$.

Claim 7.5 *We have $SS(F \boxtimes G) \subset C(F, G)$.*

Sketch of the proof.

0) For $a \in \mathbb{R}$. Define functors

$$\mathbf{cut}_{t<a}, R_{>a}, \mathbf{cut}_{\geq a}, R_{\leq a} : \mathcal{Q}_\varepsilon \to \mathcal{Q}_\varepsilon$$

as follows. Set:

— $\mathbf{cut}_{t<a} e_b = e_b$ if $b \leq a - \varepsilon$;

—$\mathbf{cut}_{t<a} e_b = \mathrm{Cone}\, e_b \to e_a$[-1] if $a - \varepsilon < b \leq a$;

— $\mathbf{cut}_{t<a} e_b = 0$ if $b > a$;

—$\mathbf{cut}_{t \geq a} e_b = 0$ if $b \leq a - \varepsilon$;

—$\mathbf{cut}_{t \geq a} e_b = e_a$ if $a - \varepsilon < b \leq a$;

—$\mathbf{cut}_{t \geq a} e_b = e_b$ if $b > a$.

— $R_{>a}e_b = e_b$ if $b > a$;

— $R_{>a}e_b = 0$ if $b \leq a$;

—$R_{\leq a}e_b = 0$ if $b > a$;

—$R_{\leq a}e_b = e_b$ if $b \leq a$.

These functors extend to functors $\mathrm{sh}_\varepsilon(X) \to \mathrm{sh}_\varepsilon(X)$. One has

$$\mathbf{cut}_{\geq a}\mathbb{A}_{[K,f]} \approx \mathbb{A}_{[K;\max(a,f)]}; \quad R_{\leq a}\mathbb{A}_{[K,f]} = \mathbb{A}_{K';f},$$

where $K' = \{x \in K | f(x) \leq a\}$.

We have natural transformations $\mathbf{cut}_{t<a} \to \mathrm{Id} \to \mathbf{cut}_{t\geq a}$; $R_{>a} \to \mathrm{Id} \to R_{\leq a}$. whose compositions are 0. The complexes

$$0 \to \mathbf{cut}_{t<a}F \to F \to \mathbf{cut}_{t\geq a}F \to 0; \quad 0 \to R_{>a}F \to F \to R_{\leq a}F \to 0$$

are acyclic for every $F \in \mathrm{psh}_\varepsilon(X)$.

We have $\mathbf{cut}_{t<a}F \bullet_X R_{\leq -a}G \sim 0$; $\mathbf{cut}_{t\geq a} \bullet_X R_{>-a}G \sim 0$ for all $F, G \in \mathrm{psh}_\varepsilon(X)$. Hence, the induced maps

$$\mathbf{cut}_{t<a}F \bullet_X R_{>-a}G \to \mathbf{cut}_{t<a} \bullet_X G$$

and

$$\mathbf{cut}_{t<a}F \bullet_X R_{>-a}G \to F \bullet R_{>-a}G$$

are homotopy equivalences. We have

$$\mathbf{cut}_{t<a}F \bullet_X G \approx F \bullet_X R_{>-a}G \tag{32}$$

Similarly, we get

$$\mathbf{cut}_{t\geq a}F \bullet_X G \approx F \bullet_X R_{\leq -a}G. \tag{33}$$

Whenever $a \leq b$ we have a natural transformation $\mathbf{cut}_{<a} \to \mathbf{cut}_{<b}$. Let $\mathbf{cut}_{a\leq t,b} := \mathrm{Cone}\,\mathbf{cut}_{<a} \to \mathbf{cut}_{<b}$.

Let us also denote $T_c : Q_\varepsilon \to Q_\varepsilon$; $T_c a = a + c$.

1) Let $P := (x_0, p_0, y_0, q_0, t_0) \notin C(F, G)$. Let us show that $F \boxtimes G$ is nonsingular at P. Let f be a smooth function on X and g on Y such that $f(x_0) = 0$, $g(y_0) = -t_0$, $d_{x_0}f = -p_0$, $d_{y_0}g = -q_0$. Let $h : X \times Y \to \mathbb{R}$ so that $h(x, y) = f(x) + g(y)$. We then have

$$\mathbb{A}_{[\Delta_X;f_\delta]} \boxtimes \mathbb{A}_{[\Delta_Y;g_\Delta]} \approx \mathbb{A}_{[\Delta_{X\times Y},h_\Delta]}.$$

Let $F' := F *_X \mathbb{A}_{[\Delta_X;f_\Delta]}$, $G' := G *_Y \mathbb{A}_{[\Delta_Y;g_\Delta]}$, $(F \boxtimes G)' := (F \boxtimes G) * \mathbb{A}_{[\Delta_{X\times Y};h_\Delta]}$. We then have $F' \boxtimes G' \approx (F \boxtimes G)'$.

It now follows that $\mathrm{SS}(F') = T_f\mathrm{SS}F$, $\mathrm{SS}(G') = T_g\mathrm{SS}G$, and $\mathrm{SS}(F \boxtimes G) = T_h\mathrm{SS}(F \boxtimes G)$. It also follows that $C(F', G') = T_hC(F, G)$. We therefore have $P' = (x_0, 0, y_0, 0, 0) \notin C(F', G')$ and it suffices to prove that $P' \notin \mathrm{SS}(F' \boxtimes G')$.

Therefore, the problem reduces to showing that if $P = (x_0, 0, y_0, 0, 0) \notin C(F, G)$, then $P \notin \mathrm{SS}(F \boxtimes G)$. We assume below that $P \notin C(F, G)$.

2) There exist neighborhoods U of $(x_0,0)\in T^*X$,V of $(y_0,0)\in T^*Y$, and $\delta>0$, such that whenever $(p,t_1)\in \mathrm{SS}F$ and $(q,t_2)\in \mathrm{SS}G$ with $p\in U$ and $q\in V$, there must be $|t_1+t_2|>\delta$.

3) Let $A=\{t\in\mathbb{R}|\exists p\in U:(p,t)\in \mathrm{SS}F\}$; $B=\{t\in\mathbb{R}|\exists q\in V:(q,t)\in \mathrm{SS}G\}$. It follows that $\mathrm{dist}(A,-B)>\delta$.

4) Let $t\in\mathbb{R}$. It follows that either $[t-\delta/2,t+\delta/2]\cap A=\emptyset$ or $[t-\delta,t+\delta/2]\cap -B=\emptyset$. In the first case call $[t-\delta/2,t-\delta/2]$ *an A-interval*, and t *an A-point*. Otherwise, call $[t-\delta/2,t-\delta/2]$ *a B-interval*, and t *a B-point*.

5) Let $f^{\mathbf{k}}(x,y)$ be an $U\times V\times(-\delta/4,\delta/4)$-lense, to be denoted by ℓ.

6) Let $a,b\in\mathbb{R}$ satisfy $a+b\geq-\delta/2$.

Suppose $[a-\delta/4,a+3\delta/4]$ is an A-interval. Let us show that

$$(R_{>a}F\boxtimes R_{>b}H)\bullet_{X\times Y}\mathbb{A}_\ell\sim 0 \tag{34}$$

for every $H\in\mathrm{psh}_\varepsilon(X)$, hence for G.

It suffices to check it for $H=[W,c]$, where $W\subset X$ is an open subset and $c\in\mathbb{R}$. The statement then follows automatically for $c\leq b$ as $R_{>b}H=0$ in this case.

7) Consider the case $c>b$. We have $R_{>b}[W,c]=[W,c]$. We have

$$(R_{>a}F\boxtimes[W,c])\bullet\mathbb{A}_\ell=(R_{>a}F\bullet_X T_c\mathbb{A}_\ell)(W)=(R_{>a}F\bullet_X\mathbb{A}_{T_c\ell})(W)\approx(F\bullet_X\mathbf{cut}_{<-a}\mathbb{A}_{T_c\ell})(W),$$

where we have used (32). We have

$$\mathbf{cut}_{<-a}\mathbb{A}_{T_c\ell}\approx\mathbb{A}_{\ell'},$$

where

$$\ell'=\{\min(f^{\mathbf{k}}(x)+c,-a)\}=\min\ell,\ell_{-a},$$

where ℓ_{-a} is the lense $f^1=f^2=-a$. We have

$$-a-3\delta/4\leq b-\delta/4\leq\min(f^{\mathbf{k}}(x)+c,-a)\leq -a.$$

Thus, ℓ' is a generalized $U\times V\times(-a-3\delta/4,-a+\delta/4)$-lense so that

$$F\bullet_X\mathbb{A}_{\ell'}\sim 0,$$

as was required.

8) Consider now the case when $(a-\delta/4,a+3\delta/4)$ is a B-interval.

Then we replace $R_{>a}F$ with $[W,c]$, where $c>a$. We are to prove

$$([W,c]\boxtimes R_{>b}G)\bullet_{X\times Y}\mathbb{A}_\ell\sim 0,$$

where ℓ is a $U\times V\times(-\delta/4,\delta/4)$-lense.

Similar to above, we have

$$([W,c]\boxtimes R_{>b}G)\bullet_{X\times Y}\mathbb{A}_\ell\approx(R_{>b}G\bullet_X\mathbb{A}_{T_c\ell})(W)\approx(G\bullet_X\mathbf{cut}_{<-b}\mathbb{A}_{T_c\ell})(W)\approx(G\bullet_X\mathbb{A}_{\ell'})(W),$$

where
$$\ell' = \min(f^{\mathbf{k}} + c, -b).$$

We have
$$a - \delta/4 \leq \min(f^{\mathbf{k}} + c, -b) \leq -b \leq a + 3\delta/4.$$

and ℓ' is a $U \times V \times (a - \delta/4, a + 3\delta/4)$-lense.

As $(a - \delta/4, a + 3\delta/4$ is a B-interval, G is nonsingular on $V \times (-(a - \delta/4), -(a + 3\delta/4))$ so that $G \bullet_Y \mathbb{A}_{\ell'} \sim 0$, as was required.

9) Let $a + b \leq -\delta/2$. Let $a_1 = -b - \delta/4, b_1 = b - \delta/4$. We have $b_1 \leq b$; $a_1 \geq a + \delta/2 - \delta/4 \geq a$. We then have the following acyclic complex:

$$0 \to R_{>a_1} F \boxtimes R_{>b} G \to R_{>a} F \boxtimes R_{>b} G \oplus R_{>a_1} F \boxtimes R_{>b_1} G \to R_{>a} F \boxtimes R_{>b_1} G$$
$$\to R_{a<t\leq a_1} F \boxtimes R_{b_1<t\leq b} G \to 0.$$

As $a_1 + b = -\delta/4$, we have
$$(R_{a<t\leq a_1} F \boxtimes R_{b_1<t\leq b} G) \bullet \mathbb{A}_\ell \sim 0.$$

Indeed, as $a_1 + b = -\delta_4$, the natural map

$$(R_{a<t\leq a_1} F \boxtimes R_{b_1<t\leq b} G) \to R_{\leq -\delta/4}(R_{a<t\leq a_1} F \boxtimes R_{b_1<t\leq b} G)$$

is a homotopy equivalence so that we have

$$(R_{a<t\leq a_1} F \boxtimes R_{b_1<t\leq b} G) \bullet \mathbb{A}_\ell \approx (R_{\leq -\delta/4}(R_{a<t\leq a_1} F \boxtimes R_{b_1<t\leq b} G)) \bullet \mathbb{A}_\ell \approx (R_{a<t\leq a_1} F \boxtimes R_{b_1<t\leq b} G) \bullet \mathbf{cut}_{\geq \delta/4} \mathbb{A}_\ell.$$

, Finally, $\mathbf{cut}_{\geq \delta/4} \mathbb{A}_\ell \approx \mathbb{A}_{\ell''}$, where $\ell'' = \max(\delta/4, f^{\mathbf{k}}) = \delta/4$ so that $\mathbb{A}_{\ell''} \sim 0$. which implies the statement.

Next, $a_1 + b_1 = -\delta/2$ and $a_1 + b = -\delta/4 \geq -\delta/2$, we have (by 7) and 8)):

$$(R_{>a_1} F \boxtimes R_{>b_1} G) \bullet \mathbb{A}_\ell \sim 0;$$

$$(R_{>a_1} F \boxtimes R_{>b} G) \bullet \mathbb{A}_\ell \sim 0.$$

Thus, if $a + b \leq -\delta/2$ and $(R_{>a} F \boxtimes R_{>b} G) \bullet \mathbb{A}_\ell \sim 0$, then $(R_{>a} F \boxtimes R_{>b-\delta/4} G) \bullet \mathbb{A}_\ell \sim 0$. Taking into account 7), 8), it now follows by induction that $(R_{>a} F \boxtimes R_{>b} G) \bullet \mathbb{A}_\ell \sim 0$, whenever $a + b = -\delta/2 - N\delta/4$, $N \geq 0$.

9) We have
$$\operatorname{hocolim}_{N\to\infty}(R_{>-\delta/4-N\delta/8} F \boxtimes R_{>-\delta/4-N\delta/8} G) \xrightarrow{\sim} F \boxtimes G,$$

which implies the statement.

7.5.1 Singular support of $\mathbb{A}_{[X,f]}$

Let $f : X \to \mathbb{R}$ be a smooth function. Set $\mathcal{L}_f := \{(x, -d_x f, f(x)|x \in X\} \subset T^*X \times \mathbb{R}$.

Proposition 7.6 *We have* $SS\mathbb{A}_{[X,f]} \subset \mathcal{L}_f$.

Sketch of the proof As follows from Sec 6.0.15, 7.4.2, it suffices to consider the case $f = 0$. Next, it suffices to consider the case $X = \mathbb{R}^n$ which reduces to the case $n = 1$ by virtue of the previous section.

Let $(x_0, p_0, t_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} = T^*\mathbb{R}\times\mathbb{R}$, where $(p_0, t_0) \neq (0, 0)$. Choose $\delta > 0$ so that $\max(|p_0|, |t_0|) > 2\delta$.

Let $U = \{(x, p, t)|\ |x - x_0|, |p - p_0|, |t - t_0| < \delta\}$. Let $\ell = \{f^{\mathbf{k}}\}$ be a U-lense on $\mathbb{R}$ supported on $|x - x_0| \leq \delta$.

We have

$$\mathbb{A}_{[\mathbb{R},0]} \bullet \mathbb{A}_\ell \approx \operatorname{Cone} \mathbb{A}_{\mathbb{R}}(K_0) \to \mathbb{A}_{\mathbb{R}}(K_1),$$

where $K_{\mathbf{k}} = \{x|\ |x| \leq \delta,\ f^{\mathbf{k}}(x) \leq 0\}$. *Case 1.* $|t_0| > 2\delta$. As $|f^{\mathbf{k}}(x) + t_0| < \delta$, $f^{\mathbf{k}}(x)$ are of the same sign for all $\mathbf{k}$ and all x, $|x| \leq \delta$ so that $K_1 = K_0$.

Case 2. $|p_0| > 2\delta$. As $|f^{\mathbf{k}}(x)' + p_0| < \delta$, $f^{\mathbf{k}}(x)'$ are of the same sign for all $\mathbf{k}$ and all x, $|x| \leq \delta$. Therefore, $K_{\mathbf{k}} = [f^{\mathbf{k}}(-\delta), f^{\mathbf{k}}(-\delta)]$. So that the arrow $\mathbb{A}_{\mathbb{R}}(K_0) \to \mathbb{A}_{\mathbb{R}}(K_1)$ is homotopy equivalent to the identity arrow $\mathbb{A} \to \mathbb{A}$, whence the statement.

7.5.2 SS$\mathbb{A}_{[\overline{U},0]}$, where U has a smooth boundary

Let $U \subset X$ be a domain with a smooth boundary. Let f be a smooth function in a neighborhood of U. For $x \in X$ set $n_x \subset T^*_x X$ be defined as follows: $n_x = 0$ if $x \in U$; n_x is the closed ray consisting of all inner normal vectors at x to U if x is a boundary point of U; $n_x = \emptyset$ otherwise. Set

$$\Sigma := \bigcup_{x\in X} n_x.$$

Proposition 7.7 *We have*

$$SS\mathbb{A}_{[\overline{U},0]} \subset \Sigma \times \{0\} \subset T^*X \times \mathbb{R}.$$

Sketch of the proof Choose an increasing sequence of smooth functions $f_n(x)$ such that $f_n(x) \to \infty$ for all $x \notin \overline{U}$ and $f_n(x) = 0$ for all $x \in \overline{U}$.

We then have

$$\operatorname{hocolim}_{n\to\infty} \mathbb{A}_{\{(t,x)|t\geq f_n(x)\}} \to \mathbb{A}_{[\overline{U},0]}.$$

Let $p \in T^*X \times \mathbb{R}\backslash\Sigma \times 0$. It follows that there for every neighborhood V of p there exists an N such that

$$\mathrm{SS}\mathbb{A}_{[X,f_n(x)]} = \mathcal{L}_{f_n} \cap V = \emptyset.$$

This implies the statement.

7.5.3 $\mathbf{SS}\mathbb{A}_{[U,0]}$

Proposition 7.8 *We have*

$$SS\mathbb{A}_{U,0]} \subset (\Sigma \times \{0\})^a \subset T^*X \times \mathbb{R}.$$

Sketch of the proof Apply the previous Proposition to $X \backslash U$.

7.5.4 Inverse image under closed embedding

Let $i : Y \to X$ be a closed embedding. Let S be a closed subset of $T^*X \times \mathbb{R}$. Define a closed subset $C'_Y S := S \hat{+} T^*_Y X|_Y \subset T^*X|_Y \times \mathbb{R}$, where $\hat{+}$ is the Whitney sum. Let $C_Y S \subset T^*Y \times \mathbb{R}$ be the image of $C'_Y S$ under the projection $T^*X|_Y \to T^*Y$. In local coordinates: let y be coordinates on Y and (y, x) on X. A point $(y_0, q_0, t_0) \in C_Y S$ iff there exists a sequence $(y_n, q_n, x_n, p_n, t_n) \in S$, where $(y_n, q_n, x_n, t_n) \to (y, q, 0, t)$ and $|x_n| p_n \to 0$.

Let $S \in \mathrm{sh}_\varepsilon(X)$. Then $\mathrm{SS} i^{-1} S \in C_Y \mathrm{SS} S$.

Sketch of the proof. 1) Let us introduce local coordinates (x, y) so that Y is given by the equation $x = 0$. Suppose $(0, \eta_0, t_0) \notin C_Y \mathrm{SS}S$. We need to show that $i^{-1}S$ is non-singular at $(0, \eta_0, t_0)$. By change of variable $t \mapsto t - t_0 - (\eta_0, y) - 1$, we reduce the problem to the case $\eta_0 = 0$, $t_0 = -1$.

Thus $(0, 0, -1) \notin C_Y \mathrm{SS}S$. This implies that there exists $\delta > 0$ such that $(x, \omega, y, \eta, t) \notin \mathrm{SS}S$, whenever

$$|x| < \delta,\ |y| < \delta,\ |\eta| < \delta,\ |t + 1| < \delta, |\omega||x| < \delta.$$

Denote this set by W

2) Lemma. For each $r_0 > 0$ there exists a smooth non-decreasing function $g_{r_0} : [0, \infty) \to [0, 1]$ such that

a) there exists $\delta > 0$ such that $g(x) = 0$ for all $x \in [0, \delta]$.

b) $g(r_0) = 1$, in particular $g(r) = 1$ for all $r \geq r_0$,

c) $|rg'(r)| < 1/2$ for all $r \geq 0$.

d) $g_{r_0}(x) \geq g_{r_1}(x)$ whenever $r_0 \leq r_1$.

3) Let $f^{\mathbf{k}}(y)$ be a W'-lense on Y, where $W' = \{(y, \eta, t) | |y| < \delta, |\omega| < \delta, |t - 1| < \delta$, supported on the set $|y| \leq \delta$. Set

$$\phi^{\mathbf{k}}_{r_0}(x, y) = (f^{\mathbf{k}}(y) - 1 - \delta)(1 - g(|x|)) + 1 + \delta$$

. Let us show that $\{\phi^{\mathbf{k}}_{r_0}\}$ is a W^a-lense supported on the set $K_{r_0} := \{(x, y) |\ |x| \leq r_0, |y| \leq \delta\}$.

a) it is clear that $\phi^1_{r_0} = \phi^2_{r_0}$ away from K_{r_0};

b) $1 + \delta > \phi_{r_0} > (1 - \delta - 1 - \delta) + 1 + \delta = 1 - \delta$;

c)

$$|x| \cdot |d_x \phi^{\mathbf{k}}_{r_0}| = |x| \cdot |f^{\mathbf{k}}(y) - 1 - \delta| \cdot |g'(|x|)| < |x| \cdot |g'(|x|)| \cdot 2\delta \leq \delta;$$

d)

$$|d_y \phi^{\mathbf{k}}_{r_0}| = |d_y f^{\mathbf{k}}(y)| \cdot |1 - g| < \delta.$$

We have $\phi^{\mathbf{k}}_{r_0}(x,y) \geq \phi^{\mathbf{k}}_{r_1}(x,y)$ if $r_0 \leq r_1$. Furthermore

$$\lim_{r_0 \downarrow 0} \phi^{\mathbf{k}}_{r_0}(x,y) = \psi^{\mathbf{k}}(x,y),$$

where $\psi(x,y) = 1 + \delta$ if $x \neq 0$ and $\psi^{\mathbf{k}}(0,y) = f(y)$. Let $\ell_{r_0} := \{\phi^{\mathbf{k}}_{r_0}\}$.
We therefore have a homotopy equivalence

$$0 \sim \mathrm{hocolim}_{r_0 \downarrow 0} F \bullet_X \mathbb{A}_{\ell_{r_0}} \approx F \bullet_X i_! \mathbb{A}_\ell \approx i^{-1} F \bullet_Y \mathbb{A}_\ell.$$

This shows the statement.

7.5.5 Direct image under closed embedding

Let $i : Y \to X$ be a closed embedding Let $F \in \mathrm{sh}_\varepsilon(Y)$. For every $y \in Y$, let $p_y : T^*_y X \to T^*_y Y$ be the projection.

Proposition 7.9 *We have*

$$SSi_! F \subset \{(y,\omega,t) | y \in Y; (y, p_y(\omega), t) \in SS(F)\}$$

.

Sketch of the proof Let $\ell = \{f^{\mathbf{k}}\}$ be a lense on X. We have

$$i_! F \bullet \mathbb{A}_\ell \approx F \bullet \mathbb{A}_{i^{-1}\ell},$$

where $i^{-1}\ell = \{f^{\mathbf{k}}|_Y\}$.

7.5.6 Direct image under open embedding

Let $U \subset X$ be a domain with a smooth boundary. Let Σ be the same as in Sec. 7.5.2. Let $F \in \mathrm{sh}_\varepsilon(U)$. Let $j : U \to X$ be the embedding.

Proposition 7.10 *We have*

$$SS(j_! F) \subset SS(F) \hat{+} \Sigma^a.$$

Sketch of the proof By change of coordinates one reduces the case to $U \subset \mathbb{R}^n$, where U is a hyperplane $x^0 > 0$. Let us denote $y := (x^1, x^2, \ldots, x^n)$ and $x := x^0$. Let $p \notin \mathrm{SS}(F) \hat{+} \Sigma^a$, w.l.o.g. we may assume $p = (x_0, 0, -1) \in T^*\mathbb{R}^n \times \mathbb{R}$. Therefore, there exists $\delta > 0$ such that F is non-singular on the open subset $W \subset T^*\mathbb{R}^n \times \mathbb{R}$ conisting of all points

$$(y, x, a, \eta, t) \in \mathbb{R}_{>0} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R},$$

where

$$0 < y < \delta; |x| < \delta;\ \delta > a > -\frac{\delta}{y};\ |\eta| < \delta;\ |t+1| < \delta.$$

Let $\Omega \subset T^*\mathbb{R}^n \times \mathbb{R}$ consist of all points (y, x, b, ω) of the form

$$|y| < \delta, |x| < \delta, |b| < \delta, |\omega| < \delta, |t + 1| < \delta.$$

$\ell = \{f^{\mathbf{k}}\}$ be an Ω^a-lense supported on the set $|y| \leq \delta, |x| \leq \delta$.

Choose smooth functions for each $r > 0$: $\sigma_r : \mathbb{R} \to [0, 1]$, satisfying:

—$\sigma_r(x) = 0$, $x \leq 0$; $\sigma_{r_0}(x) = 1$ for $x \geq r_0$;

— $\sigma_r(x) = 0$ in a neighborhood of 0;

— $\sigma_{r_1}(x_1) \leq \sigma_{r_2}(x_2)$ if $r_1 \geq r_2$ and $x_1 \leq x_2$;

—$x\sigma_r'(x) \leq 1/2$ for all x and all r.

Let us define new lenses

$$F_r^{\mathbf{k}}(y, x) = (f^{\mathbf{k}}(y, x) - 1 + \delta)\sigma_r(y) + 1 - \delta.$$

Let $\ell_r := \{F_r^{\mathbf{k}}\}$. We have

— ℓ_r is supported on a compact within U;

— ℓ_r is an Ω-lense. Indeed:

$$1 - \delta \leq F^k k \leq 1 + \delta;$$

$$|d_x F^{\mathbf{k}}| = |d_x f^{\mathbf{k}}|\sigma(y) \leq |d_x f^{\mathbf{k}}| < \delta;$$

$$-\delta y < \min(y d_y f^{\mathbf{k}}, 0) < y d_y F^{\mathbf{k}} < 1/2|f^{\mathbf{k}} - 1 + \delta| < \delta.$$

We also have $\lim_{r \downarrow 0} F_r^{\mathbf{k}}(y, x) = f^{\mathbf{k}}(y, x)$ for all $(y, x) \in U$. The statement now follows.

7.5.7 Proper direct image

let $f : X \to Y$ be a proper map of smooth manifolds. Let $F \in \mathrm{sh}_\varepsilon(X, C)$ and $\mathrm{SS}F \subset T$. Let $f(T) \subset T^*Y \times \mathbb{R}$ be the set consisting of all points (x, ω, t), where there exists a $y \in p^{-1}x$ such that $(y, f^*\omega, t) \in T$.

Proposition 7.11 *We have $SSf_!F \subset f(T)$.*

7.5.8 Direct image along $\mathbb{R}^n$

Let $p : X \times \mathbb{R}^n \to X$ be the projection. Let $F \in \mathrm{sh}_\varepsilon(X \times \mathbb{R}^n)$ and let $\mathrm{SS}F \subset T \subset T^*X \times T^*\mathbb{R}^n \times \mathbb{R}$. Let $P : T^*X \times T^*\mathbb{R}^n \times \mathbb{R} \to T^*X \times (\mathbb{R}^n)^* \times \mathbb{R}$ be the projection and let $I : T^*X \times \mathbb{R} \to T^*X \times (\mathbb{R}^n)^* \times \mathbb{R}$ be the embedding onto $T^*X \times 0 \times \mathbb{R}$. Let $f(T) := I^{-1}\overline{P(T)}$.

Proposition 7.12 *We have $SSF \subset f(T)$.*

7.5.9

Let X be a smooth manifold with a marked point x_0. Let $\mathcal{S}_X \subset \mathrm{sh}_\varepsilon(X)$ be the full sub-category consisting of all objects F such that $\mathrm{SS}F \subset T^*X \times \{0\}$.

We have functors $\mathrm{sh}_\varepsilon(X) \overset{F}{\to} \mathrm{sh}_(X) \overset{G}{\to} \mathrm{sh}_\varepsilon(X)$, where $F(S) = S \bullet_{\mathbf{pt}} \mathbb{A}_{t\geq 0}$; $G(T) := T \boxtimes \mathbb{A}_{t\geq 0}$.

Proposition 7.13 *The functors F, G are mutually inverse equivalences of categories.*

7.5.10

Let X be a simply-connected manifold with a marked point x_0. Let $\mathrm{Loc}(X) \subset \mathrm{sh}_q(X)$ be the full sub-category consisting of all objects supported on $T^*_X X \times 0$. Let $\mathrm{const}(X)$ be the full sub-category consisting of all $F \in \mathrm{const}(X)$ satisfying $F_{x_0} \in \mathbb{A}\text{-mod} \subset \mathbf{GZ}$.

Proposition 7.14 *1) We have $Hom(F, G) \in \mathbf{GZ}_{\geq 0}$.*

2) The through map

$$\tau_{\leq 0} Hom(F, G) \to Hom(F|_{x_0}; G|_{x_0})$$

is a homotopy equivalence.

7.5.11 Sheaves constant along $\mathbb{R}^n$

Let $p : X \times \mathbb{R}^n \to X$ be the projection. Let $\mathcal{C} \subset \mathrm{sh}_\varepsilon(X \times \mathbb{R}^n)$ be the full sub-category of objects F, where

$$\mathrm{SS}(F) \subset T^*X \times T^*_{\mathbb{R}^n}\mathbb{R}^n \times \mathbb{R}.$$

Proposition 7.15 *The category $\mathcal{C}$ consists of all objects F homotopy equivalent to objects of the form $G \boxtimes \mathbb{A}_{\mathbb{R}^n}$, $G \in \mathrm{sh}_\varepsilon(X)$.*

7.5.12 Fourier transform

Let $E = \mathbb{R}^n$ with the standard euclidean pairing $\phi : E \times E \to \mathbb{R}$. Let $\mathcal{F} \in \mathrm{sh}_q(E \times E)$, $\mathcal{F} = \mathbb{A}_{[E\times E, \phi]}$. Let $\mathcal{F}^t = \mathbb{A}_{[E\times E, -\phi]}[n]$.. Let $R : T^*E \times \mathbb{R} \to T^*E \times \mathbb{R}$, where $R(q, p, t) = (p^\vee, -q, t + \langle p, q\rangle)$, where $\vee : E^* \to E$ is induced by the pairing. Let $a : E \to E$ be given by $a(v) = -v$.

Let $\mathbb{F}, \mathbb{F}^t : \mathrm{sh}_q(E) \to \mathrm{sh}_q(E)$, $\mathbb{F}(G) := G *_E \mathcal{F}$; $\mathbb{F}^t(G) := G *_E \mathcal{F}^t$.

Proposition 7.16 *1)We have a zig-zag termwise homotopy equivalences $\mathbb{F}\mathbb{F}^t \approx Id$; $\mathbb{F}^t\mathbb{F} \approx Id$; $\mathbb{F}^t \approx a_!\mathbb{F}[-n]$;*

2)$SS(G *_E \mathbf{F}) \subset R(SS(G))$.

7.5.13 Fourier transform of convolution

Let E_1, E_2, E_3 be real vector spaces. Let $K \in \mathrm{sh}_\varepsilon(E_1|E_2, C)$; $L \in \mathrm{sh}_\varepsilon(E_2|E_3, C)$. Let $a_2 : E_2 \times E_3 \to E_2 \times E_3$, $a_2(v, w) = (-v, w)$.

Proposition 7.17 *We have*

$$\mathbb{F}(K *_{E_2} L) \approx \mathbb{F}K *_{E_2^*} a_{2!}\mathbf{F}L;$$

The proof is straightforward.

Let now $K \in \mathrm{sh}_\varepsilon(E_1|E_2; C)$ and $F \in \mathrm{sh}(E_2, C)$. Let $a : E_1 \to E_1$ be given by $a(v) = -v$.

Corollary 7.18 *We have 1)*

$$\mathbb{F}K^! F \sim a_!(\mathbb{F}K)^! \mathbb{F}F.$$

2) The natural map

$$((\mathbb{F}K)^! \mathbb{F}F) *_{E_1} \mathbb{F}K \to \mathbb{F}F$$

is homopy equivalent to

$$((\mathbb{F}K)^! \mathbb{F}F) *_{E_1^*} \mathbb{F}K \approx a_! \mathbb{F}K^! F *_{E_1^*} \mathbb{F}F \approx \mathbb{F}K^! F *_{E_1} K \to \mathbb{F}F.$$

Indeed, 1) follows from the above proposition and 2) follows from the fact that $\mathbb{F}$ is a homotopy equivalence of categories, therefore preserves pairs of adjoint functors.

7.6 Comparison of the two inverse images

Let $i : Y \to X$ be a closed embedding. Let $m = \dim Y$; $n + m = \dim X$. Let $F \in \mathrm{sh}_\varepsilon(X)$. Set $D_Y := i^! \mathbb{A}_X$. We have a natural map $i_! D_Y \to \mathbb{A}_X$. Let $\Delta_X : X \to X \times X$ be the diagonal embediding. We have an induced map $\Delta_{X!} i_! D_Y \to \Delta_! \mathbb{A}_X$. We now have an induced map

$$F *_X \Delta_! i_! D_Y \to F *_X \Delta_! \mathbb{A}_X \approx F.$$

Let $\delta : Y \to X \times Y$ be the diagonal embedding. We have

$$i_! F *_X \delta_! D_Y \approx F *_X \Delta_! i_! D_Y,$$

whence induced maps

$$i_! F *_X \delta_! D_Y \to F;$$
$$F *_X \delta_! D_Y \to i^! F. \tag{35}$$

7.6.1 Theorem: formulation

Let $U \subset T^*X \times \mathbb{R}$ be a conic open subset containing $T^*_Y X \times \mathbb{R}$, where conic means stable under positive dilation of fibers of the bundle $T^*X \times \mathbb{R} \to X \times \mathbb{R}$.

Theorem 7.19 *Suppose $SS(F) \cap \overline{U} \subset T^*X \times \mathbb{R}$ is proper over $X \times \mathbb{R}$. Then the map (35) is a homotopy equivalence.*

The rest of the subsection is devoted to the proof.

7.6.2 Reduction to the flat case

The statement is local in Y. Let $y_0 \in Y$. Choose a pre-compact neighborhood V of y_0 endowed with a diffeomorphism $\phi : \overline{V} \cong B_n \times B_m \subset \mathbb{R}^n \times \mathbb{R}^m$, where $B_n \subset \mathbb{R}^n$ is the unit ball centered at 0 and $\phi(Y \cap \overline{V}) = 0 \times B_m$. We have an identification $T^*X|_U = U \times \mathbb{R}^n \times \mathbb{R}^m$. It follows that there exist an open cone $C \subset \mathbb{R}^n \times \mathbb{R}^m$, $\overline{C} \supset \mathbb{R}^n \times 0$, and a compact subset $K \subset \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\phi(U \cap T^*V \times \mathbb{R}) \supset V \times C \times \mathbb{R}$$

and

$$\mathrm{SS}(F) \cap T^*V \times C) \subset T^*V \times K.$$

One can choose diffeomorphisms $h : \mathbb{R}^n \cong \mathbf{int}B_n$; $h_m : \mathbb{R}^m \cong \mathbf{int}B_m$ and an open cone $A \subset \mathbb{R}^n \times \mathbb{R}^m$, $\mathbb{R}^n \times 0 \subset \overline{A}$, satisfying: $(h_n \times h_m)^*(\mathbf{int}B_n \times \mathbf{int}B_m \times C) \supset \mathbb{R}^n \times \mathbb{R}^m \times A$, where $(h_n \times h_m)^* : T^*(\mathbf{int}B_n \times \mathbf{int}B_m) \to T^*(\mathbb{R}^n \times \mathbb{R}^m)$ is the induced map.

3) The problem reduces to the case $X = \mathbb{R}^n \times \mathbb{R}^m$, $Y = 0 \times \mathbb{R}^m$, $\mathrm{SS}F \cap \mathbb{R}^n \times \mathbb{R}^m \times \overline{A}$ is compact, where $A \subset \mathbb{R}^n \times \mathbb{R}^m$ is an open cone, $0 \times \mathbb{R}^m \subset \overline{A}$.

Let (x, y) be local coordinates on $\mathbb{R}^n \times \mathbb{R}^m$. Let (x, ω, y, η) be coordinates on $T^*(\mathbb{R}^n \times \mathbb{R}^m)$. There exists a $C > 0$ such that

$$A \supset \{(\omega, \eta) | 0 < C|\eta| < |\omega|\}.$$

There exists a $D > 0$ such that F is non-singular on the set

$$\{(x, \omega, y, \eta) | \max(D, C|\eta|) < |\omega|\}.$$

Denote $H := \{(\omega, \eta) | \eta \neq 0; \max(D, C|\eta|) < \omega\}$. Let

$$\Sigma := \mathbb{R}^n \times \mathbb{R}^m \backslash H,$$

$$\Sigma = \{(\omega, \eta) |\ |g| \leq \max(D, C|\eta|)\}$$

7.6.3 Applying the Fourier tranform

Let us apply Fourier transform (7.5.12).

1) We have $\mathbb{F}F$ is supported on Σ.

2) The properties of Fourier transform imply that the map

$$\mathbb{F}\delta_{Y!}D_Y *_X F \to \mathbb{F}F$$

is homotopy equivalent to the following map

$$\mathbb{A}_{[\{(x_1,y_1,x_2,y_2)|x_1=x_2\},0]} *_{\mathbb{R}^n \times \mathbb{R}^m} \mathbb{F}F \to \mathbb{A}_{\{(x_1,y_1,x_2,y_2)|x_1=x_2,y_1=y_2\},0]} *_{\mathbb{R}^n \times \mathbb{R}^m} \mathbb{F}F$$

which is homotopy equivalent to the natural map

$$p^{-1}p_!\mathbb{F}F \to \mathbb{F}F,$$

same as in Sec. 5.16.3. The map (35) is then equivalent to the induced map

$$p_!\mathbb{F}F \to p_*\mathbb{F}F.$$

which is a homotopy equivalence because p is proper on the support of $\mathbb{F}F$.

8 Action of $\mathbf{Sp}(2N)$

Let G be the universal cover of $\mathrm{Sp}(2N)$. Let $V=\mathbb{R}^{2N}$ be the standard symplectic vector space with the coordinates (q,p) and let $E=\mathbb{R}^N$ so that $V=T^*E$. The group $\mathrm{Sp}(2N)$, hence G, acts on V.

8.1 Graph of the G-action on T^*E

. Let $a: T^*E\to T^*E$ be the antipode map $(q,p)\mapsto(q,-p)$. Let $\Gamma\subset G\times V\times V$ consist of all points of the form $\{(g,v,gv^a)|\ g\in \mathrm{Sp}(2N); v\in V\}$. It follows that there exists a unique Legendrian sub-manifold $\mathcal{L}\subset T^*(G\times E\times E)\times\mathbb{R}$ which — diffeomorphically projects onto Γ under the projection

$$T^*(G\times E\times E)\times\mathbb{R}\to G\times T^*(E\times E)\times\mathbb{R}.$$

— contains all the points of the form $(e,v,v^a,0)$, where e is the unit of G and $v\in V$.

Let $\mathcal{C}$ be the full sub-category of $\mathrm{sh}_\infty(G\times E\times E)$ consisting of all objects F satisfying:

— there exists a homotopy equivalence $F|_{e\times E\times E}\sim\mathbb{A}_{[\Delta_E],0}$, where $\Delta_E\subset E\times E$ is the diagonal.

— $\mathrm{SS}(F)\subset\mathcal{L}$.

We have a functor $\mathcal{C}\to\mathbb{A}$-mod, $F\mapsto F|_{(0,0,0)}$

Theorem 8.1 *This functor is a weak equivalence.*

Sketch of the proof. *Part 1: Let us construct at least one object* $\mathbb{S}$ *of* $\mathcal{C}$ satisfying $F|_{0,0,0}=\mathbb{A}$.

1) For an open subset $U\subset G$, let $\mathcal{L}_U\subset T^*(U\times E\times E)\times\mathbb{R})$ be the restriction of $\mathcal{L}$. Let $\mathcal{C}_U$ be the full sub-category of $\mathrm{sh}_q(U\times E\times E)$ consisting of all objects F such that $\mathrm{SS}(F)\subset\mathcal{L}_U$ and there exists a homotopy equivalence $F|_{e\times E\times E}\sim\mathbb{A}_{[\Delta_E,0]}$.

2) Let $\mathcal{U}$ be a small enough geodesically convex neighborhood of unit in $\mathrm{Sp}(2N)$ satisfying: for each $g\in\mathcal{U}$ we have: (q,p') is a non-degenerate system of coordinates, where $(q',p')=g(q,p)$. $\mathcal{U}$ lifts uniquely to G, to be denoted by the same letter.

3) We will freely use the notation from Sec. 7.5.12. Let

$$R_1: T^*E\times T^*E\times\mathbb{R}\to T^*E\times T^*E\times\mathbb{R},$$

be defined by $R_1(u_1,u_2,t)=(u_1,R^{-1}(u_2,t))$, where R as in Sec 7.5.12. Let $\mathcal{C}'_U\subset\mathrm{sh}_q(U\times E\times E)$ consist of all objects F such that

— there exists a homotopy equivalence $F|_{e\times E\times E}\sim\mathbf{F}'$.

— $\mathrm{SS}F\subset R_1(\mathcal{L}_U)$.

It follows that the functor $G\mapsto G*_E\mathbf{F}$ induces a homotopy equivalence of categories $\mathcal{C}'_U\to\mathcal{C}_U$.

4) The Legendrian manifold $R_1\mathcal{L}_\mathcal{U}\subset T^*(G\times E\times E)\times\mathbb{R}$ projects uniquely onto the base $G\times E\times E$, therefore, $R_1\mathcal{L}_\mathcal{U}$ is of the form $\mathcal{L}_f$ for some smooth function f on $G\times E\times E$.

Let $\mathcal{A}\subset\mathrm{sh}_q(G\times E\times E)$ be the full sub-category of objects F satisfying:

— $\mathrm{SS}(F)\subset T^*_{U\times E\times E}U\times E\times E\times 0$;

— there exists a homotopy equivalence $F|_{e\times E\times E}\sim \mathbb{A}_{E\times E}$.

It follows that $\mathcal{A}$ is the category consisting of all objects homotopy equivalent to $\mathbb{A}_{[U\times E\times E,0]}$.

According to Sec. 7.4.2, the convolution with $\mathbb{A}_{\Delta_E,f}$ gives a homotopy equivalence of categories $\mathcal{A}\to \mathcal{C}'_U$.

Fix an object $S_{\mathcal{U}}\in \mathcal{C}_{\mathcal{U}}$ along with a homotopy equivalence

$$S_{\mathcal{U}}|_{0\times E\times E}\sim \mathbb{A}_{[\Delta_E,0]}.$$

5) For $h\in\mathcal{U}$, set $S_h:=S_{\mathcal{U}}|_{h\times E\times E}$. Every $g=G$ can be written as $g=g_1g_2\cdots g_n$, where $g_i,g_i^{-1}\in\mathcal{U}$. Set $S_{g_1,\ldots,g_n}=S_{g_1}*_E S_{g_2}*_E\cdots *_E S_{g_n}$.

For each g, choose an object $S_{g\mathcal{U}}$ which is homotopy equivalent to one of $S_{g_1,\ldots,g_n}*_E S_{\mathcal{U}}$ for $g_1\cdots g_n=g$. Observe that the objects $S_{g_1,\ldots,g_n}$ and $S_{g'_1,\ldots,g'_m}$, where $g_1\cdots g_n=g'_1\cdots g'_m=g$ are homotopy equivalent. It suffices to show that

$$S_{g_1,\cdots,g_m,(g'_m)^{-1},\cdots,(g'_1)^{-1}}\sim \mathbb{A}_{[\Delta_E,0]}$$

that is $S_{g_1g_2\cdots g_n}=\mathbb{A}_{\Delta_E,0}$ whenever $g_1g_2\cdots g_n=e$. As U is geodesically closed, there is a unique shortest geodesic line joining $g_1\cdots g_k$ and $g_1\cdots g_{k+1}$. We will thus get a broken geodesic line starting and terminating at e. As G is simply connected, this line can be contracted to a point. By possibly adding intermediate points, one can reduce the problem to the case when there exist smooth paths $h_k:[0,1]\to U$ such that $h_1(t)\cdots h_n(t)=e$, $h_k(1)=e, h_k(0)=g_k$ for all k. Let $S_k\in \mathrm{sh}([0,1]\times E\times E)$, $S_k:=h_k^{-1}S_{\mathcal{U}}$. Consider

$$\Sigma:=S_1*_E S_2*_E\cdots *_E S_n\in \mathrm{sh}_q(I^n\times E\times E)|_{\Delta_I\times E\times E},$$

where $\Delta_I\subset I^n$ is the diagonal.

It follows that

$$\Sigma_{1\times E\times E}\sim\mathbb{A}_{[\Delta_E,0]};\quad \Sigma_{0\times E\times E}\sim S_{g_1,g_2,\cdots,g_n}.$$

Next, the singular support estimate shows that Σ is locally constant along Δ_I, which implies the statement.

6) Choose a covering $G=\bigcup_n g_n\mathcal{U}$. Let $I\in\mathbf{Cov}_G$ be the poset consisting of all non-empty intersections $g_{i_1}\mathcal{U}\cap\cdots g_{i_k}\mathcal{U}$. Each element of I is geodesically convex. It follows that all the restictions $S_{g_{i_l}\mathcal{U}}|_{g_{i_1}\mathcal{U}\cap\cdots\cap g_{i_k}\mathcal{U}}$ are homotopy equivalent. Indeed, choose a point $h\in g_{i_1}\mathcal{U}\cap\cdots\cap g_{i_k}\mathcal{U}$; 4) implies that there is a homotopy equivalence of restrictions $S_{g_{i_l}\mathcal{U}}|_{h\times E\times E}$ with S_h. The statement now follows from 4).

For every $A\in I$, $A=g_{i_1}\mathcal{U}\cap g_{i_2}\mathcal{U}\cap\cdots\cap g_{i_k}\mathcal{U}$, choose an object $S_A\in\mathcal{C}_A$ to be homotopy equivalent to each of the restrictions $S_{g_{i_l}\mathcal{U}}|_{A\times E\times E}$.

7) For each $V\in I$ let $j:V\to G$ be the embedding. Let $T_V:=j_{\mathcal{V}!}S_{\mathcal{V}}$.

8) Whenever $A\subset B$, $A,B\in I$, we have a homotopy equivalence $\mathbb{A}\sim \mathrm{Hom}(T_A,T_B)$. Let $r_{AB}:T_A\to T_B$ be the image of $1\in\mathbb{A}$.

9) We have $r_{BC}r_{AB}$ is homotopy equivalent to $E_{ABC}r_{AC}$ for some $E_{ABC}\in\mathbb{A}^\times$.

10) E_{ABC} is a 2-cocycle on I. Since $H^2(G,\mathbb{A}^\times)=0$, E_{ABC} is exact. Therefore, wlog we can assume that $E_{ABC}=1$.

11) Denote $\mathcal{J}(A,B) := \tau_{\leq 0}\mathrm{Hom}(T_A,T_B)$. We have a functor $\mathcal{J} \to I$ which is a homotopy equivalence of categories so that we have the constant functor $Z : \mathcal{J}^{\mathbf{op}} \to I^{\mathbf{op}} \to \mathbf{GZ}$, $Z(A) = \mathbb{A}$ for all A.

Finally, we set $\mathbb{S} := S_G := \mathcal{Z} \otimes^L_{\mathcal{J}^{\mathbf{op}}} S$.

Part 2. Uniqueness The convolution with $\mathbb{S}$ gives a pair of quasi-inverse maps between $\mathcal{C}_G$ and the full sub-category of objects $S \in \mathrm{sh}_q(G \times E \times E)$ with $\mathrm{SS}\mathbb{S} \subset T^*_G G \times T^*_E(E \times E) \times \{0\}$, where there exists an isomorphism

$$S|_{e\times E\times E} \sim \mathbb{A}_{\{(e,e,0)|e\in E\}}.$$

The latter category, hence the initial one, satisfies $\mathrm{Hom}(F,G) \in \mathbf{GZ}_{\geq 0}$ for every pair of objects. Passing to $\tau_{\geq 0}$ yields the statement.

8.1.1 The object $\mathbb{S}$

Fix an object $\mathbb{S} \in \mathcal{C}$ endowed with a homotopy equivalence $\mathbb{S}|_{0,0,0} \to \mathbb{A}$.

9 Objects supported on a symplectic ball

9.1 Projector onto the ball

Let $i_0 : \mathbb{R}/2\pi\mathbb{Z} \to \mathrm{Sp}(2N)$ be a one-parametric subgroup consisting of all transformations

$$q' = q\cos(2a) + p\sin(2a);$$

$$p' = -q\sin(2a) + p\cos(2a).$$

Let $i : \mathbb{R} \hookrightarrow G$ be the lifting. Denote $\mathcal{A} := i(\mathbb{R})$. Let $\mathcal{T} \in \mathrm{sh}_q(\mathcal{A} \times E \times E)$ be the restriction of $\mathbb{S}$. The object $\mathcal{T}$ is microsupported within the set

$$\Sigma = \Sigma_0 \cup \{(a, -(q^2+p^2), q, -p, q', p', -S(q,p,a))|(q,p) \in V; a \in \mathbb{R}, \sin(2a) \neq 0\} \subset T^*\mathcal{A} \times T^*E \times T^*E \times \mathbb{R} \quad (36)$$

where

$$\Sigma_0 = \{(\pi n, -(q^2+p^2), q, -p, q, p, 0)|(q,p) \in V, n \in \mathbb{A}\} \cup \{(\pi(\frac{1}{2}+n); -(q^2+p^2), q, -p, -q, -p, 0)|(q,p) \in V, n \in \mathbb{A}\};$$

$$S(q,p,a) = \frac{\cos(2a)(q^2+(q')^2) + 2qq'}{2\sin(2a)}.$$

Let $\mathcal{B} = \mathbb{R}$ with the coordinate b. Let $p_{\mathcal{B}} : \mathcal{B} \times E \times E \to E \times E$ be the projection. Set

$$\mathcal{P}_R := p_{\mathcal{B}!}\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a,b)\in\mathcal{A}\times\mathcal{B}|b<R^2\},-ab]}[1] \in \mathrm{sh}_q(E \times E).$$

Let

$$\Delta_{a\leq 0} := \{(a,a)|a \leq 0\} \subset \mathcal{A} \times \mathcal{A}.$$

We have

$$\mathcal{P}_R \sim p_{\mathcal{A}!}\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\Delta_{a\leq 0},-aR^2]},$$

where $p_{\mathcal{A}} : \mathcal{A} \times E \times E \to E \times E$ is the projection.

9.1.1 The map $\alpha : T_{-\pi R^2}\mathcal{P}_R[2N] \to \mathcal{P}_R$

We have a homotopy equivalence

$$T^a_{-\pi}\mathcal{T}[-2N] \sim \mathcal{T},$$

where $T^a_{-\pi}$ is the translation along $\mathcal{A}$ by $-\pi$ units.

Thus, we have a map

$$\begin{aligned}\mathcal{P}_R \sim p_{\mathcal{A}!}((T^a_{-\pi}\mathcal{T}) *_{\mathcal{A}} \mathbb{A}_{[\Delta_{a\leq 0},-aR^2]}[-2N]) \sim p_{\mathcal{A}!}(\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_2\leq 0,a_1=a_2+\pi\},-a_2R^2]})[-2N] \\ \sim p_{\mathcal{A}!}(\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_1\leq \pi;a_1=a_2\},\pi R^2-a_1R^2]})[-2N] \\ \sim T_{\pi R^2}p_{\mathcal{A}!}(\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_1\leq \pi;a_1=a_2\},-a_1R^2]})[-2N] \\ \to T_{\pi R^2}p_{\mathcal{A}!}(\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_1\leq 0;a_1=a_2\},-a_1R^2]})[-2N] \sim T_{\pi R^2}\mathcal{P}_R[-2N].\end{aligned}$$

This map can be rewritten as

$$\alpha : T_{-\pi R^2}\mathcal{P}_R[2N] \to \mathcal{P}_R.$$

9.1.2 Hom$(T_c\mathcal{P}_R;\mathcal{P}_R)$

Let $(\nu-1)\pi R^2 < c \leq \nu\pi R^2$, where $\nu \in \mathbb{Z}$. Let $G_c := \mathrm{Hom}(T_c\mathcal{P}_R;\mathcal{P}_R)$. Then

$$G_c \sim \mathbb{Z}[-2N\nu] \text{ if } \nu \geq 0, \quad G_c = 0 \text{ if } \nu > 0.$$

The natural map $G_{\nu\pi R^2} \to G_c$ is a homotopy equivalence. The generator of $G_{\nu\pi R^2}$, $\nu < 0$ is given by α^{*n}.

The map $\mathcal{P}_R \to \mathbb{A}_{[\Delta_E,0]}$ induces a homotopy equivalence

$$\mathrm{Hom}(T_c\mathcal{P}_R;\mathcal{P}_R) \to \mathrm{Hom}(T_c\mathcal{P}_R;\mathbb{A}_{[\Delta_E,0]}).$$

9.1.3 $\mathcal{P}_R$ is a projector

We have a natural map

$$\mathbf{pr} : \mathcal{P}_R \to \mathbb{A}_{[\Delta_E,0]}. \tag{37}$$

Let $\mathcal{C}_R \subset \mathrm{sh}_q(E)$ be the full subcategory of objects supported away from $\mathring{B}_R \times \mathbb{R} \subset T^*E \times \mathbb{R}$. Let $\mathrm{sh}_q[\mathring{B}_R] \subset \mathrm{sh}_q(E)$ be the left orthogonal complement to $\mathcal{C}_R$. We have $\mathcal{P}_R *_E F \in \mathrm{sh}_q[\mathring{B}_R]$; $\mathrm{Cone}\,\mathcal{P}_R *_E F \to F \in \mathcal{C}_R$ so that $\mathcal{P}_R$ gives a semi-orthogonal decomposition.

9.1.4 Generalization

Denote by $\mathrm{sh}_\varepsilon[T^*X \times \mathring{B}_R \times \mathbb{R}] \subset \mathrm{sh}_\varepsilon(X \times E)$ be the left orthogonal complement to the full category of objects supported away from $T^*X \times \mathring{B}_R \times \mathbb{R}$. The convolution with $\mathcal{P}_R$ gives a semi-orthogonal decomposition.

9.1.5 The object $\gamma = \operatorname{Cone}\alpha$

Let $\gamma := \operatorname{Cone}\alpha$. We have

$$\gamma \sim \mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_1=a_2;-\pi R^2<a_1\leq 0\},-aR^2]}$$

We have a homotopy equivalence

$$E_c := \operatorname{Hom}(T_c\gamma, \mathcal{P}_R) \stackrel{\sim}{\to} \operatorname{Hom}(T_c\gamma; \mathbb{A}_{[\Delta_E,e_0]})$$

We have

$$E_c = (\operatorname{Cone} G_c \to G_{c-\pi R^2}[-2N])[-1],$$

where the map is incuced by the multiplication by α.

Therefore,

—$E_c = \mathbb{A}[-2N-1]$, $0 < c \leq \pi R^2$;

—$E_c = 0$ otherwise.

9.1.6 Singular support of γ

We have

$$\mathrm{SS}\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_1=a_2,-\pi R^2<a_1\leq 0\},-aR^2]} \subset \{(a, R^2+k, q, -p, q', p', t-aR^2) \in \Sigma| -\pi < a < 0\} \cup S,$$

where Σ is as in (36) and

$$S = \{(-\pi, R^2+k, q, -p, q, p, -\pi R^2)|k \leq -p^2-q^2\} \cup \{(0, R^2+k, q, -p, q, p, 0)|k \leq -p^2-q^2\}.$$

Therefore, we have

$$\mathrm{SS}\gamma \subset \{(q, -p, q', p', -aR^2-S(a,q,q'))|p^2+q^2 = R^2; -\pi < a < 0\} \cup \{(q, -p, q, p, -\pi R^2)|q^2+p^2 \leq R^2\} \cup \{(q, -p, q, p, 0)|q^2+p^2 \leq R^2\}.$$

It follows that $0 \leq -aR^2 - S(a, q, q') \leq \pi R^2$ if $-\pi < a < 0$.

9.1.7 Singular support of $\mathcal{P}$

Similarly, one can find

$$\mathrm{SS}\mathcal{P} \subset \{(q, -p, q', p', -aR^2 - S(a, q, q')|p^2 + q^2 = R^2; a < 0\} \cup \{(q, -p, q, p, 0)|q^2 + p^2 \leq R^2\}.$$

9.1.8 Singular support of $\mathrm{Cone}\,\mathcal{P} \to \mathbb{A}_{[\Delta_E,0]}$

We have

$$\mathrm{Cone}(\mathcal{P} \to \mathbb{A}_{[\Delta_E,0]}) \approx p_{\mathcal{A}!}\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_1=a_2,a_1\leq 0\},-aR^2]}$$

so that

$$\mathrm{SS}\mathcal{T} *_{\mathcal{A}} \mathbb{A}_{[\{(a_1,a_2)|a_1=a_2,a_1\leq 0\},-aR^2]} \subset \{(a,R^2+k,q,-p,q',p',t-aR^2) \in \Sigma | a < 0\} \cup S',$$

where Σ is as in (36) and

$$S' = \{(0,R^2+k,q,-p,q,p,0)|k \geq -p^2-q^2\}.$$

Therefore,

$$\mathrm{SS}\,\mathrm{Cone}(\mathcal{P} \to \mathbb{A}_{[\Delta_E,0]}) \subset \{(q,-p,q',p',-aR^2-S(a,q,q')|p^2+q^2=R^2; a<0\} \cup \{(q,-p,q,p,0)|q^2+p^2 \geq R^2\}.$$

9.1.9 Corollaries

Corollary 9.1 *We have*

$$\mathrm{Cone}(\mathcal{P} \to \mathbb{A}_{[\Delta_E,0]}) \bullet \mathbb{A}_{[\mathbf{pt},c]} \approx 0;$$

$$\mathrm{Cone}(\mathcal{P} \boxtimes \mathcal{P} \to \mathbb{A}_{[\Delta_E\times\Delta_E,0]} \bullet \mathbb{A}_{[\mathbf{pt},c]} \approx 0.$$

for all $c \leq 0$*.*

Corollary 9.2 *Let* $F \in \mathrm{sh}(E\times E)$*. Then the natural maps*

$$\mathit{Hom}(\mathbb{A}_{\Delta_E\times\Delta_E};F) \xrightarrow{\sim} \mathit{Hom}(\mathcal{P}\boxtimes\mathcal{P}; F\boxtimes\mathbb{A}_{[\mathbf{pt},0]});$$

$$\mathit{Hom}(\mathbb{A}_{\Delta_E\times\Delta_E};F) \xrightarrow{\sim} \mathit{Hom}(T_{2\pi R^2}\gamma\boxtimes\gamma[-4N]; F\boxtimes\mathbb{A}_{[\mathbf{pt},0]})$$

are homotopy equivalences.

9.1.10 Convolution of γ with itself

We have a homotopy equivalence

$$\gamma *_E \gamma \sim \gamma \oplus T_{-\pi R^2}\gamma[2N].$$

Denote by $\mu : \gamma *_E \gamma \to \gamma$ the projection.

We now have the following homotopy equivalence

$$\mathrm{Hom}(T_c\gamma, \mathbb{A}_{[\Delta_E,0]}) \xrightarrow{\mu} \mathrm{Hom}(T_c\gamma *_E \gamma; \mathbb{A}_{[\Delta_E,0]}),$$

for all c except those in $(\pi R^2, 2\pi R^2]$.

In particular, for $0 < c \leq \pi R^2$, we have:

$$\mathrm{Hom}(T_c\gamma *_E \gamma; \mathbb{A}_{[\Delta_E,0]}) \sim \mathbb{A}[-2N-1];$$

For $c \leq 0$, the above expresssion is homotopy equivalent to 0.

Let $\Lambda \in \mathrm{sh}_q(\mathbf{pt})$; $\Lambda = \mathrm{Cone}(\mathbb{A}_{[\mathbf{pt},-\pi R^2]} \to \mathbb{A}_{[\mathbf{pt},0]})$.

We have a chain of homotopy equivalences

$$\mathrm{Hom}(\gamma; \mathbb{A}_{\Delta_E} \boxtimes \Lambda) \overset{\mu}{\sim} \mathrm{Hom}(\gamma *_E \gamma; \mathbb{A}_{\Delta_E} \boxtimes \Lambda) \sim \mathbb{A}[-2N].$$

In particular, we have a homotopy equivalence

$$\mathrm{Hom}(\gamma, \Lambda \boxtimes \mathbb{A}_{\Delta_E}[2N]) \sim \mathbb{A}.$$

Let

$$\nu : \gamma \to \Lambda \boxtimes \mathbb{A}_{\Delta_E}[2N] \tag{38}$$

be the generator.

One also has a map $\varepsilon : \Lambda \boxtimes \mathbb{A}_{\Delta_E} \to \gamma$ which has a homotopy unit property with respect to μ, the through map

$$\gamma \sim \mathbb{A}_{\Delta_E} *_E \gamma \to \Lambda \boxtimes \mathbb{A}_{\Delta_E} *_E \gamma \to \gamma *_E \gamma \to \gamma$$

is homotopy equivalent to the Identity.

The induced map

$$\mathrm{Hom}(\gamma, \Lambda \boxtimes \mathbb{A}_{\Delta_E}) \overset{\varepsilon}{\to} \mathrm{Hom}(\gamma, \gamma) \tag{39}$$

is a homotopy equivalence. The map ν on the LHS corresponds to Id on the RHS.

9.1.11 Lemma on $\nu \boxtimes \nu$

Consider the following maps

$$\gamma \boxtimes \gamma \overset{\nu\boxtimes\nu}{\to} \Lambda \boxtimes \mathbb{A}_{\Delta_E} \boxtimes \Lambda \boxtimes \mathbb{A}_{\Delta_E}[4N] \to \Lambda \boxtimes \mathbb{A}_{\Delta_E \times \Delta_E}[4N]; \tag{40}$$

$$\gamma \boxtimes \gamma \overset{\overline{\mu}}{\to} p_{14}^{-1}\gamma \boxtimes p_{23}^{-1}\mathbb{A}_{\Delta_E} \overset{\nu}{\to} \Lambda \boxtimes p_{14}^{-1}\mathbb{A}_{\Delta_E} \boxtimes p_{23}^{-1}\mathbb{A}_{\Delta_E}[3N] \to \mathbb{A}_{\Delta_E \times \Delta_E}[4N]. \tag{41}$$

Here the maps $\overline{\mu}$ is obtained from μ by conjugation. The last arrow is the generator of

$$\mathrm{Hom}(p_{23}^{-1}\mathbb{A}_{\Delta_E} \otimes p_{14}^{-1}\mathbb{A}_{\Delta_E}; \mathbb{A}_{\Delta_E \times \Delta_E}[N]).$$

Lemma 9.3 *The maps (40) and (41) are homotopy equivalent.*

Sketch of the proof One reformulates the statement as follows:

By the conjugacy, the map ν correponds to a homotopy equivalence

$$\xi : \Lambda \to \gamma *_{E^2} \mathbb{A}_{\Delta_{[\mathbb{R}^n,0]}}[n]$$

The problem reduces to showing that the map

$$\Lambda \to \Lambda \approx (\gamma \boxtimes \gamma) *_{E^4} \mathbb{A}_{\Delta \times \Delta}[2n] \to (\gamma \boxtimes \gamma) *_{E^4} \mathbb{A}_{(v_1,v_2,v_3,v_4) \in E^4 | v_1 = v_4; v_2 = v_3}[n] \approx (\gamma *_E \gamma) *_{E^2} \mathbb{A}_{\Delta}[n] \to \gamma *_{E^2} \mathbb{A}_{\Delta}[n] \tag{42}$$

is homotopy equivalent to

$$\Lambda\otimes\Lambda\to\Lambda\to\gamma *_{E^2}\mathbb{A}_\Delta[n]. \tag{43}$$

According to Sec. 35 we have a homotopy equivalence,

$$\gamma *_{E_2}\mathbb{A}_\Delta[n]\cong \mathrm{Hom}(\mathbb{A}_\Delta;\gamma).$$

The map ξ rewrites as $\xi':\Lambda\to\mathrm{Hom}(\mathbb{A}_\Delta;\gamma)]$ which produces a map $e:\Lambda\otimes\mathbb{A}_\Delta\to\gamma$.

The map (42) rewrites as

$$\Lambda\otimes\Lambda\to\mathrm{Hom}(\mathbb{A}_\Delta;\gamma)\otimes\mathrm{Hom}(\mathbb{A}_\Delta;\gamma)\to\mathrm{Hom}(\mathbb{A}_\Delta;\gamma *_E\gamma)\to\mathrm{Hom}(\mathbb{A}_\Delta;\gamma).$$

The map (43) rewrites as

$$\Lambda\otimes\Lambda\to\Lambda\to\mathrm{Hom}(\mathbb{A}_\Delta;\gamma).$$

Homotopy equivalence of the two maps follows from the following maps being homotopy equivalent:

$$\Lambda\otimes\mathbb{A}_\Delta *_E\Lambda\otimes\mathbb{A}_\Delta\stackrel{e*e}{\to}\gamma *_E\gamma\to\gamma$$

and

$$\Lambda\otimes\Lambda\to\Lambda\to\gamma.$$

The latter statement follows from Sec. 9.1.10.

9.1.12 γ as an object of $\mathrm{sh}_{\pi R^2}(E\times E)$

It follows that γ is supported within the set $E\times E\times[-\pi R^2;0]$. Therefore, γ determines an object of $\mathrm{sh}_{\pi R^2}(E\times E)$, to be denoted by Γ.

Using the bar-resolution for $\Gamma *_E\Gamma$, we see that it is glued of $\gamma *_E\Lambda^{*_E^n} *_E\gamma$. We therefore have the following homotopy equivalences (all the hom's are in $\mathrm{sh}_{\pi R^2}(E\times E)$:

$$\mathrm{Hom}(\Gamma;\mathbb{A}_{\Delta_E})\stackrel{\xi}{\sim}\mathrm{Hom}(\Gamma *_E\Gamma;\mathbb{A}_{\Delta_E})\sim\mathbb{A}[-2N].$$

9.2 Study of the cateory $\mathrm{sh}_q(F\times E\times E)[T^*F\times \mathbf{int}B_R\times\mathbf{int}B_R\times\mathbb{R}]$

9.2.1 The category $\mathcal{A}_I$

Let $I\subset R$ be an open subset. Denote by $\mathcal{A}_I$ the full sub-category of

$$\mathrm{sh}_q(F\times E\times E)[T^*F\times \mathbf{int}B_R\times\mathbf{int}B_R\times\mathbb{R}]$$

consisting of all objects X, where

$$\mathrm{SS}(X)\cap T^*F\times\mathbf{int}B_R\times\mathbf{int}B_R\times I=\emptyset.$$

9.2.2 Study of $\mathcal{A}_{(a,\infty)}$

Let $F \in \mathcal{A}_{(a,\infty)}$.

We have a natural map

$$F * (P_R \boxtimes P_R) \to (R_{\leq a}F) * (P_R \boxtimes P_R),$$

where $R_{\leq a}$ is as in the proof of Claim 7.5.

Lemma 9.4 *The above map is a homotopy equivalence.*

Sketch of the proof Equivalently, we are to show

$$(R_{>a}F) * (P_R \boxtimes P_R) \sim 0.$$

We have

$$\operatorname{hocolim}_{c\downarrow a} R_{>c}F \xrightarrow{\sim} R_{>a}F,$$

therefore, it suffices to show that

$$R_{>c}F * (P_R \boxtimes P_R) \sim 0, \quad c > a.$$

As P_R is supported within $B_R \times B_R \times [0, \infty)$, we further reformulate:

$$(R_{>c}F) * (P_R \boxtimes P_R) \sim 0. \tag{44}$$

Let us study $\mathrm{SS}R_{>c}F$. As $F \in \mathcal{A}_I$, F is non-singular on the set

$$\Omega\{(f, \eta, v_1, \zeta_1, v_2, \zeta_2, t) |\ t > a; \quad |v_1|, |v_2| < R\}.$$

Let $\ell\{f^{\mathbf{k}}\}$ be an Ω^a-lense. According to (32), we have

$$(R_{>c}F) \bullet \mathbb{A}_\ell \approx F \bullet \tau_{\leq -c}\mathbb{A}_\ell \approx F \bullet \mathbb{A}_{\ell_{-c}},$$

where $\ell_{-c} = \{\min(-c, f^{\mathbf{k}})\}$. This implies that $R_{>c}F$ is non-singular on Ω, which implies (44).

9.2.3 Study of $\mathcal{A}_{(-\infty,a)}$

Lemma 9.5 *Let $F \in \mathcal{A}_{(-\infty,a)}$. Then $\tau_{<a}F \sim 0$.*

Sketch of the proof It suffices to show that $R_{\leq c}F \sim 0$ for all $c < a$. Similar to the previous Lemma, we deduce that $R_{\leq c}F$ is non-singular on the set

$$T^*F \times \mathbf{int}B_R \times \mathbf{int}B_R \times \mathbb{R}.$$

Next, we have homotopy equivalences

$$R_{\leq c} \approx R_{\leq c}(F * (P_R \boxtimes P_R)) \xrightarrow{\sim} R_{\leq c}(R_{\leq c}F * (P_R \boxtimes P_R)) \sim 0.$$

This proves the statement.

9.2.4 Study of $\mathcal{A}_{\mathbb{R}\backslash a}$

Let $b_R \subset E$ be the open ball of radius R centered at 0. We have functors

$$\alpha : \mathrm{sh}(F \times b_R \times b_R) \to \mathcal{A}_{\mathbb{R}\backslash a},$$

where

$$\alpha(S) = (S \boxtimes \mathbb{A}_{[\mathbf{pt},a]}) * (P_R \boxtimes P_R);$$
$$\beta : \mathcal{A}_{\mathbb{R}\backslash a} \to \mathrm{sh}(F \times b_R \times b_R),$$
$$\beta(T) = T \bullet \mathbb{A}_{\mathbf{pt},a}.$$

Proposition 9.6 *The functors α, β establish homotopy inverse homotopy equivalences of categories.*

Sketch of the proof Let $S \in \mathcal{A}_{\mathbb{R}\backslash a}$. According to the two previous subsections we have homotopy equivalences:

$$S \approx R_{\le a} S * (P_R \boxtimes P_R) \approx (\tau_{\ge a} R_{\le a} S) * (P_R \boxtimes P_R) \approx ((S \bullet \mathbb{A}_{t \ge a}) \boxtimes \mathbb{A}_{[\mathbf{pt},a]}) * (P_R \boxtimes P_R),$$

which implies the statement.

9.2.5 $\mathbf{SS}(\alpha(F))$

Proposition 9.7 *Let C be a closed conic subset of $T^*F \times T^*b_R \times T^*b_R$. $F \in \mathcal{A}_{\mathbb{R}\backslash a}$ and suppose $SS(F) \cap T^*F \times B_R \times B_R \times a \subset C$. Then $SS(\alpha(F) \boxtimes \mathbb{A}_{[\mathbf{pt},a]}) \subset C \times a$.*

9.2.6 The category $\mathcal{A}_{\mathbb{R}\backslash a,\Delta}$

Let $\alpha : B_R \to B_R$ be the antipode map, $\alpha(q,p) = (q,-p)$. Let

$$\Delta^\alpha = \{(\alpha(v), v) | v \in \mathbf{int} B_R\} \subset \mathbf{int} B_R \times \mathbf{int} B_R.$$

Let $\mathcal{A}_{\mathbb{R}\backslash a,\Delta} \subset \mathcal{A}_{\mathbb{R},\backslash a}$ be the full sub-category of objects X where

$$\mathrm{SS}(X) \cap T^*F \times \mathbf{int} B_R \times \mathbf{int} B_R \times \mathbb{R} \subset T^*_F F \times T^*_{\Delta^\alpha}(\mathbf{int} B_R \times \mathbf{int} B_R) \times a.$$

Let $A_F \subset \mathrm{sh}(F \times b_R \times b_R)$ be the full sub-category of objects T where

$$\mathrm{SS}(T \boxtimes \mathbb{A}_{[\mathbf{pt},a]}) \subset T^*_{F \times \Delta_{b_R}}(F \times b_R \times b_R \times a). \tag{45}$$

According to the previous subsection, we have a homotopy equivalence

$$\beta : A_F \to \mathcal{A}_{\mathbb{R}\backslash a,\Delta}.$$

Furthermore, let $\mathrm{Loc}(F) \subset \mathrm{sh}(F)$ be the full sub-category of objects T where

$$\mathrm{SS}(T \boxtimes \mathbb{A}_{[\mathbf{pt},a]}) \subset T^*_F F \times a.$$

Let $\gamma : \mathrm{Loc}(F) \to A_F$ be given by

$$\gamma(S) = S \boxtimes \mathbb{A}_{\Delta_{b_R}}.$$

Lemma 9.8 *γ is a homotopy equivalence of categories.*

Therefore,

Proposition 9.9 *the functor $\zeta := \beta\gamma : \mathrm{Loc}(F) \to \mathcal{A}_{\mathbb{R}\backslash a,\Delta}$ is a homotopy equivalence of categories.*

9.2.7 The category $\mathcal{C}_\mathcal{I}$

Let $\mathrm{sh}_q(F\times\mathbb{R}^n\times\mathbb{R}^n)[T^*F\times\mathbf{int}B_R\times T^*\mathbb{R}^n\times\mathbb{R}]$ be the full sub-category of $\mathrm{sh}_q(F\times\mathbb{R}^n\times\mathbb{R}^n)$ consisting of all objects F which are left orthogonal to all objects non-singular on $T^*F\times\mathbf{int}B_R\times T^*\mathbb{R}^n\times\mathbb{R}$, same as in Sec 9.1.4.

Below we will study the full sub-category

$$\mathcal{C}_\mathcal{I}\subset \mathrm{sh}_q(F\times\mathbb{R}^n\times\mathbb{R}^n)[T^*F\times\mathbf{int}B_R\times T^*\mathbb{R}^n\times\mathbb{R}]$$

consisting of all objects T satisfying $\mathrm{SS}(T)\cap T^*F\times\mathbf{int}B_R\times T^*\mathbb{R}^n\times\mathbb{R}\subset\mathcal{L}$.

9.2.8 Main Theorem

Let A_F be a category as in (45).

Theorem 9.10 *We have a homotopy equivalence between the categories $\mathcal{C}_\mathcal{I}$ and A_F.*

The proof of this theorem occupies the rest of the subsection.

1) Extend I to $F\times[-1,1]\times B_R$ as follows. For $t\in[-1,1]\backslash 0$, set

$$J(f,t,x)=\frac{I(f,tx)-I(f,0)}{t}+I(f,0).$$

This map extends uniquely to a smooth map $J:F\times[-1,1]\times B_R\to T^*\mathbb{R}^n$. The grading of I extends uniquely to a grading $\mathcal{J}$ of J.

Let $K=J|_{F\times 0}$. It follows that K is a family of linear symplectomorphisms of $T^*\mathbb{R}^n$ restricted to B_R. The grading $\mathcal{J}$ determines uniquely a map

$$\mu:F\to\overline{\mathbf{Sp}}(2N)\times\mathbb{R}. \tag{46}$$

2) For every $(f,t)\in F\times B_R$ we have a Hamiltonian vecor field on B_R, namely $\dfrac{dJ(f,t)}{dt}$. Let $H_{(f,t)}$ be a smooth function on B_R correpsonding to this vector field and satisfying $H_{(f,t)}(0)=0$. It follows that $H:F\times I\times B_R\to\mathbb{R}$ is a smooth function. It extends to a smooth function on $F\times I\times T^*E$ whose support projects properly onto $F\times I$.

3) Let $\chi:\mathbb{R}\to[-1,1]$ be a non-decreasing smooth function such that $\chi(t)=-1$ for all $t\le -1$, $\chi(t)=1$ for all $t\ge 1$, and $\chi(0)=0$. Let $K(f,t)=J(f,\chi(t))$ and $h(f,t,v)=H(f,\chi(t),v)\chi'(t)$ so that $h(f,t,-)$ is the Hamiltonian function of the vector field $\dfrac{dK(f,t)}{dt}$. It follows that there exists a unique family of symplectomorphisms $M:F\times\mathbb{R}\times E\to E$ such that

a) $M|_{F\times 0}$ is the family of linear symplectomorphisms coinciding with $J|_{F\times 0}=K_{F\times 0}$;

b) $\dfrac{dM(f,t)}{dt}$ is the Hamiltonian vector field of $h(f,t,-)$.

It also follows that $M|_{F\times\mathbb{R}\times B_R}=K$

4) The family M defines a Legendrian sub-manifold $\mathcal{L}_M\subset T^*(F\times E\times E\times\mathbb{R})$ such that $\mathcal{L}_M\cap T^*F\times B_R\times T^*E\times\mathbb{R}=\mathcal{L}_K$.

5) According to the theorem of Guillermou-Kaschiwara-Schapira, there exists a quantization of $\mathcal{L}_M$: an object $Q \in \mathrm{sh}_q(F \times E \times E)$ such that $\mathrm{SS}Q \subset \mathcal{L}_M$ and $Q|_{t=0} = \mu^{-1}\mathbf{S}$, where μ is as in (46).

6) Similarly, one defines a quantization Q' of the family M^{-1} of inverse symplectomorphisms.

7) Let $\Delta : F \times E \times E \to F \times I \times F \times I \times E \times E$ be the following embedding

$$\Delta(f, v_1, v_2) = (f, 1, f, 1, v_1, v_2).$$

We have endofunctors

$$S \mapsto S *_{F\times E} \Delta_! Q; \quad S \mapsto S *_{F\times E} \Delta_! Q'$$

of $\mathrm{sh}_q(F \times E \times E)$ which descends to homotopy inverse homotopy equivalences between $\mathcal{C}_O$ and $\mathcal{C}_I$, where $O : F \times B_R \to B_R \stackrel{\iota}{\to} E$ is the constant family, where ι is the standard embedding.

By definition, $\mathcal{C}_O = \mathcal{A}_{\mathbb{R}\backslash 0,\Delta}$. By Proposition 9.9 we have a homotopy equivalence $\zeta : \mathrm{Loc}(F) \to \mathcal{C}_O$. We thus have constructed a zig-zag homotopy equivalence between $\mathrm{Loc}(F)$ and $\mathcal{C}_I$. Denote by $\mathcal{P}_I \in \mathcal{C}_I$ the object corresponding to $\mathbb{A}_F \in \mathrm{Loc}(F)$.

9.2.9 Inverse functor

We have $\mathcal{P}_I \in \mathrm{sh}_q(F \times b_R \times E)$.

Let $I' : F \times B_R \to T^*E$ be given by $I'(f, v) = \alpha I(f, \alpha(v))$, where $\alpha : T^*E \to T^*E$, $\alpha(q, p) = \alpha(q, -p)$. Let $\mathcal{Q}_I := \sigma_! \mathcal{P}_{I'} \in \mathrm{sh}_q(F \times E \times b_R)$, where $\sigma : b_R \times E \to E \times b_R$ is the permutation.

Let $\Delta_F : F \to F \times F$ be the diagonal embedding.

Proposition 9.11 *We have*

$$\mathcal{Q}_I *_{F\times E} \Delta_{F!}\mathcal{P}_I \approx \mathbb{A}_F \boxtimes \mathcal{P}_R \in \mathrm{sh}_q(F \times b_R \times b_R).$$

9.2.10

Let $\pi : T^*F \to F$ be the projection. Let $G_I \subset T^*F \times T^*E$ be an open subset defined as follows

$$G_I = \{(\phi, v) | v \in I(\pi(f) \times \mathbf{int} B_R)\}.$$

Let us also define functors

$$\mathbb{P} : \mathrm{sh}_q(F \times b_R)[T^*F \times \mathbf{int} B_R] \to \mathrm{sh}_q(F \times E)[G_I]; \quad \mathbb{Q} : \mathrm{sh}_q(F \times E)[G_I] \to \mathrm{sh}_q(F \times b_R)[T^*F \times \mathbf{int} B_R],$$

where

$$\mathbb{P}(S) = S *_{F\times b_R} \Delta_{F!}\mathcal{P}_I; \quad \mathbb{Q}(T) = T *_{F\times E} \Delta_{F!}\mathcal{Q}_I.$$

Proposition 9.12 *The functors* $\mathbb{P}, \mathbb{Q}$ *establish homotopy mutually inverse homotopy equivalences between the categories* $\mathrm{sh}_q(F \times b_R)[T^*F \times \mathbf{int} B_R]$ *and* $\mathrm{sh}_q(F \times E)[G_I]$.

9.2.11 Lemma on $\mathcal{P}_I, \mathcal{Q}_I$

We abbeviate $\mathcal{P} := \mathcal{P}_I$, $\mathcal{Q} := \mathcal{Q}_I$.

We have natural maps

$$\alpha : \Delta_F^{-1}(\mathcal{Q} \circ_{b_R} \mathcal{P}) \to \mathbb{A}_{F\times\Delta_E};$$

$$\beta : \Delta_F^{-1}(\mathcal{P} \circ_E \mathcal{Q}) \to \mathbb{A}_{F\times\Delta_E}.$$

We therefore have a pair of induced maps

$$\mathrm{Id} \circ \alpha, \beta \circ \mathrm{Id} : \mathcal{P} \circ_E \mathcal{Q} \circ_E \mathcal{P} \to \mathcal{P}. \tag{47}$$

which are homotopy equivalent and likewise for the pair:

$$\alpha \circ \mathbb{A}_{F\times\Delta_E}, \mathbb{A}_{F\times\Delta_E}\mathrm{Id}\circ : \mathcal{Q} \circ \mathcal{P} \circ \mathcal{Q} \to \mathcal{Q}. \tag{48}$$

One gets the following corollary from (47), (48).

A. Let

$$p_1, p_2 : F \times E \times E \times E \times E \to F \times E \times E$$

be projections, where

$$p_i(\phi, e_1, f_1, e_2, f_2) = (\phi, e_i, f_i).$$

The maps α, β induce, by the conjugacy, maps

$$A : p_1^{-1}\mathcal{P} \otimes p_2^{-1} \to \mathcal{Q} \to \mathbb{A}_{\{f_1=e_2\}}[N];$$

$$B : p_1^{-1}\mathcal{Q} \otimes p_2^{-1} \to \mathcal{P} \to \mathbb{A}_{\{f_1=e_2\}}[N].$$

Let $p_j^i : (F \times (E \times E)^2)^2 \to F \times E \times E$ be projections, where

$$p_j^i(\phi^1, e_1^1, f_1^1, e_2^1, f_2^1, e_1^2, f_1^2, e_2^2, f_2^2) = (\phi^i, e_j^i, f_j^i).$$

We have the following maps

$$\mathbf{A} : (p_1^1)^{-1}\mathcal{P} \otimes (p_2^1)^{-1}\mathcal{Q} \otimes (p_1^2)^{-1}\mathcal{P} \otimes (p_2^2)^{-1}\mathcal{Q} \stackrel{A\boxtimes B}{\longrightarrow} \mathbb{A}_{\{f_1=e_2, e_1=f_2, f_3=e_4, e_3=f_4\}}[2N]$$
$$\to \mathbb{A}_{\{f_1=e_2, e_1=f_2=e_3=f_4, f_3=e_4\}}[2N]$$

$$\mathbf{B} : (p_1^1)^{-1}\mathcal{P} \otimes (p_2^1)^{-1}\mathcal{Q} \otimes (p_1^2)^{-1}\mathcal{P} \otimes (p_2^2)^{-1}\mathcal{Q} \stackrel{A\boxtimes B}{\longrightarrow} \mathbb{A}_{\{\phi^1=\phi_2, f_1=e_4, e_1=f_4, f_2=e_3, e_2=f_3\}}[2N]$$
$$\to \mathbb{A}_{\{f_1=e_4, e_1=f_4, f_2=e_3, e_2=f_3\}}[2N][2N]$$

We also have a map

$$\delta : \mathbb{A}_{\{f_1=e_2, e_1=f_2=e_3=f_4, f_3=e_4\}}[2N] \to \mathbb{A}_{\{f_1=e_4, e_1=f_4, f_2=e_3, e_2=f_3\}}[2N][2N]$$

As follows from (**??**), we have a homotopy equivalence:

$$\mathbf{B} \sim \delta \mathbf{A}.$$

This can be rewritten as follows: we have an object

$$\mathcal{P} \circ_F \mathcal{Q} \circ_{F\times E} \mathcal{P} \circ_F \mathcal{Q} \in \mathrm{sh}_q(E^6)$$

The map **A** induces a map

$$\mathcal{A} : \mathcal{P} \circ_F \mathcal{Q} \circ_{F\times E} \mathcal{P} \circ_F \mathcal{Q} \to \mathbb{A}_{e_1=e_6, e_2=e_3, e_4=e_5}[-3N];$$

The map **B** induces a map

$$\mathcal{B} : \mathcal{P} \circ_F \mathcal{Q} \circ_{F\times E} \mathcal{P} \circ_F \mathcal{Q} \to \mathbb{A}_{e_1=e_6, e_2=e_5, e_3=e_4}[-2N]$$

we also have a map

$$\delta_1 : \mathbb{A}_{e_1=e_6, e_2=e_3, e_4=e_5} \to \mathbb{A}_{e_1=e_6, e_2=e_5, e_3=e_4}[N]$$

and we have a homotopy equivalence

$$\mathcal{B} \sim \delta_1 \mathcal{A}. \tag{49}$$

Let $\gamma \in \mathrm{sh}_{\pi R^2}(E\times E)$ and $\gamma \to \mathbb{A}_{e_1=e_2} \otimes \Lambda[2N]$. be as in Sec. 9.1.5.

Let $\mathcal{F} \in \mathrm{sh}_{\pi R^2}(E\times E)$ be such that $\mathrm{SS}\mathcal{F} \subset V \times V$.

We then have the following maps

$$\begin{aligned} &\mathcal{P} \circ_F \mathcal{Q} \circ_{F\times E} \mathcal{P} \circ_F \mathcal{Q} \circ_{E^6} (p_{23}^{-1}\gamma \otimes p_{45}^{-1}\gamma \otimes p_{16}^{-1}\mathcal{F}) \\ &\qquad \sim \mathbb{A}_{e_1=e_6, e_2=e_3, e_4=e_5}[-3N] \circ_{E^6} (p_{23}^{-1}\gamma \otimes p_{45}^{-1}\gamma \otimes p_{16}^{-1}\mathcal{F}) \\ &\qquad\qquad \sim \Lambda \otimes (\mathbb{A}_{e_1=e_6} \circ_{E\times E} \mathcal{F})[-3N+4N-2N] \end{aligned} \tag{50}$$

$$\begin{aligned} &\mathcal{P} \circ_F \mathcal{Q} \circ_{F\times E} \mathcal{P} \circ_F \mathcal{Q} \circ_{E^6} (p_{23}^{-1}\gamma \otimes p_{45}^{-1}\gamma \otimes p_{16}^{-1}\mathcal{F}) \\ &\qquad \tilde{\to} \mathbb{A}_{e_1=e_6, e_2=e_5, e_3=e_4}[-2N] \circ_{E^6} (p_{23}^{-1}\gamma \otimes p_{45}^{-1}\gamma \otimes p_{16}^{-1}\mathcal{F}) \\ &\qquad\qquad \to \mathbb{A}_{e^1=e^4, e^2=e^3} \circ_{E^4} (p_{23}^{-1}\gamma \otimes p_{14}^{-1}\mathcal{F})[-2N] \\ &\qquad\qquad\qquad \to \mathbb{A}_{e^1=e^4} \circ_{E^2} (\mathcal{F})[-2N+2N-N] \end{aligned} \tag{51}$$

As follows from (49) and Sec 9.1.11

Lemma 9.13 *the maps (50) and (51) are homotopy equivalent.*

9.3 Pair of consequitive families

Let $u : F \times B_r \to B_R$, $v : F \times B_R \to E$ be graded families of symplectic embeddings. Let $w : F \times B_r \to E$ be defined by $w(f,b) = v(f, u(f,b))$. The gradings define liftings $g_u : F \times B_r \to \overline{\mathbf{Sp}}(2N)$; $g_v : F \times B_R \to \overline{\mathbf{Sp}}(2N)$ of the corresponding differential maps.

Let $g_w : F \times B_r \to \overline{\mathbf{Sp}}(2N)$ be given by $g_w(f,b) = g_v(f, u(f,b))g_u(f,b)$. It follows that g_w lifts the differential map $F \times B_r \to E$ determined by w. Therefore, g_w is a grading of w.

Proposition 9.14 *We have a homotopy equivalence* $\mathbb{P}_v \circ \mathbb{P}_u \tilde{\to} \mathbb{P}_w$.

Sketch of the proof As above, let us extend the family v to a family

$$v_t : F \times [-1,1] \times B_R \to E,$$

where

$$v_t(f,t,b) = \frac{v(f,tb) - v(f,0)}{t} + v(f,0).$$

Let $w_t : F \times [-1,1] \times B_r \to E$, where $w_t(f,t,b) = v_t(f,t,u(f,b))$. The gradings from v and w extend to v_t, w_t. We will show that there exists a homotopy equivalence

$$\mathbb{P}_{v_t} \circ \mathbb{P}_u \tilde{\to} \mathbb{P}_{w_t}. \tag{52}$$

Restriction to $t = 1$ will then show the Proposition.

To show the existence of (52), it suffices to establish the homotopy equivalence of the restriction to $t = 0$. Observe that v_0 comes from a family of linear symplectomorphisms $F \to \mathrm{Sp}(2N)$ whose grading defines a lifting $V_0 : F \to \overline{\mathbf{Sp}}(2N)$. Let $V \in \mathrm{sh}_\infty(F \times \times E \times E)$ be the corresponding object. We have a homotopy equivalence

$$\mathbb{P}_{v_0} \circ \mathbb{P}_u \sim V \circ \mathbb{P}_u$$

so the problem reduces to establishing a homotopy equivalence $V \circ \mathbb{P}_u \tilde{\to} \mathbb{P}_{v_0 u}$.

In a similar way (via considering the family u_t), one reduces the problem to the case when the family u is linear. The grading then defines an object $U \in \mathrm{sh}_\infty(F \times E \times E)$. Similarly, the linear family $v_0 u$, along with its grading, defines an object $W \in \mathrm{sh}_\infty(F \times E \times E)$.

Next, we have homotopy equivalences $U \circ \mathbb{P}_{B_r} \tilde{\to} \mathbb{P}_u$; $W \circ \mathbb{P}_{B_r} \tilde{\to} \mathbb{P}_{v_0 u}$ so that the problem reduces to establishing a homotopy equivalence

$$V \circ U \tilde{\to} W,$$

which follows from Sec 8.

9.4 Mobile families

9.4.1 Definition

Let $U \subset T^*E$ be an open subset let $j : U \to T^*E$ be the corresponding open embedding. Let $I : U \times B_R \to T^*E$ be a family of symplectic embeddings, where we assume $I|_{U \times 0} = j$.

The family I defines a Lagrangian sub-manifold

$$L_I \subset T^*U \times \mathbf{int} B_R \times T^*E.$$

Set $F = E \oplus E^*$.

We have a natural identification $T^*U = U \times F$. For each $\xi \in U$ let $L_\xi := T^*_\xi U \times \mathbf{int} B_R \times T^*E \cap L_I \subset F \times \mathbf{int} B_R \times T^*E$. Let $P_\xi \subset F \times \mathbf{int} B_R$ be the image of L_ξ under the projection along T^*E Call I *mobile* if for every ξ, P_ξ is a graph of an embedding $\mathbf{int} B_R \to F$.

9.4.2 Main proposition

We have objects $\mathcal{P}_I, \mathcal{Q}_I \in \mathrm{sh}_q(U \times E \times E)$. Let $p_1, p_2 : U \times E \times E \times E \times E \to U \times E \times E$ be the projections

$$p_1(u, e_1, f_1, e_2, f_2) = (u, e_1, f_1); \quad p_2(u, e_1, f_1, e_2, f_2) = (u, e_2, f_2).$$

Consider

$$R_I := p_1^{-1}\mathbb{P}_I \circ p_2^{-1}\mathbb{Q}_I.$$

Let $i : E^3 \to E^4; p : E^3 \to E^2$ be given by $i(a,b,c) = (a,b,b,c)$; $p(a,b,c) = (a,c)$. According to the previous subsection, we have a map

$$p_! i^{-1} R_I \to \mathbb{A}_{[U \times \Delta_E, 0]}$$

where $\Delta_E \subset E \times E$ is the diagonal.

By the conjugacy, we have a map

$$R_I \to \mathbb{A}_{[U \times \Delta_{14} \times \Delta_{23}, 0]}[N],$$

where $N = \dim E$ which, in turn, gives rise to a map

$$\alpha : \pi_{U!} R_I \to \mathbb{A}_{[\Delta_{14} \times \Delta_{23}, 0]}[-N],$$

where $\pi_U : U \times E^4 \to E^4$ is the projection along U.

Let $V \subset U$ be an open subset satisfying: for every $u \in U$, if $I(u \times B_R) \cap V \neq \emptyset$, then $I(u \times B_R) \subset U$.

Let $p_i : T^*E^4 \to T^*E$ be the projections $i = 1, 2, 3, 4$. Let $p_{ij} := p_i \times p_j : T^*E^4 \to T^*E^2$.

Proposition 9.15 *Let $A, B \in \mathrm{sh}_q(E \times E)$ and assume that $SSA \subset B_R \times B_R \times \mathbb{R}$; $SSB \subset V$. Then* $H := (\mathrm{Cone}\, \alpha) *_{E^4} (p_{23}^{-1} A \circ p_{14}^{-1} B) \sim 0$.

Sketch of the proof. Let us define a family of symplectic embeddings

$$J : U \times (-1, 1) \times B_R \to T^*E$$

by means of dilations, same as above. One then defines an object $\pi_{U!} R_J \in \mathrm{sh}_q((-1,1) \times E^4)$, a map

$$\alpha_J : \pi_{U!} R_J \to \mathbb{A}_{[(-1,1) \times \Delta_{14} \times \Delta_{23}, 0]}[-N],$$

and an object

$$H_J := (\mathrm{Cone}\, \alpha_J) *_{E^4} (p_{23}^{-1} A \otimes p_{14}^{-1} B) \in \mathrm{sh}_q((-1,1)).$$

Singular support estimate (see below) shows that

$$\mathrm{SS} H_J \subset T_I^* I \times \mathbb{R}.$$

Therefore, it suffices to show that $H_J|_0 \sim 0$, in other words, the problem reduces to the case when I is a family of linear symplectic embeddings. The latter case can be reduced to the case when every embedding is a parallel transfer which is straightforward.

Estimate of SSH_J. It suffices to show that

$$\mathrm{SS}(\pi_{U!} R_J *_{E^4} (A \boxtimes B)) \subset T^*_{(-1,1)}(-1,1).$$

Let us identify

$$T^*(U \times \mathbb{R} \times E^4) \times \mathbb{R} = (U \times \mathbb{R}) \times (F \oplus \mathbb{R}) \times F^4 \times \mathbb{R}.$$

We have

$$\mathrm{SS}(R_J) \subset \{(\tau, \eta_J(\tau, v_1) - \eta_J(\tau, v_2), v_1^a, J(\tau, v_1), v_2^a, J(v_2)) | \tau \in U \times \mathbb{R}, v_i \in F, |v_i| < R\} \times \mathbb{R}$$
$$\cup \{(\tau, \zeta, v_1, w_1, v_2, w_2) || v_1|, |v_2| \leq R; \max(|v_1|, |v_2|) = R.\} \times \mathbb{R}.$$

Consider now $\mathrm{SS}(R_J *_{E \times E} A)$. As $\mathrm{SS}(A) \subset \{(v_1, v_2) || v_1|, |v_2| < R\}$, it follows that

$$\mathrm{SS}(R_J *_{E \times E} A) \subset \{(\tau, \eta_J(\tau, v_1) - \eta_J(\tau, v_2), J(\tau, v_1), J(\tau, v_2)) || v_1|, |v_2| < R\} \times \mathbb{R}.$$

Let us estimate

$$\mathrm{SS}((R_J *_{E \times E} A) *_{E \times E} B).$$

It follows that there exists a compact subset $K \subset U$ such that

$$\mathrm{SS}((R_J *_{E \times E} A) *_{E \times E} B) \subset \{(\tau, \eta_J(\tau, v_1) - \eta_J(\tau, v_2)) | \tau \in K \times (-1, 1), |v_1|, |v_2| < R\} \times \mathbb{R}.$$

Namely, one can choose $K = \overline{\{u \in U | I(u, B_R) \cap V \neq \emptyset\}}$.

Let now $\tau = (u, x) \in U \times (-1, 1)$. We have $\eta_J(\tau, v) \in F \oplus \mathbb{R}$. Let $f(\tau, v)$ be the F-component and $x(\tau, v)$ be the $\mathbb{R}$-component. Let us now estimate

$$\mathrm{SS}(\pi_{U!}(R_J *_{E^4} (A \boxtimes B))).$$

As π_U is proper on the support of $R_J *_{E^4} (A \boxtimes B)$, the singular support in question is determined by the condition $f(\tau, v_1) - f(\tau, v_2) = 0$. As the family I is mobile, this condition implies $v_1 = v_2$, which implies $\eta_J(\tau, v_1) - \eta_J(\tau, v_2) = 0$ and

$$\mathrm{SS}(\pi_{U!}(R_J *_{E^4} (A \boxtimes B))) \subset T^*_{(-1,1)}(-1, 1) \times \mathbb{R}.$$

10 Tree operads and multi-categories

10.1 Planar/cyclic trees

Let us introduce a notation for a tree $\mathbf{t}$. Denote by $\mathbf{inp}(\mathbf{t})$ the set of inputs of $\mathbf{t}$, $V_{\mathbf{t}}$ the set of inner vertices of $\mathbf{t}$, for $v \in V_{\mathbf{t}}$, denote by E_v the set of inputs of v. Let $p_{\mathbf{t}}$ be the principal vertex of $\mathbf{t}$.

10.1.1 Planar trees

Define *a planar tree* as a tree with a total order on every set E_v; we then have an induced total order on $\mathbf{inp}(\mathbf{t})$.

We have a unique identification of ordered sets $E_v = \{1, 2, \ldots, n_v\}$, where $n_v = \# E_v$; $\mathbf{inp}_{\mathbf{t}} = \{1, 2, \ldots, n_{\mathbf{t}}\}$, where $n_{\mathbf{t}} = \# \mathbf{inp}_{\mathbf{t}}$.

10.1.2 Cyclic trees

Define *a cyclic tree* as a tree with a total order on every set E_v, $v \neq p_{\mathbf{t}}$, and a cylic order on $p_{\mathbf{t}}$. We then have an induced cycic order on $\mathbf{inp_t}$, in particular, we assume $\mathbf{inp_{pt}} \neq \emptyset$.

A rigid cyclic tree is a cyclic tree along with identifications $E_{p_{\mathbf{t}}} = \{1, 2, \ldots, n_{p_{\mathbf{t}}}\}$; $\mathbf{inp}_{p_{\mathbf{t}}} = \{1, 2, \ldots, n_{\mathbf{t}}\}$ which agree with the cyclic order on both sets.

10.1.3 Inserting trees into a tree

Let $\mathbf{t}$ be a planar tree. Let $\mathbf{t}_v$, $v \in V_{\mathbf{t}}$ be planar trees where $n_{\mathbf{t}_v} = n_v$. One then can insert the trees $\mathbf{t}_v$ into $\mathbf{t}$. Denote the resulting tree by $\mathbf{t}\{\mathbf{t}_v\}_{v \in V_{\mathbf{t}}}$.

Similarly, let $\mathbf{t}$ be a rigid cyclic tree. Let $\mathbf{t}_v$, $v \in V_{\mathbf{t}} \backslash p_{\mathbf{t}}$ be planar trees with $n_{\mathbf{t}_v} = n_v$; let $\mathbf{t}_{p_{\mathbf{t}}}$ be a rigid cyclic tree with $n_{\mathbf{t}_{p_{\mathbf{t}}}} = n_{p_{\mathbf{t}}}$. One then can define a similar insertion, to be denoted by $\mathbf{t}\{\mathbf{t}_v\}_{v \in V_{\mathbf{t}}}$.

10.1.4 Isomorphism classes of trees

Let **trees** be the set of isomorphism classes of planar trees and $\mathbf{trees}^{\mathrm{cyc}}$ be the set of isomorphism classes of rigid cyclic trees. Let $\mathcal{A}$ be a SMC enriched over **ground**. $\mathcal{T}(\mathcal{A})$ be the **ground**-category of all families of objects in $\mathcal{A}$ parameterized by $\mathbf{trees} \sqcup \mathbf{cyctrees}$.

Let also $\mathbf{trees}_n \subset \mathbf{trees}$ be the subset consisting of all isomorphism classes of trees with $n_{\mathbf{t}} = n$ and likewies for $\mathbf{cyctrees}_n$. The above defined insertions are defined on the level of isomorphism classes.

10.1.5 Famililes parameterized by isomorphism classes of trees

Let $\mathcal{A}$ be a $\bigoplus$-closed SMC. Let $\mathcal{T}(\mathcal{A})$ be a category, enriched over sets, whose every object is a family of objects $X_{\mathbf{t}} \in \mathcal{A}$, $\mathbf{t} \in \mathbf{trees} \sqcup \mathbf{cyctrees}$. Let $X, Y \in \mathcal{T}(\mathcal{A})$. Let us define a new family $X \circ Y \in \mathcal{T}(\mathcal{A})$ as follows:

$$X \circ Y(\mathbf{T}) = \bigoplus_{\mathbf{T} = \mathbf{t}\{\mathbf{t}_v\}_{v \in V_{\mathbf{t}}}} X(\mathbf{t}) \otimes \bigotimes_{v \in V_{\mathbf{t}}} Y(\mathbf{t}_v).$$

This way, $\mathcal{T}(\mathcal{A})$ becomes a monoidal category. The unit object $\mathbf{unit} \in \mathcal{T}(\mathcal{A})$ is defined by setting $\mathbf{unit}(\mathbf{t}) = \mathbf{unit}_{\mathcal{A}}$ for all isomorphism classes of planar trees with one vertex (corollas) and all isomorphism classes $\mathbf{t}$ of rigid cyclic trees with one vertex and matching numberings of E_p and $\mathbf{inp_t}$. Otherwise, $\mathbf{unit}(\mathbf{t}) = 0$.

10.2 Collections of functors

Let $\mathcal{C}$ be a **GZ**-category tensored over $\mathcal{A}$.

Let us define a category over **Sets**, $\mathcal{F}(\mathcal{C})$, as follows

$$\mathcal{F}(\mathcal{C}) := \prod_{n=0}^{\infty} \mathbf{swell}(\mathcal{C}^n \otimes \mathcal{C}^{\mathbf{op}}) \times \prod_{n=1}^{\infty} \mathcal{C}^n$$

so that an object $F \in \mathcal{F}(\mathcal{C})$ is a collection of objects $F^{[n]} \in \mathbf{swell}(\mathcal{C}^{\otimes n} \otimes \mathcal{C}^{\mathbf{op}})$, $n \geq 0$, and $F^{(n)} \in \mathbf{swell}(\mathcal{C}^{\otimes n})$, $n \geq 1$.

Let $\mathbf{t}$ be a planar tree. Define an object

$$F(\mathbf{t}) \in \mathbf{swell}(\mathcal{C}^{\otimes n_{\mathbf{t}}} \otimes \mathcal{C}^{\mathbf{op}}).$$

A) Let $h : \mathcal{C}^{\mathbf{op}} \otimes \mathcal{C} \to \mathbf{ground}$ be the hom functor.

B) We have an equivalence of categories

$$\left(\bigotimes_{v \in V_{\mathbf{t}}} (\mathcal{C}^{\otimes n_v} \otimes \mathcal{C}^{\mathbf{op}}) \cong \bigotimes_{v \in V_{\mathbf{t}} \backslash p_{\mathbf{t}}} \mathcal{C} \otimes \mathcal{C}^{\mathbf{op}} \right) \otimes (\mathcal{C}^{\otimes n_{\mathbf{t}}} \otimes \mathcal{C}^{\mathbf{op}}),$$

coming from the bijection

$$\bigsqcup_{v \in V_{\mathbf{t}}} E_v \cong V_{\mathbf{t}} \sqcup \mathbf{inp}_{\mathbf{t}} \backslash p_{\mathbf{t}}$$

which associates to an edge its target.

As a result we have a through map

$$\circ_{\mathbf{t}} : \left(\bigotimes_{v \in V_{\mathbf{t}}} \mathbf{swell}(\mathcal{C}^{\otimes n_v} \otimes \mathcal{C}^{\mathbf{op}}) \to \mathbf{swell} \bigotimes_{v \in V_{\mathbf{t}} \backslash p_{\mathbf{t}}} \mathcal{C} \otimes \mathcal{C}^{\mathbf{op}} \right) \otimes (\mathcal{C}^{\otimes n_{\mathbf{t}}} \otimes \mathcal{C}^{\mathbf{op}}) \to \mathbf{swell}(\mathcal{C}^{\otimes n_{\mathbf{t}}} \otimes \mathcal{C}^{\mathbf{op}}).$$

C) Set $F(\mathbf{t}) := \circ_{\mathbf{t}}(\bigotimes_{v \in V_{\mathbf{t}}} F^{[n_v]}$.

Let now $\mathbf{t}$ be a rigid cyclic tree. Define a functor $F(\mathbf{t}) \in \mathbf{swell}(\mathcal{C}^{n_{\mathbf{t}}})$ in a similar way. Let

$$\circ_{\mathbf{t}} : \mathcal{C}^{\otimes n_{p_{\mathbf{t}}}} \otimes \bigotimes_{v \in V_{\mathbf{t}} \backslash p_{\mathbf{t}}} (\mathcal{C}^{\otimes n_v} \otimes \mathcal{C}^{\mathbf{op}}) \to \mathcal{C}^{\otimes n_{\mathbf{t}}}$$

be defined similar to above and set

$$F(\mathbf{t}) := \circ_{\mathbf{t}}(F^{(n_{p_{\mathbf{t}}})} \otimes \bigotimes_{v \in V_{\mathbf{t}} \backslash p_{\mathbf{t}}} F^{[n_v]}.$$

10.3 Schur functors

Suppose $\mathcal{C}$ is tensored over $\mathcal{A}$. Let $X \in \mathcal{T}(\mathcal{A})$ and $F \in \mathcal{F}(\mathcal{C})$. Define an object $\mathbb{S}_X(F) \in \mathcal{F}(\mathcal{C})$ as follows

$$\mathbb{S}_X(F)^{[n]} := \bigoplus_{\mathbf{t} \in \mathbf{trees}_n} \mathbf{t}(F); \quad \mathbb{S}_X(F)^{(n)} = \bigoplus_{\mathbf{t} \in \mathbf{cyctrees}_n} \mathbf{t}(F).$$

We have natural isomorphisms

$$\mathbb{S}_X \mathbb{S}_Y F \cong \mathbb{S}_{X \circ Y} F; \quad \mathbb{S}_{\mathbf{unit}} F \cong F.$$

In fact, we have a $\mathcal{T}(\mathcal{A})$-action on $\mathcal{F}(\mathcal{C})$.

10.4 Tree operads

A tree operad in $\mathcal{T}(\mathcal{A})$ *is the same as a unital monoid in* $\mathcal{T}(\mathcal{A})$*.*

10.4.1 A tree operad triv

Let $\mathbf{triv} \in \mathcal{T}(\mathcal{A})$ *be given by* $\mathbf{triv}(\mathbf{t}) = \mathbf{unit}_{\mathcal{A}}$ *for all* $\mathbf{t}$*.*

10.4.2 Endomorphism tree operad

Let $\mathcal{C}$ *be enriched and tensored over* $\mathcal{A}$*. Let* $F, G \in \mathcal{F}(\mathcal{C})$*. Consider a functor* $H_{F,G} : \mathcal{T}(\mathbf{swell}\,\mathcal{A}) \to \mathbf{Sets}$*,*

$$H_{F,G}(X) = Hom(\mathbb{S}_X F; G)$$

The functor $H_{F,G}$ *is representable. Denote the representing object by* $\mathcal{H}_{F,G}$*. We have (*$\mathbf{t}$ *is planar):*

$$\mathcal{H}_{F,G}(\mathbf{t}) = \underline{Hom}_{\mathbf{swell}(\mathcal{C}^{\otimes n_{\mathbf{t}}} \otimes \mathcal{C}^{\mathbf{op}})}(F(\mathbf{t}); G^{[n_{\mathbf{t}}]});$$

if $\mathbf{t}$ *is a rigid cyclic tree, we have:*

$$\mathcal{H}_{F,G}(\mathbf{t}) = \underline{Hom}_{\mathbf{swell}\,\mathcal{C}^{\otimes n_{\mathbf{t}}}}(F(\mathbf{t}); G^{(n_{\mathbf{t}})}).$$

Set $\mathrm{End}_F := \mathcal{H}_{F,F}$*. We have a natural tree operad structure on* End_F*. Furthermore, we have an* $\mathrm{End}_F - \mathrm{End}_G$*-bi-module structure on* $\mathcal{H}_{F,G}$ *(where we interpret tree operads* $\mathrm{End}_F, \mathrm{End}_G$ *as monoids in* $\mathcal{T}(\mathbf{swell}\,\mathcal{A})$*.*

10.4.3 Quasi-contracible tree operads

Let now $\mathcal{A} = \mathbf{pt}$ *so that* $\mathbf{swell}\,\mathcal{A} = \mathbf{GZ}$*. Call a tree operad* $\mathcal{O} \in \mathcal{T}(\mathbf{GZ})$ pseudo-contractible *if*

1) $\mathcal{O}(\mathbf{t}) \in \mathbf{GZ}$ *admits a truncation for every* $\mathcal{O}(\mathbf{t})$*. We therefore have an induced tree operad structure on* $\tau_{\leq 0}\mathcal{O}$ *and a map of tree operads* $\tau_{\leq 0}\mathcal{O} \to \mathcal{O}$*.*

2) Every object $\tau_{\leq 0}\mathcal{O}$ *admits a trunctation* $\tau_{\geq 0}$*, to be denoted* $H^0\mathcal{O}(\mathbf{t})$ *which is a finitely generated free* $\mathbb{A}$*-module; we have an induced map of tree operads* $\tau_{\leq 0}\mathcal{O} \to H^0\mathcal{O}$*. We require this map to be a term-wise homotopy equivalence.*

A quasi-contractible tree operad *is a pseudo-contractible operad* $\mathcal{O}$ *endowed with a map of tree operads* $\mathbf{triv} \to H^0(\mathcal{O})$*.*

In this case there exists a splitting of the map $\tau_{\leq 0}\mathcal{O}(\mathbf{t}) \to H^0\mathcal{O}(\mathbf{t})$*, hence a pull-back of the diagram*

$$\mathbf{triv} \to H^0(\mathcal{O}) \leftarrow \tau_{\leq 0}\mathcal{O},$$

to be denoted by $\mathbf{triv}_{\mathcal{O}}$ *so that we have a diagram*

$$\mathbf{triv} \overset{\sim}{\leftarrow} \mathbf{triv}_{\mathcal{O}} \to \mathcal{O}.$$

Let $\mathcal{O}_1, \mathcal{O}_2$ *be quasi-contractible operads and* $\mathcal{M}$ *a* $\mathcal{O}_1 - \mathcal{O}_2$*-bi-module. Call* $\mathcal{M}$ pseudo-contractible *if there exist truncations* $\tau_{\leq 0}\mathcal{M}(\mathbf{t})$ *and* $\tau_{\geq 0}\tau_{\leq 0}\mathcal{M}(\mathbf{t}) =: H^0\mathcal{M}(\mathbf{t})$*, where each* $H^0\mathcal{M}(\mathbf{t})$ *is a finitely generated free* $\mathbb{A}$*-module.*

A quasi-contractible $\mathcal{O}_1 - \mathcal{O}_2$-bi-module $\mathcal{M}$ is a pseudo-contractible $\mathcal{O}_1 - \mathcal{O}_2$-bi-module $\mathcal{M}$ endowed with a map

$$(\mathbf{triv}, \mathbf{triv}, \mathbf{triv}) \to (H^0\mathcal{O}_1, H^0\mathcal{M}, H^0\mathcal{O}_2)$$

of triples: a pair of tree-operads and their bi-module.

Similar to above, we have a pull-back of the diagram

$$(\mathbf{triv}, \mathbf{triv}, \mathbf{triv}) \leftarrow (\tau_{\leq 0}\mathcal{O}_1, \tau_{\leq 0}\mathcal{M}, \tau_{\leq 0}\mathcal{O}_2) \to (H^0\mathcal{O}_1, H^0\mathcal{M}, H^0\mathcal{O}_2),$$

to be denoted by $(\mathbf{triv}_{\mathcal{O}_1}, \mathbf{triv}_{\mathcal{M}}; \mathbf{triv}_{\mathcal{O}_2})$ so that we have a diagram

$$(\mathbf{triv}, \mathbf{triv}, \mathbf{triv}) \stackrel{\sim}{\leftarrow} (\mathbf{triv}_{\mathcal{O}_1}, \mathbf{triv}_{\mathcal{M}}, \mathbf{triv}_{\mathcal{O}_2}) \to (\mathcal{O}_1, \mathcal{M}, \mathcal{O}_2).$$

10.5 Pull backs from $\mathcal{F}(\mathcal{D})$ to $\mathcal{F}(\mathcal{C})$

Let $\mathcal{A}$ have internal hom. Let $\mathcal{C}, \mathcal{D}$ be categories enriched over $\mathcal{A}$. Let $G \in \mathcal{F}(\mathcal{D})$. Let $L \in \mathbf{swell}(\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D})$.

Consider the following functor $H : \mathcal{F}(\mathcal{C})^{\mathbf{op}} \to \mathbf{Sets}$ as follows.

1) We have functors

$$e_L : \mathcal{C}^{\otimes n} \otimes \mathcal{C}^{\mathbf{op}} \otimes (\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D})^{\otimes n} \to \mathcal{D}^{\otimes n} \otimes \mathcal{C}^{\mathbf{op}},$$

via using the hom-functor $\mathcal{C}^{\otimes n} \otimes (\mathcal{C}^{\mathbf{op}})^{\otimes n} \to \mathbf{GZ}$, as well as

$$f_L : \mathcal{D}^{\otimes n} \otimes \mathcal{D}^{\mathbf{op}} \otimes \mathcal{C}^{\mathbf{op}} \otimes \mathcal{D} \to \mathcal{D}^{\otimes n} \otimes \mathcal{C}^{\mathbf{op}}.$$

via the hom functor $\mathcal{D}^{\mathbf{op}} \otimes \mathcal{D} \to \mathbf{GZ}$.

Similarly, one defines a cyclic version:

1)

$$e_L^{\mathrm{cyc}} : \mathcal{C}^{\otimes n} \otimes (\mathcal{C}^{\mathbf{op}} \otimes \mathcal{D})^{\otimes n} \to \mathcal{D}^{\otimes n};$$

2) Set

$$H^{[n]}(F) := \mathrm{Hom}(e_L(F^{[n]} \otimes \mathcal{L}^{\otimes n}; G^{[n]});$$
$$H^{(n)}(F) := \mathrm{Hom}(e_L^{\mathrm{cyc}}(F^{(n)} \otimes \mathcal{L}^{\otimes n}; G^{(n)}).$$

Set

$$H(F) = \prod_{n \geq 0} H^{[n]}(F) \times \prod_{n > 0} H^{(n)}(F).$$

It follows that the functor H is representable. Denote the representing object by $L^{-1}G$.

Let $X \in \mathcal{T}(\mathcal{A})$. We have a natural map $\mathbb{S}_X L^{-1}G \to L^{-1}\mathbb{S}_X G$.