CHAPTER 1: WAVE EQUATION ON A RIEMANNIAN MANIFOLD

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Contents

1. Introduction

These Lectures are concerned with eigenfunctions of the Laplacian Δ of a Riemannian manifold (M, g). The Laplacian of (M, g) is given locally by

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right), \tag{1.1}$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$ and $\sqrt{g} := \sqrt{\det[g_{ij}]}$. Since g is usually understood, we often write the Laplacian as Δ. The eigenvalue problem (or Helmholtz equation) is,

$$(\Delta_g + \lambda^2)\varphi_\lambda = 0. \tag{1.2}$$

When M is compact, there exists[1] an orthonormal basis $\{\varphi_j\}_{j \geq 0}$ of $L^2(M)$ of eigenfunctions,

$$\Delta_g \varphi_j = -\lambda_j^2 \varphi_j, \qquad \langle \varphi_j, \varphi_k \rangle_{L^2(M)} := \int_M \varphi_j \varphi_k \, dV_g = \delta_{jk} \tag{1.3}$$

enumerated in increasing order of the eigenvalues

$$0 = \lambda_0^2 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \tag{1.4}$$

are repeated according to multiplicity. Eigenfunctions φ_λ represent modes of vibration of M or stationary states in quantum mechanics. We are mainly interested in the high frequency behavior of the orthonorma basis sequence of eigenfunctions $\{\varphi_j\}_{j=0}^\infty$ as $\lambda_j \to \infty$. The correspondence principle of quantum mechanics states that the high frequency behavior is related to classical mechanics on (M, g), i.e. to the geodesic flow $G^t : T^*M \to T^*M$ on the cotangent bundle.

The study of eigenfunctions splits up into Local and Global techniques. Local techniques study eigenfunctions on small balls $B(p, r) = B_r(p)$ on M, often of wavelength scale $r = \frac{C}{\lambda}$. After re-scaling, the eigenfunctions behave

[1]Existence follows from the spectral theorem for compact self-adjoint operators applied to Δ^{-1}. We will be taking it for granted.

almost like harmonic functions on the scaled ball. Global techniques exploit the wave equation. The key player is the half-wave propagator

$$U(t) = e^{it\sqrt{-\Delta}} : L^2(M) \to L^2(M).$$

It is a unitary group of operators which possess the eigenfunction expansion,

$$U(t,x,y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_j(x)\varphi_j(y). \tag{1.5}$$

The sum converges in (and only in) the sense of distributions. Unlike the the heat kernel, $U(t,x,y)$ is a singular distribution representing the amplitude of the wave created by propagating a pulse (delta-function) δ_y. The wave front after time t is the distance sphere $S_t(y) = \{x : r(x,y) = t\}$, where $r(x,y)$ is the Riemannian distance.

Except in special model cases such as the flat torus or sphere, eigenfunctions are very difficult to analyse individually or asymptotically. The principal method to determine their asymptotic behavior it is to study the wave kernel (1.5) and its singularities. Since it is a linear combination of all the eigenfunctions, special techniques must be developed to sift out individual ones to the extent possible. Often, this means to study the asymptotics as $\lambda \to \infty$ of the 'dual' spectral projections kernels,

$$\Pi_{[0,\lambda]}(x,y) = \sum_{j:\lambda_j \leq \lambda} \varphi_j(x)\varphi_j(y). \tag{1.6}$$

The individual terms $\varphi_j(x)\varphi_j(y)$ represent jumps in the Weyl sum (1.6) (when the eigenvalues are multiple, one sums over all j with the given eigenvalue.)

A smooth verion of (1.6) is

$$\rho(\sqrt{-\Delta} - \lambda)(x,y) = \sum_j \rho(\lambda - \lambda_j)\varphi_j(x)\varphi_j(y), \tag{1.7}$$

where $\rho \in \mathcal{S}(\mathbb{R})$ has a compactly supported Fourier transform $\hat{\rho}(t)$. By Fourier inversion,

$$\rho(\sqrt{-\Delta} - \lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} e^{-it\sqrt{-\Delta}} dt. \tag{1.8}$$

The series (1.7) converges absolutely and uniformly and is a C^∞ kernel on $\mathbb{R} \times M \times M$. This follows from Weyl's law that

$$N(\lambda) := \#\{j : \lambda_j \leq \lambda\} = \frac{Vol(M,g)|B_n|}{(2\pi)^n} We\lambda^n + O(\lambda^{n-1}), \tag{1.9}$$

where B_n is the unit ball in $\mathbb{R}^n$ and $|B_n|$ is its Euclidean volume. [2]

The fact that wave fronts are distance spheres suggests that singularities of solutions of the wave equation propagate along geodesics of (M,g). Geodesics on M are projections under $\pi : T^*M \to M$ of phase space

[2]Convergence of (1.7) is a HW exercise, assuming the Weyl law.

geodesics, phase space being the cotangent bundle T^*M. Phase space geodesics are orbits $G^t(x,\xi) = \gamma_{x,\xi}(t)$ of the geodesic flow, namely the Hamiltonian flow on T^*M of the Hamiltonian $H(x,\xi) = |\xi|_g$, where $|\xi|_g^2 = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j$. Geometers usually define geodesics using the Hamiltonian $|\xi|_g^2$ but the singularities of the wave equation propagate along orbits of H.

By its nature, the Global Harmonic analysis of eigenfunctons is interdisciplinary: it seeks to connect dynamics of the geodesic flow and harmonic analysis of eigenfunctions. Hence it a short mini-course it is impossible to cover the background in both areas. Since our purpose is to explain how to connect dynamics of G^t and harmonic analysis, we will have to take results on each side as a 'black box' and only present the highlights at the heart of the connections between the two areas.

The focus of these lectures is on two aspects of eigenfunctions:

- Quantum limit measures of the sequence of eigenfunctions [3]
- Nodal sets $\mathcal{N}_{\varphi_\lambda}$ of eigenfunctions, especially upper bounds on the nodal surface measure $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda})$. The emphasis is on real analytic metrics, where eigenfunctions may be analytically continued to the complexification of M (called a Grauert tube) and where complex analysis techniques may be applied.

1.1. **Contents of Lecture 1.** In Lecture 1, we study wave kernels on Riemannian manifolds. We begin with three model cases:

- Euclidean wave kernels, especially in dimension 3: Propagators, Poisson kernel, and fundamental solution. Relation to spherical means. Fourier transform formula.
- Flat torus: periodization.
- Sphere $\mathbb{S}^n$.
- Hyperbolic space $\mathbb{H}^n$. Discrete subgroups and hyperbolic quotients. Periodization.

We then consider wave kernels (1.5) on general Riemannian manifolds for small times and define and construct *parametrices*. We briefly review the Hadamard-Riesz parametrix construction on parametrices.. In some sense it is the generalization of the spherical means approach in the Euclidean case.

[3]Also called semi-classical measures, microlocal defect measures, Wigner distributions.

2. General notation

Let (M, g) be any complete Riemannian manifold. Let Δ_g be its Laplacian and let $\Box = \frac{\partial^2}{\partial t^2} - \Delta_g$.

The Cauchy problem for the wave equation on $\mathbb{R} \times M$ is the initial value problem (with Cauchy data f, g)

$$\begin{cases} \Box u(t,x) = 0, \\ \\ u(0,x) = f, \;\; \frac{\partial}{\partial t} u(0,x) = g(x), \end{cases}.$$

The solution operator of the Cauchy problem (the "propagator") is the wave group,

$$\mathcal{U}(t) = \begin{pmatrix} \cos t\sqrt{\Delta} & \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}} \\ \\ \sqrt{\Delta}\sin t\sqrt{\Delta} & \cos t\sqrt{\Delta} \end{pmatrix}.$$

The solution of the Cauchy problem with data (f, g) is $\mathcal{U}(t) \begin{pmatrix} f \\ g \end{pmatrix}$.

- Even part $C(t) := \cos t\sqrt{\Delta}$ which solves the initial value problem

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta)u = 0 \\ u|_{t=0} = f \qquad \frac{\partial}{\partial t} u|_{t=0} = 0 \end{cases} \tag{2.1}$$

- Odd part $S(t) = \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$ is the operator solving

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta)u = 0 \\ u|_{t=0} = 0 \qquad \frac{\partial}{\partial t} u|_{t=0} = g \end{cases} \tag{2.2}$$

The forward half-wave group is the solution operator of the Cauchy problem

$$(\frac{1}{i}\frac{\partial}{\partial t} - \sqrt{-\Delta})u = 0, \quad u(0,x) = u_0.$$

The solution is given by

$$u(t,x) = U(t)u_0(x),$$

with

$$U(t) = e^{it\sqrt{-\Delta}} = C(t) + i\sqrt{-\Delta}S(t)$$

the unitary group on $L^2(M)$ generated by the self-adjoint elliptic operator $\sqrt{-\Delta}$. $U(t)$ is more complicated than $E(t), S(t)$ because of the $\sqrt{-\Delta}$ factor (a pseudo-differential operator of order 1). As will be reviewed below, $C(t), S(t)$ have finite propagation speed but $U(t)$ does not, because $\sqrt{-\Delta}$ is a non-local operator (does not preserve supports of functions). Yet, $U(t)$ is more useful for applications. $C(t), S(t)$ were studied classically by Hadamard [H] and Riesz [R], among many others, but $U(t)$ may have first been studied by Hormander [Ho68], Chazarain [Ch74] and Duistermaat-Guillemin [DG75].

All functions of Δ are defined by the spectral theorem. That is, if $\Delta\varphi = -\lambda^2\varphi$ then $F(\Delta)\varphi = F(-\lambda^2)\varphi$. For instance, $\sqrt{-\Delta}\varphi = \lambda\varphi$.

We refer to any of the operators $\cos t\sqrt{\Delta}$, $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$, $e^{it\sqrt{-\Delta}}$ as "propagators" since they take initial data and evolve it in time.

2.1. **Fundamental solutions.** A fundamental solution of the wave equation is a solution of

$$\Box E(t,x,y) = \delta_0(t)\delta_x(y).$$

The right side is the Schwartz kernel of the identity operator on $\mathbb{R} \times M$.

There exists a unique fundamental solution which is supported in the forward conoid

$$C_+ = \{(t,x,y) : t > 0, t^2 - r^2(x,y) > 0\}.$$

called the advanced (or forward) propagator. It is given by

$$E_+(t) = H(t)\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}, \tag{2.3}$$

where $H(t) = \mathbf{1}_{t\geq 0}$ is the Heaviside step function. It is well-defined for any curved globally hyperbolic spacetime, while Cauchy problems and propagators require a choice of "Cauchy hypersurface" like $\{t = 0\}$.

3. Euclidean wave kernels

In this section, we review the exact formulae for the propagators and fundamental solution and the Poisson kernel in Euclidean $\mathbb{R}^n$.

As above, we wish to find exact solution operators for the Cauchy problem

$$\begin{cases} \Box u = 0, \\ u(x,0) = \varphi(x), \\ u_t(x,0) = \psi(x). \end{cases} \tag{3.1}$$

for the homogeneous wave equation. We define the solution operators

$$S(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}, \quad C(t) = S'(t) = \cos t\sqrt{-\Delta}$$

of the homogeneous wave equation (3.1).

There are several methods to obtain explicit formulae for these propagators.

- Using the spherical means operator L_r.
- Using the Fourier transform.

The spherical means operator is defined by

$$L_r f(x) = \int_{S^*_x M} f(x + r\xi) dS(\xi), \tag{3.2}$$

Intuitively, it should be related to the wave equation because wave fronts are distance spheres $S_t(x) = \partial B(x,t)$ where $B(x,t)$ is the ball of radius $|t|$ around x.

A key point is that $[L_r, \Delta] = 0$ in Euclidean space. This is also true for $\mathbb{H}^n, \mathbb{S}^n$ but it is very rarely true on a Riemannian manifold. We will take advantage of this symmetry to express $C(t), S(t)$ in terms of L_t.

It is not necessarily the case that if $[A,B] = 0$ then $A = F(B)$ for some F. But this is the case for L_r: On Euclidean space $\mathbb{R}^n$ there is a classical explicit formula

$$L_r u(x) = W_m(ir\sqrt{\Delta})u(x), \quad (W_m(z) = \Gamma(\frac{m}{2})(\frac{2}{z})^{\frac{m}{2}-1} J_{\frac{m-2}{2}}(z))$$

In the lowest dimensions, they become

$$L_r = \begin{cases} J_0(r\sqrt{-\Delta})u(x), & n = 2 \\ \\ \frac{\sin(r\sqrt{-\Delta})}{r\sqrt{-\Delta}} u(x), & n = 3. \end{cases}$$

In Section 3.9 we review the so-called Pizzetti formula giving a Taylor expansion of $L_r \simeq I + \frac{r^2}{2n}\Delta + \cdots$. Note that $J_0(x) = 1 - \frac{x^2}{4} + \cdots$ and $\frac{\sin x}{x} \simeq 1 - x^2/3! + \cdots$.

The general formula on Euclidean $\mathbb{R}^n$ is given by

Proposition **1.** *Let $u(x,t)$ be the solution of* (3.1). *Then,*

$$\begin{aligned} u(x,t) \quad &= \quad \frac{1}{\gamma_n}\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)}\varphi(y)dS(y)\right) \\ & \quad + \frac{1}{\gamma_n}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)}\psi(y)dS(y)\right) \end{aligned}$$

where $\gamma_n = 1 \cdot 3 \cdot 5 \cdots (n-2)$, *or in operator terms*

$$\begin{cases} S(t) = \frac{1}{\gamma_n}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}} t^{n-2} L_t, \\ \\ C(t) = \frac{1}{\gamma_n}\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}} t^{n-2} L_t, \end{cases}$$

The explicit formula for $U(t) = \exp(-it\sqrt{-\Delta})$ in terms of spherical means involves $\sqrt{-\Delta} \cdot L_t$. The calculus of Fourier integral operators allows one to make sense of this and give formulae, but because $\sqrt{-\Delta}$ is non-local we do not expect an averaging operator over the sphere $S_t(x) = \partial B(x,t)$. But we may expect it differs from such an operator by a smoothing operator (an operator with a smooth Schwartz kernel).

3.1. **Darboux-Euler formula.** Let us make the abbreviation

$$\bar{u}(t,r;x) := L_r u(x,t),$$

and consider it for a solution $u(x,t)$ of the homogeneous wave equation. We claim that for each fixed x it is a solution of the Darboux-Euler equation

$$\begin{cases} \bar{u}_{tt} - \bar{u}_{rr} - \frac{n-1}{r}\bar{u}_r = 0, \quad 0 < r < \infty, t \geq 0, \\ \bar{u}(r,0;x) = \bar{\varphi}(x;r), \quad \bar{u}_t(r,0;x) = \bar{\psi}(r;x). \end{cases} \tag{3.3}$$

Proof.

$$\begin{aligned} \bar{u}(r,t;x) &= ⨍_{\partial B_r(x)} u(y,t)dS(y) \\ &= ⨍_{\partial B_1(0)} u(x+ry,t)dS(y). \end{aligned}$$

Hence

$$\begin{aligned} \bar{u}_r(r,t;x) &= ⨍_{\partial B_1(0)} \nabla u(x+ry,t)\cdot y dS(y) \\ &= ⨍_{\partial B_r(x)} \nabla u(y,t)\cdot \tfrac{y-x}{r} dS(y) \\ &= ⨍_{\partial B_r(x)} \tfrac{\partial u}{\partial \nu} dS(y) \\ &= \tfrac{1}{\mathcal{H}^{n-1}(S_r(x))} \int_{\partial B_r(x)} \tfrac{\partial u}{\partial \nu} dS(y) \\ &= \tfrac{1}{\mathcal{H}^{n-1}(S_r(x))} \int_{B_r(x)} \Delta u(y,t)dy \\ &= \tfrac{1}{\mathcal{H}^{n-1}(S_r(x))} \int_{B_r(x)} u_{tt}(y,t)dy. \end{aligned}$$

Then

$$\begin{aligned} \bar{u}_r(r,t;x) &= \tfrac{1}{\mathcal{H}^{n-1}(S_r(x))} \int_{B_r(x)} u_{tt}(y,t)dy \\ \Longrightarrow & \quad (r^{n-1}\bar{u}_r(r,t;x))_r = \tfrac{1}{\mathcal{H}^{n-1}(S_1(x))} \int_{\partial B_r(x)} u_{tt}(y,t)dS(y) \\ &= r^{n-1}⨍_{\partial B_r(x)} u_{tt}(y,t)dS(y) = r^{n-1}\bar{u}_{tt}(r,t;x). \end{aligned}$$

It follows that

$$(r^{n-1}\bar{u}_r(r,t;x))_r = r^{n-1}\bar{u}_{tt}(r,t;x)$$

or equivalently

$$(n-1)r^{n-2}\bar{u}_r + r^{n-1}\bar{u}_{rr} = r^{n-1}\bar{u}_{tt}$$

and dividing by r^{n-1} gives the Darboux formula.

∎

3.2. **Proof of Propsition 1 in dimension 3.** In dimension 3, Proposition 1 says:

$$u(x,t) = \frac{1}{\gamma_3}\frac{\partial}{\partial t}\left(⨍_{\partial B(x,t)} \varphi(y)dS(y)\right) + \frac{1}{\gamma_3}\left(t⨍_{\partial B(x,t)} \psi(y)dS(y)\right)$$

The first term is $C(t)$ and the second is $S(t)$.

If we set $n = 3$ in (3.3), we get the equation

$$\bar{u}_{tt} - \bar{u}_{rr} - \frac{2}{r}\bar{u}_r = 0.$$

We now prove the formula in Proposition 1 from this.

It is actually HW Exercise 1. Try to do it yourself. The trick is to multiply the spherical means $\bar{u}$ by r and reduce to a 1 D wave equation. The proof given below is the solution to this exercse

The equation is equivalent to

$$\begin{cases} \frac{\partial^2}{\partial t^2}(r\bar{u}) - \frac{\partial^2}{\partial r^2}(r\bar{u}) = 0 \\ r\bar{u}_{t=0} = r\bar{\varphi}, \;\; \partial_t(r\bar{u})|_{t=0} = r\bar{g}. \end{cases} \tag{3.4}$$

This is a 1D wave equation which can be solved by d'Alembert's formula:

$$\begin{aligned} r\bar{u}(x,r,t) &= \tfrac{1}{2}[(r+t)\bar{f}(x,r+t) + (r-t)\bar{f}(x,r-t)] \\ &+ \tfrac{1}{2}\int_{r-t}^{r+t} \tau\bar{g}(x,\tau)d\tau. \end{aligned} \tag{3.5}$$

Now divide by r and take the limit as $r \to 0$ to get

$$\begin{aligned} u(x,t) &= t\bar{g}(x,t) + \partial_t(t\bar{f}(x,t)) \\ &= \tfrac{1}{4\pi t}\int_{|y-x|=t} g(y)dS(y) + \tfrac{\partial}{\partial t}\left(\tfrac{1}{4\pi t}\int_{|y-x|=t} f(y)dS(y)\right). \end{aligned} \tag{3.6}$$

3.3. **Kirchhoff formula.** If $\varphi \in C^1$, then we may perform the differentiation and obtain a simpler formula, known as Kirchoff's formula:

Proposition **2.** *The solution of* (3.1) *is given by*

$$u(x,t) = \frac{1}{4\pi t^2}\int_{S_t(x)} [\varphi(y) + \nabla\varphi(y)\cdot(y-x) + t\psi(y)]dS(y).$$

HW Exercise 2 is to prove this formula.

3.4. **Dimension two.** The standard method for solving the wave equation on $\mathbb{R}^2$ is to increase the dimension by one to $\mathbb{R}^3$ and pulling back the solution $u(x_1, x_2, t)$ on $\mathbb{R}^2 \times \mathbb{R}$ to a solution $\tilde{u}(x_1, x_2, x_3, t)$ on $\mathbb{R}^2 \times \mathbb{R}$ of the Cauchy problem with pulled back data which is independent of the third coordinate. Thus, the solution is given by

$$\tilde{u}(x_1, x_2, 0, t) = \fint_{\partial B_t(\bar{x})} \left[\tilde{\varphi}(y) + \nabla\tilde{\varphi}(y)\cdot(y-x) + t\tilde{\psi}(y)\right] dS(y).$$

Here $B_t(\bar{x})$ is the ball of $\mathbb{R}^3$ of radius t around $\bar{x} = (x_1, x_2, 0)$. But if F is any function independent of the third coordinate,

$$\begin{aligned} ⨍_{\partial B_t(\bar{x})} F(y) dS(y) &= \tfrac{1}{4\pi t^2} \int_{\partial B_t(\bar{x})} F(y) dS(y) \\ &= \tfrac{1}{4\pi t^2} \int_{\partial B_t(x)} F(y)(1 + |\nabla \sqrt{\Gamma}|^2)^{\frac{1}{2}} dy, \end{aligned}$$

where $B_t(x)$ is the ball of radius t around $x \in \mathbb{R}^2$ and $\Gamma(y) = (t^2 - |x-y|^2)$. Some elementary calculations then give

$$u(x,t) = \frac{1}{2\pi t^2} \int_{B_t(x)} \frac{t\varphi(y) + t^2\psi(y) + t\nabla\varphi(y)\cdot(y-x)}{\sqrt{t^2 - |x-y|^2}} dy.$$

The method of descent is universal. Given any even dimensional (M^n, g) we form the product $(M^n \times \mathbb{R}, g \oplus dx_{n+1}^2)$ and solve the wave equation on the product space with data pulled back from M^n.

3.5. **Poisson kernel formula for $U(t) = \exp it\sqrt{-\Delta}$ in the Euclidean case.** The half-wave propagator is constructed on $\mathbb{R}^n$ by the Fourier inversion formula,

$$U(t,x,y) = \int_{\mathbb{R}^n} e^{i\langle x-y, \xi\rangle} e^{it|\xi|} d\xi. \tag{3.7}$$

The Poisson kernel (extending functions on $\mathbb{R}^n$ to harmonic functions on $\mathbb{R}_+ \times \mathbb{R}^n$) is the half-wave propagagor at positive imaginary times $t = i\tau$ $(\tau > 0)$,

$$\begin{aligned} U(i\tau, x, y) &= \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} e^{-\tau|\xi|} d\xi \\ &= \tau^{-n} \left(1 + (\tfrac{x-y}{\tau})^2\right)^{-\frac{n+1}{2}} = \tau \left(\tau^2 + (x-y)^2)\right)^{-\frac{n+1}{2}}. \end{aligned} \tag{3.8}$$

In the case of $\mathbb{R}^n$, the Poisson kernel analytically continues to $t + i\tau, \zeta = x + ip \in \mathbb{C}_+ \times \mathbb{C}^n$ as the integral

$$U(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|} e^{i\langle \xi, x+ip-y\rangle} d\xi, \tag{3.9}$$

which converges absolutely for $|p| < \tau$. If we substitute $\tau \to \tau - it$ and let $\tau \to 0$ we get the formula

$$U(t,x,y) = C_n \lim_{\tau \to 0} it((t+i\tau))^2 - r(x,y)^2)^{-\frac{n+1}{2}}, \tag{3.10}$$

for a constant C_n depending only on the dimension. The limit is taken in the sense of distributions and is then written

$$U(t,x,y) = C_n \;\; it \; ((t+i0))^2 - r(x,y)^2)^{-\frac{n+1}{2}}, \tag{3.11}$$

Background on distributions is given in the Appendix.

3.6. **Fourier formula.** The wave kernels in $\mathbb{R}^n$ may be expressed as Fourier integrals. We illustrate this only for the half-wave propagator $U(t) = \exp(it\sqrt{-\Delta})$, since we did not give a spherical means formula for it. Since

$$\delta_y(x) = \delta(x-y) = \int_{\mathbb{R}^n} e^{2\pi i\langle x-y,\xi\rangle} d\xi$$

the kernel of $U(t)$ is $U(t)\delta_y(x)$ which is

$$U(t,x,y) = \int_{\mathbb{R}^n} e^{2\pi i\langle x-y,\xi\rangle} e^{it|\xi|} d\xi.$$

If one puts the integral in polar coordinates $\xi = r\omega$, one gets

$$U(t,x,y) = \int_0^\infty \int_{S^{n-1}} e^{2\pi ir\langle x-y,\omega\rangle} e^{itr} r^{n=1} dr d\omega.$$

The spherical integral

$$J_{\frac{n-2}{2}}(r|x-y|) = \int_{S^{n-1}} e^{2\pi ir\langle x-y,\omega\rangle} d\omega$$

is a Bessel function. Hence we get

$$U(t,x,y) = \int_0^\infty J_{\frac{n-2}{2}}(r|x-y|) e^{itr} r^{n-1} dr.$$

One could go further with this caculation, e.g. $r^{n-1}e^{itr} = D_t^{n-1} e^{itr}$ so that

$$U(t,x,y) = D_t^{n-1} \int_0^\infty J_{\frac{n-2}{2}}(r|x-y|) e^{itr} dr.$$

3.7. **Fundamental solution.** In view of (2.3), an explicit formula for $S(t)$ induces one for the forward fundamental solution and in dimension 3 it says that

$$E^+ * \psi(x,t) = \frac{H(t)}{4\pi} \left(\frac{1}{t} \int_{\partial B(x,t)} \psi(y) dS(y) \right).$$

Another way to write this is that

$$E_+(t,x) = \frac{\delta(t-r)}{4\pi r}.$$

Above, we thought of the propagator as a 1-parameter family of operators on $\mathbb{R}^3$ indexed by t, but now we think of the kernels as distributions on $M \times \mathbb{R}$. In this section, we give another derivation that uses the theory of distributions rather than 'advanced calculus' and the Darboux-Euler formula from [HoI, GeSh, F]. It is based on pullbacks of distributions under submersions. The submersion in question is

$$Q(x,t) := t^2 - |x|^2 : \mathbb{R}^{3+1} \backslash \mathcal{N} \to \mathbb{R} \backslash \{0\},$$

where

$$\mathcal{N} = \{(x,t) : Q(x,t) = 0\}$$

is the null cone. Note that 0 is a critical value of Q and that $(0,0)$ is a critical point of Q. Hence $\mathcal{N}$ is a critical level set.

Away from the critical point it makes sense to define the pullback

$$Q^*\delta_0 = \delta_0(t^2 - |x|^2)$$

of the 1D delta function δ_0 at 0. In general, measures and distributions cannot be pulled back under maps. However, the theory of distributions gives it a meaning when the map is a submersion [HoI, GeSh, F]. $\delta(Q)$ is simply the 'Leray measure' on $Q^{-1}(0)$ or conditional measure on this level set. It is the measure supported on $Q^{-1}(0)$ with Gelfand-Leray form

$$Q^*\delta_0 = \frac{dxdt}{dQ}. \tag{3.12}$$

One may define its integral against a test function $\varphi \in C_c^\infty(\mathbb{R}^{3+1})$ by

$$\langle \delta_0(Q), \varphi \rangle = \int_{Q=0} \varphi \frac{dxdt}{dQ} = \int_{Q=0} \varphi \frac{dS}{|\nabla Q|}, \tag{3.13}$$

where dS is the Riemannian surface measure $\iota_\nu dxdt$ where $\nu = \frac{\nabla Q}{|\nabla Q|}$ is the unit normal.

In probability texts, the same formula is derived as follows: Let

$$\varphi_Q(t) := \frac{\partial}{\partial t} \int_{Q<t} \varphi dxdt.$$

Then

$$\langle \delta(Q), \varphi \rangle := \varphi_Q(0).$$

In the case of $Q = t^2 - |x|^2$, (3.13) gives

Lemma **3.** *$\delta_0(Q)$ is the following measure:*

$$\langle \delta_0(Q), \varphi \rangle = \frac{1}{2} \int_{\mathbb{R}^3} \varphi(x, |x|) \frac{dx}{|x|} + \frac{1}{2} \int_{\mathbb{R}^3} \varphi(x, -|x|) \frac{dx}{|x|}.$$

We denote the first term by $\delta_+(\varphi)$ and the second by $\delta_-(\varphi)$. Here, the first term corresponds to the upper half of the light cone where $t = |x|$ and the second term correspons to the bottom half. For the first term, we parametrize the light cone by $x \to (x, |x|)$. The Gelfand-Leray form is $\frac{dxdt}{d(t^2-|x|^2)}$ and we eliminate the variable t using $d(t^2 - |x|^2) = 2tdt - 2x \cdot dx$. The Gelfand -Leray form, is the unique form (when restricted to $Q^{-1}(0)$) satisfying $dQ \wedge \frac{dxdt}{dQ} = dxdt$ and clearly this is true $\frac{dxdt}{2tdt} = \frac{1}{2t} dx = \frac{1}{2|x|} dx$ on $Q^{-1}(0)$.

The first term above is therefore

$$\langle E^+, \varphi \rangle := \frac{1}{2} \int_{\mathbb{R}^3} \varphi(x, |x|) \frac{dx}{|x|}.$$

Similarly for the second. QED

Proposition **4.** *The following distributions on $\mathbb{R}^{3+1}$ are the forward/backward fundamental solutions:*

$$E^+(t,x) = \frac{\delta(t-r)}{4\pi r}, \quad E^-(t,x) = \frac{\delta(t+r)}{4\pi r}.$$

That is,

$$\Box E^+ = 2\pi\delta_0. \tag{3.14}$$

Hence E^+ is a fundamental solution supported in the forward light cone. Similarly for E^- in the backward light cone.

Proof. The next observation is:

LEMMA **5.** $\Box\delta_0(Q) = 0$ *on* $\mathbb{R}^{3+1}\backslash\{0\}$.

Here, as usual, $\Box = \frac{\partial^2}{\partial t^2} - \Delta$ is the d'Alembertian.

Proof. We compute by the chain rule as if δ_0 were a function. Note that

$$\Box f(Q) = \nabla \cdot \nabla f(Q) = \nabla \cdot f'(Q)\nabla Q = f''(Q)\nabla Q \cdot \nabla Q + f'(Q)\Box Q,$$

where the dot product is Lorentzian. Now in dimension 3+1, $\nabla Q \cdot \nabla Q = 4Q$ and $\Box Q = 8$. In any dimension n, if $f(t)$ homogeneous is of degre a,

$$\Box f(Q) = g(Q), \quad g(t) := 2nf'(t) + 4tf''(t) = (2n + 4(a-1))f'(t).$$

Here we use that $tf''(t) = (a-1)f'(t)$. When $a = \frac{2-n}{2}$ the right side is zero.

Now suppose $f = \delta_0$. Then all derivatives of f are supported at 0 and f is homogeneous of degree $-1 = \frac{2-4}{2}$, the right side is zero.

∎

Since E^+, E^- have disjoint supports it follows that $\Box E^\pm = 0$ on $\mathbb{R}^{3+1}\backslash\{0\}$. It follows that

$$\Box E^\pm = P(D)\delta_0$$

since all distributions supported at 0 are of this form, where $P(D)$ is a constant coefficient PDO. Now we just consider homogeneities to determine that $P(D)$ must be a constant c: Write $E^+ = \delta_+(Q)$.

- $\delta_+(Q)$ is homogeneous of degree -2.
- $\Box\delta_+(Q)$ is homogenous of degree -4.
- δ_0 is homogeneous of degree -4.
- $D^\alpha\delta_0$ is homogeneous of degree $-4 - |\alpha|$.

It follows that $\alpha = 0$ and $P(D) = c$. By using a test function $\varphi = \rho(t)$ one finds that $c = 2\pi$. (Left to reader). Hence, we proved the Proposition.

∎

3.8. **Higher dimensions.** For general $\mathbb{R}^{(n-1)+1}$, one has

$$\Box f(Q) = 4Q(f''(Q)\Box Q + 2nf'(Q)).$$

Proposition **6.** *The forward fundamental solutions on (spacetime) $\mathbb{R}^n$ are*

$$\begin{cases} E^+ = \frac{1}{2\pi^m}\, \delta_+^{(m-1)}(Q), & n = 2m+2, \\ \\ E^+ = \frac{H(t)}{2\pi^{m-\frac{1}{2}}\Gamma(\frac{3}{2}-m)} Q_+^{-m+\frac{1}{2}}, & n = 2m+1. \end{cases}$$

As in the case $n = 3+1$, the most important step is to prove:

Lemma **7.** $\Box f(Q) = 0$ *on* $\mathbb{R}^{(n-1)+1}\backslash\{0\}$ *if*

$$\begin{cases} f(t) = \delta^{(n-\frac{3}{2})}(t), & n \text{ even}, \\ \\ t_+^{-\frac{n}{2}+1}, & n \text{ odd}. \end{cases}$$

We omit the proofs, which may be found in [GeSh, HoI].

3.9. **Pizzetti formula.** On Euclidean $\mathbb{R}^n$, there exists an exact formula known as Pizzetti's formula,

$$L_r = P_r(\Delta) := \Gamma(\frac{n}{2}) \sum_{k=0}^{\infty} (\frac{r}{2})^{2k} \frac{1}{k!\Gamma(\frac{n}{2}+k)} \Delta^k \tag{3.15}$$

which is valid on real analytic functions. The intial expansion has the form,

$$L_r = I + \frac{\Delta}{2n} r^2 + \sum_{k=2}^{\infty} P_k(\Delta) r^{2k}.$$

Let $J_\nu(z)$ be the Bessel function

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\Gamma(k+1)} (\frac{z}{2})^{2k+\nu}.$$

When $0 > \lambda > \lambda_0(B)$ with $B = B_r(x)$ then the eigenfunction of eigenvalue λ can be expressed as the ball mean

$$u(x) = \frac{1}{V_r 2^{\frac{n}{2}} \Gamma(\frac{n}{2}+1)} \tau_{\frac{n}{2}(\sqrt{\lambda} r)} \int_{B_r(x)} u(y) dV(y). \tag{3.16}$$

Here, $\tau_\alpha(z) = z^{-\alpha} J_\alpha(z)$. Also $W_\alpha(z) = \Gamma(\frac{\alpha}{2}) 2^{\frac{\alpha}{2}-1} \tau_{\frac{\alpha}{2}-1}(z)$.

There is a similar related formula for the ball means operator. It follows that if u is a harmonic function on $\mathbb{R}^n$ then $L_r u = P_r(0) u = u$ and similarly $M_r u = u$. A second identity is that

$$\int_{B_r(x)} \Delta u(y) dy = r^{n-1} \frac{d}{dr} \left(r^{1-n} \int_{S_r(x)} u(y) dS(y) \right) = \omega_n r^{n-1} \frac{d}{dr} L_r u(x).$$

Here $\omega_n = |S^{n-1}|$. If $\Delta u \geq 0$, i.e. if u is subharmonic, then $L_r u(x)$ increases with r. It follows that $u(x) \leq L_r u(x)$ for all r. By integrating the inequality in r, one also has $u(x) \leq M_r u(x)$.

The identity (3.15) has many repercussions for harmonic and subharmonic functions on $\mathbb{R}^n$, and also for the wave equation on $\mathbb{R}^n$. Although L_r and $S(r)$ are different functions of Δ, their Taylor expansions agree in the first two terms when $t = \frac{r}{\sqrt{n}}$. A deeper fact is that both L_r (or M_r) and $S(t)$ are Fourier integral operators associated to the same canonical relation, namely the union of the graph of the geodesic flow G^t and of G^{-t}. This is true for small $|t|$ or r on any Riemannian manifold. Therefore there exists an elliptic pseudo-differential operator $A(t, D_t, x, D_x)$ on $\mathbb{R}_t \times \mathbb{R}^n$ so that $S(t) = AL_t$. The Hadamard parametrix method gives an explicit constrution of $S(t)$ in terms of operations on the spherical means operator on any manifold without conjugate points.

A classic book on the relations between the wave equation and spherical means is F. John [J].

4. $\mathbb{S}^n$

In this section we follow the exposition in [T1], which gives a quick calculation of the half-wave propagator e^{itA} for $A = \sqrt{-\Delta + (\frac{n-1}{2})^2}$ using no more than the Poisson integral formula for a Euclidean ball $B \subset \mathbb{R}^{n+1}$.

4.1. **Spectral theory of Δ.**

First we need some spectral theory of the Laplacian $\Delta_{\mathbb{S}^n}$. Its eigenfunctions are known as 'spherical harmonics'. Spherical harmonics are eigenfunctions of $\Delta_{\mathbb{S}^n}$. Let $P(x) = P(x_1, \ldots, x_{n+1})$ be a polynomial on $\mathbb{R}^{n+1}$, then recall

- P is a homogeneous of degree k if $P(rx) = r^k P(x)$. We denote the space of such polynomials by $\mathcal{P}_\ell$. A basis is given by the monomials
$$x^\alpha = x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}, \quad \text{where } |\alpha| = \alpha_1 + \cdots + \alpha_{n+1} = \ell.$$
- P is a harmonic if $\Delta_{\mathbb{R}^{n+1}} P(x) = 0$. We denote the space of harmonic homogeneous polynomials of degree ℓ by $\mathcal{H}_\ell$.
- The restriction to $\mathbb{S}^n$ of a harmonic homogeneous (of degree ℓ) polynomial is a spherical harmonic (of degree ℓ).

$$\Delta_{\mathbb{S}^n} P|_{\mathbb{S}^n} = -(\ell(\ell-1)+)\ell) P|_{\mathbb{S}^n} = -\ell(\ell+n-1) P|_{\mathbb{S}^n}.$$

This shows that the restriction of $P \in \mathcal{H}_\ell$ to the unit sphere (i.e., a spherical harmonic of degree ℓ by definition) is an eigenfunction of $\Delta_{\mathbb{S}^n}$ with eigenvalue $-\ell(\ell+n-1)$.

THEOREM **8.** *Let $\mathcal{H}_\ell$ denote the space of spherical harmonics of degree ℓ, then $L^2(\mathbb{S}^n) = \bigoplus_{\ell=0}^\infty \mathcal{H}_\ell$ is a direct sum of orthogonal subspaces of dimensions*

$$\dim \mathcal{H}_\ell = \binom{n+\ell-1}{n-1} + \binom{n+\ell-3}{n-1}.$$

In particular when $n = 3$ the eigenvalues are $-\ell(\ell+2) = -(\ell+1)^2 + 1$ and the multiplicity is ℓ^2.

The operator $\mathcal{N}$ whose eigenvalue on $\mathcal{H}_\ell$ is ℓ is known as the degree operator. Consider $\mathbb{S}^3$ where the eigenvalue of Δ is $-(\ell+1)^2+1$. Then $\Delta - I$ is a perfect square and we define

$$A = \sqrt{-\Delta + I}.$$

Then the eigenvalues of A are $\ell + 1$ so

$$\mathcal{N} = A - I.$$

In general,

$$A = \sqrt{-\Delta + (\frac{n-1}{2})^2}.$$

A key object in the theory of spherical harmonics is the orthogonal projector

$$\Pi_\ell \colon L^2(\mathbb{S}^n) \to \mathcal{H}_\ell$$

whose Schwartz kernel $\Pi_\ell(x,y)$ is defined by

$$\Pi_\ell f(x) = \int_{\mathbb{S}^n} \Pi_\ell(x,y) f(y)\, dS(y).$$

Here, f is any L^2 function on the sphere and dS is the standard surface measure. We note that the along diagonal $\Pi_\ell(x,x) = C_\ell$ is a constant because it is rotationally invariant and $O(n+1)$ acts transitively on $\mathbb{S}^n$. Indeed, by integrating we find

$$\Pi_\ell(x,x) = \frac{\dim \mathcal{H}_\ell}{Vol(\mathbb{S}^n)}.$$

4.2. **From Poisson integral to half-wave kernel.** We recall that the Poisson integral formula for the unit ball is:

$$u(x) = \int_{\mathbb{S}^n} \frac{1-|x|^2}{|x-\omega'|^2} f(\omega') dS(\omega').$$

Write $x = r\omega$ with $|\omega| = 1$ to get:

$$P(r,\omega,\omega') = \frac{1-r^2}{(1-2r\langle\omega,\omega'\rangle + r^2)^{\frac{n+1}{2}}}.$$

A second formula for $u(r\omega)$ is

$$u(r,\omega) = r^{A - \frac{n-1}{2}} f(\omega) = e^{-t(A-\frac{n-1}{2})} f(\omega),$$

where $A = \sqrt{-\Delta + (\frac{n-1}{2})^2}$ and where $t = \log \frac{1}{r}$. Thus, harmonic extension is written as an evolution equation with generator $A - \frac{n-1}{2}$.[4] This follows

[4]It resembles a heat equation but the generator is roughly $\sqrt{-\Delta}$ rather than $-\Delta$. It is a jump process rather than a continuous one.

from by writing the equation $\Delta_{\mathbb{R}^{n+1}} u = 0$ as an Euler equation:

$$\left(r^2 \frac{\partial^2}{\partial r^2} + nr\frac{\partial}{\partial r} - \Delta_{\mathbb{S}^n}\right) u = 0.$$

Another explanation is that on the space $\mathcal{H}_N$ of spherical harmonics of degree N on $\mathbb{S}^n$, the harmonic extension is simply the homogeneous extension as a polynomial of degree N, i.e. by r^N. But $A|_{\mathcal{H}_N} = N + \frac{n-1}{2}$. For instance, in dimension 2, $-\Delta|_{\mathcal{H}_N} = N(N+1) = (N+\frac{1}{2})^2 - \frac{1}{4}$, so $A - \frac{1}{2} = N$.

The Poisson operator kernel with $r = e^{-t}$ is given by

$$P(t, \omega, \omega') \quad = \quad C_n \frac{\sinh t e^{-(n-1)t}}{(\cosh t - \cos r(\omega, \omega'))^{\frac{n+1}{2}}}.$$

It follows that

$$e^{-tA} = C_n \sinh t (\cosh t - \cos r(\omega, \omega'))^{-\frac{n+1}{2}}.$$

Note that $U(t) = e^{itA}$, resp. $P(t) = e^{-tA}$ has the Schwartz kernel

$$\sum_{N=0}^{\infty} e^{it(N+\frac{n-1}{2})} \mathit{\Pi}_N(\omega, \omega'), \;\; \text{resp.} \; \sum_{\mathrm{N}=0}^{\infty} \mathrm{e}^{-\mathrm{t}(\mathrm{N}+\frac{n-1}{2})} \mathit{\Pi}_{\mathrm{N}}(\omega, \omega'),$$

where $\mathit{\Pi}_N : L^2(\mathbb{S}^n) \to \mathcal{H}_N$. Thus, the $P(t) = U(it)$ for $t > 0$. The Schwartz kernel of $U(t)$ is thus obtained by analytically continuing the Poisson kernel in time. For $t > 0$, $P(t + i\tau)$ is a smoothing operator, but its boundary value at $t = 0$ is the distributional kernel $U(\tau)$. We thus have,

Proposition **9.**

$$e^{itA} = \lim_{\varepsilon \to 0^+} C_n i \sin t (\cosh \varepsilon \cos t - i \sinh \varepsilon \sin t - \cos r(\omega, \omega')^{-\frac{n+1}{2}}$$

$$= \lim_{\varepsilon \to 0^+} C_n i \sinh(it - \varepsilon)(\cosh(it - \varepsilon) - \cos r(\omega, \omega')^{-\frac{n+1}{2}}.$$

If we formally put $\varepsilon = 0$ we obtain:

$$e^{itA} = C_n i \sin t (\cos t - \cos r)^{-\frac{n+1}{2}}.$$

This expression is singular when $\cos t = \cos r$ and only well-defined if we recall that it is the distribution boundary value above. We note that $r \in [0, \pi]$ and that it is singular on the cut locus $r = \pi$. Also, $\cos : [0, \pi] \to [-1, 1]$ is decreasing, so the wave kernel is singular when $t = \pm r$ if $t \in [-\pi, \pi]$.

When n is even, the expression appears to be pure imaginary but that is because we need to regularize it on the set $t = \pm r$. When n is odd, the square root is real if $\cos t \geq \cos r$ and pure imaginary if $\cos t < \cos r$.

We see that the kernels of $\cos tA$, $\frac{\sin tA}{A}$ are supported inside the light cone $|r| \leq |t|$. On the other hand, e^{itA} has no such support property (it has infinite propagation speed). On odd dimensional spheres, the kernels are supported on the distance sphere (sharp Huyghens phenomenon).

4.3. **Fundamental solution and Propagators on $\mathbb{S}^3$.** Given the above formula for e^{itA}, we can read off the formula for the propagators. We only record the formulae in dimension 3.

PROPOSITION **10.** *On $\mathbb{S}^3$ for $t > 0$,*

$$\frac{\sin t\sqrt{-(\Delta+1)}}{\sqrt{(-(\Delta+1)}}\delta_y(x) = \frac{\delta(t-r)}{4\pi \sin t}, \quad \cos t\sqrt{-(\Delta+1)}\delta_y(x) = \frac{\delta'(t-r)}{4\pi \sin r}$$

In the next section we give a direct proof of the analogous formula on hyperbolic space. By analogy with the Euclidean case, we define

$$Q(t,x,y) = \cos t - \cos r$$

on $\mathbb{S}^n \times \mathbb{R}$. The first formula in the Proposition is equivalent to the fact that $E = \delta(Q)$ is a fundamental solution on $\mathbb{S}^3$, the sum $E = E^+ + E^-$ of the forward and backward fundamental solutions. In fact, one may show directly that $(\Box - 1)\delta(Q) = \delta_0$ where $\Box = \frac{\partial^2}{\partial^2} - \Delta$. This is done in the Remark at the end of the next section on Hyperbolic space.

5. Wave kernel and Poisson kernel on Hyperbolic space $\mathbb{H}^n$

$\mathbb{H}^n$ is the symmetric space G/K where $G = SO(1,n)_0$ and $K = SO(n)$. In geodesic polar coordinates centered at any point y, the meteric has the form

$$g = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}}$$

and the Riemannian volume form is

$$d\,\mathrm{Vol} = C_n\ (\sinh r)^{n-1} dr d\omega$$

and the Laplace operator is

$$\Delta = \partial_r^2 + (n-1)\coth r \partial_r + \sinh r^{-2}\Delta_{S^{n-1}}.$$

Also, the gradient is

$$\nabla = \sum_{i,j} g^{ij}\frac{\partial}{\partial x_i} e_j = \frac{\partial|}{\partial r} + G\nabla_\omega.$$

In hyperbolic polar coordinates centered at the origin, the Laplacian is the operator

$$\triangle = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\frac{\partial^2}{\partial \theta^2}.$$

It is known that the spectrum of $\Delta + 1$ is $[0, \infty]$.

5.1. **Sine wave kernel.** Let $n = 3$ and consider the sine wave kernel $S(t)$ for $\Delta + 1$ on $\mathbb{H}^3$. As in the Euclidean case, let $E = E^+ + E^-$ be the fundamental solution where $E^\pm$ is the forward/backward fundamental solution.

PROPOSITION **11.** *Let $Q = \cosh t - \cosh r$. Then:*

- $E = \delta(Q)$*, where*

$$\langle \delta(Q), \varphi \rangle = C_3 [\int_{\mathbb{H}^3} \varphi(x, r) \frac{\sinh^2 r dr dt d d\omega}{\sinh r} + \int_{\mathbb{H}^3} \varphi(x, -r) \frac{\sinh^2 r dr dt d d\omega}{\sinh r}.]$$

- *For $t > 0$,*

$$E^+ = \frac{\sin t \sqrt{-(\Delta+1)}}{\sqrt{(-(\Delta+1)}} \delta_y(x) = \frac{\delta(t-r)}{4\pi \sinh t}, \quad \cos t \sqrt{-(\Delta+1)} \delta_y(x) = \frac{\delta'(t-r)}{4\pi \sinh r}$$

Proof. The Laplacian of $\mathbb{H}^3$ is

$$\Delta = x_3^2 \Delta_0 - x_3 \frac{\partial}{\partial x_3},$$

and in geodesic normal coordinates it is

$$\Delta = \partial_r^2 + 2 \coth r \partial_r + \sinh r^{-2} \Delta_{S^2}.$$

Since $\mathbb{H}^3$ is a symmetric space, the fundamental solutions $E^\pm$ is a function only of (t, r) (verify!) so a fundamental solution must solve

$$[\partial_r^2 + 2 \coth r \partial_r - 1] E = \delta_0.$$

Here, δ_0 is the delta-function with respect to the volume form, i.e.

$$\langle \delta_0, \psi \rangle = \psi(0) = \int_{\mathbb{H}^3 \times \mathbb{R}} \delta_0(t, r) \psi(r, t) \sinh^2 r dr d\omega dt.$$

Let $y \in \mathbb{H}^3$. The set $\mathcal{C}_y := \{(x, t) : \cosh r(x, y) - \cosh t = 0\} = \{(x, t) : r(x, y) = |t|\}$ is called the *characteristic conoid* based at y and will appear again later on. The conoid may be parametrized by points x of $\mathbb{H}^3$ with $t = \pm r(x, y)$ giving the upper and lower sheets of the conoid.

In the case of $Q = \cosh t - \cosh r$, (3.13) (with $d\,\mathrm{Vol} = C_3\ (\sinh r)^2 dr d\omega$ replacing dx) gives the stated expression for $\langle \delta(Q), \varphi \rangle$.
We did not cancel the common factors of $\sinh r$ to clarify the use of Lemma 3.13.

We denote the first term by $\delta_+(\varphi)$ and the second by $\delta_-(\varphi)$. The first term above is therefore

$$\delta_+(\varphi) = \langle E^+, \varphi \rangle := C_3 [\int_{\mathbb{H}^3} \varphi(x, r) \frac{\sinh^2 r dr dt d d\omega}{\sinh r}.$$

Similarly for the second. The main point is to show that

$$(\Box - 1) E^+ = 2\pi \delta_0. \tag{5.1}$$

Hence E^+ is a fundamental solution supported in the forward conoid. Similarly for E^- in the backward conoid.

We first show that

$$\Box \delta_0(Q) = C\delta(Q), \text{ on } \mathbb{H}^3\backslash\{\mathrm{y}\}. \tag{5.2}$$

As before, we use that

$$\Box f(Q) = f''(Q)\nabla Q \cdot \nabla Q + f'(Q)\Box Q,$$

where the dot product is Lorentzian. Noting that $(\cosh^2 t - \cosh^2 r) = Q(\cosh t + \cosh r)$, we have

$$\begin{cases} (i)\ \nabla Q \cdot \nabla Q & = & (\sinh t dt, -\sinh r dr) \cdot (\sinh t dt, -\sinh r dr) \\ & = & \sinh^2 t - \sinh^2 r = Q(\cosh t + \cosh r), \\ (ii)\ \Box Q & = & [\frac{\partial^2}{\partial t^2} - \partial_r^2 - 2\coth r \partial_r](\cosh t - \cosh r) \\ & = & \cosh t + \cosh r + 2\coth r \sinh r = \cosh t + 3\cosh r. \end{cases}$$

Hence,

$$\Box \delta(Q) = \delta''(Q)Q(\cosh t + \cosh r) + \delta'(Q)(\cosh t + 3\cosh r).$$

Since $t\delta''(t) = -2\delta'(t)$, we have

$$\delta''(Q)Q(\cosh t + \cosh r) = (\cosh t + \cosh r)(-2\delta'(Q) = -2Q\delta'(Q) - 4\cosh\delta'(Q)).$$

Next, write $(\cosh t + 3\cosh r) = Q + 4\cosh r$. Then,

$$\delta'(Q)(\cosh t + 3\cosh r) = Q\delta'(Q) + 4\cosh r\delta'(Q).$$

It follows that

$$\begin{array}{rcl} \Box \delta(Q) & = & -2Q\delta'(Q) - 4\cosh\delta'(Q)) + Q\delta'(Q) + 4\cosh r\delta'(Q) \\ & = & -Q\delta'(Q) = \delta(Q). \end{array}$$

This concludes the proof that $(\Box - 1)\delta(Q) = 0$ on $\mathbb{H}^3 - \{y\}$.

It follows that $(\Box - 1)\delta(Q)$ is a distribution supported at $\{y\}$ and is therefore a linear combinations of derivatives of δ_y, which we write as $\delta_0(t, r, \omega)$ in normal coordinates. If we Taylor expand the coefficients of $\Box$ around 0 in (t, r) it becomes the Euclidean $\Box$ and the homogeneity calculations in that case also imply that $(\Box - 1)\delta(Q) = c\delta_0$. The value of c can be calculated from a convenient test function, as in the Euclidean case.

$\Box$

Remark: In the case of $\mathbb{S}^3$, one analytically continues the equations above, replacing $\cosh r$ by $\cos r$ and so on. The main change is that second derivatives reverse signs of $\cos t, \cos r$. In this case, we get

$$\begin{cases} (i)\ \nabla Q\cdot\nabla Q &= \sin^2 t - \sin^2 r = -(\cos^2 t - \cos^2 r) = -Q(\cos t + \cos r), \\ (ii)\ \Box Q &= [\frac{\partial^2}{\partial t^2} - \partial_r^2 - 2\cot r\partial_r](\cos t - \cos r) \\ &= -\cos t - \cos r - 2\coth r \sinh r = -\cos t - 3\cosh r. \end{cases}$$

Since all signs reverse, we get $(\Box + 1)\delta(Q) = C_n\delta_0$.

5.2. **Poisson kernel and wave kernel.** We obtain the wave kernel on hyperbolic space by analytic continuation of the wave kernel of $\frac{\sin tA}{A}$ on the sphere:

Proposition **12.** *The Poisson kernel e^{-tA} on hyperbolic space with*

$$A = \sqrt{\Delta - (\frac{n-1}{2})^2}$$

is

$$U(i\tau, x, y) = \sinh\tau\,(\cosh(\tau + i0) - \cosh r)^{-\frac{n+1}{2}}. \tag{5.3}$$

The right side is by definition

$$\lim_{\varepsilon\to 0^+} -2C_n \operatorname{Im}(\cos(it - \varepsilon) - \cosh r)^{-\frac{n-1}{2}}.$$

Taylor [T1] proves this formula by analytic continuation of the standard formula for spheres of all radii R or equivalently, for spheres of all curvatures $K > 0$. To be more precise, we just consider the radial parts of the Laplacians of the various metrics. A ball of radius R is the dilate by R of the unit ball and the metrics are related by the dilation. The radial part of the Laplacian $\Delta_{\mathbb{S}^n(R)}$ of radius R is obtained by dilation. One then checks that the radial part of the Laplacian for hyperbolic space $\mathbb{H}^n$ is the analytic continuation in R of the radial part for $\mathbb{S}^n(R)$ when $K \to -1$. This implies that the fundamental solutions must also be analytic in the parameter K.

5.3. **Wave equation and spherical means.** On hyperbolic space, the spherical means operator is defined by

$$M_r f(x) = \fint_{S_r(x)} f(y)dS(y),$$

where dS is the Riemannian surface measure on the sphere $S_r(x)$ in the hyperbolic metric.

One then has the following formulae for the solution of the modified wave equation (cf. [He, GrN])

$$\begin{cases} (\Box + (\frac{n-1}{2})^2)u(x,t) = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{cases}$$

Let

$$N^r_{m,k} f(x) = (\frac{\partial}{\partial \cosh r})^m (M_r f(x) \sinh^k(t)).$$

Proposition **13.** *When $n \geq 3$ is odd,*

$$u(x,t) = C_n \left(\frac{\partial}{\partial t} N^t_{\frac{n-3}{2}, n-2} \varphi(x) + N^t_{\frac{n-3}{2}, n-2} \psi(x) \right),$$

where $C_n = \frac{1}{(n-2)!!}$.

When n is even,

$$\frac{1}{2} \int_0^t \frac{u(x,s) + u(x,-s)}{\sqrt{\cosh s - \cosh r}} ds = C_n N^t_{\frac{n-2}{2}, n-2} \varphi(x).$$

6. Hadamard parametrix on a general Riemannian manifold

The wave group of a Riemannian manifold is the unitary group $U(t) = e^{it\sqrt{-\Delta}}$. As above, we also write $E(t) = \cos t\sqrt{-\Delta}$ and $S(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$. We now review the construction of a Hadamard parametrix for $E(t)$ and $S(t)$. There is a similar parametrix for $U(t)$ but it is somewhat more complicated because $U(t)$ is not a function of Δ.

The basic ansatz is that

$$S(t,x,y) = \int_0^\infty e^{i\theta(r^2(x,y) - t^2)} \sum_{k=0}^\infty W_k(x,y) \theta^{\frac{d-3}{2} - k} d\theta \qquad (t < \text{inj}(M,g)) \tag{6.1}$$

where $W_0(x,y) = \Theta^{-\frac{1}{2}}(x,y)$. Here as above, $\Theta(x,y)$ is the volume density in normal coordinates. The higher coefficients are determined by transport equations, and θ^r is regularized at 0 (see below). This formula is only valid for times $t < \text{inj}(M,g)$ but using the group property of $U(t)$ it determines the wave kernel for all times. It shows that for fixed (x,t) the kernel $S(t,x,y)$ is singular along the distance sphere $S_t(x)$ of radius t centered at x, with singularities propagating along geodesics. It only represents the singularity and in the analytic case only converges in a neighborhood of the characteristic conoid.

We recall that the even part $E(t)$ of the wave kernel, $\cos t\sqrt{\Delta}$ which solves the initial value problem

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta) u = 0, \\ u|_{t=0} = f, \quad \frac{\partial}{\partial t} u|_{t=0} = 0. \end{cases} \tag{6.2}$$

Similarly, the odd part $S(t)$ of the wave kernel, $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$, is the operator solving

$$\begin{cases} (\frac{\partial}{\partial t}^2 - \Delta)u = 0, \\ u|_{t=0} = 0, \quad \frac{\partial}{\partial t}u|_{t=0} = g. \end{cases} \tag{6.3}$$

These kernels only really involve Δ and may be constructed by the Hadamard-Riesz parametrix method. As above they have the form

$$\int_0^\infty e^{i\theta(r^2-t^2)} \sum_{j=0}^\infty W_j(x,y)\theta_{reg}^{\frac{n-1}{2}-j} d\theta \quad \text{modulu } C^\infty \text{ functions}, \tag{6.4}$$

where W_j are the Hadamard-Riesz coefficients determined inductively by the transport equations

$$\begin{cases} \frac{\Theta'}{2\Theta} W_0 + \frac{\partial W_0}{\partial r} = 0, \\ 4ir(x,y)\{(\frac{k+1}{r(x,y)} + \frac{\Theta'}{2\Theta})W_{k+1} + \frac{\partial W_{k+1}}{\partial r}\} = \Delta_y W_k. \end{cases} \tag{6.5}$$

The solutions are given by:

$$\begin{cases} W_0(x,y) = \Theta^{-\frac{1}{2}}(x,y), \\ W_{j+1}(x,y) = \Theta^{-\frac{1}{2}}(x,y) \int_0^1 s^k \Theta(x,x_s)^{\frac{1}{2}} \Delta_2 W_j(x,x_s)\, ds, \end{cases} \tag{6.6}$$

where x_s is the geodesic from x to y parametrized proportionately to arc-length and where Δ_2 operates in the second variable.

A well-known formula for homogeneous distributions on $\mathbb{R}$ is:

$$\int_0^\infty e^{i\theta\sigma}\theta_+^\lambda d\lambda = ie^{i\lambda\pi/2}\Gamma(\lambda+1)(\sigma+i0)^{-\lambda-1}.$$

One has,

$$\int_0^\infty e^{i\theta(r^2-t^2)}\theta_+^{\frac{d-3}{2}-j} d\theta = ie^{i(\frac{d-1}{2}-j)\pi/2}\Gamma(\frac{d-3}{2}-j+1)(r^2-t^2+i0)^{j-\frac{d-3}{2}-2}. \tag{6.7}$$

Here there is apparently trouble when d is odd since $\Gamma(\frac{d-3}{2}-j+1)$ has poles at the negative integers.

One then uses

$$\Gamma(\alpha+1-k) = (-1)^{k+1}(-1)^{[\alpha]}\frac{\Gamma(\alpha+1-[\alpha])\Gamma([\alpha]+1-\alpha)}{\alpha+1}\frac{1}{\alpha-[\alpha]}\frac{1}{\Gamma(k-\alpha)}.$$

We note that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Here and above t^{-n} is the distribution defined by $t^{-n} = Re(t+i0)^{-n}$ (see [Be], [G.Sh., p.52,60].) We recall that $(t+i0)^{-n} = e^{-i\pi\frac{n}{2}}\frac{1}{\Gamma(n)}\int_0^\infty e^{itx}x^{n-1}dx$.

We also need that $(x+i0)^\lambda$ is entire and

$$(x+i0)^\lambda = \begin{cases} e^{i\pi\lambda}|x|^\lambda & x<0 \\ x_+^\lambda & x>0. \end{cases}$$

The imaginary part cancels the singularity of $\frac{1}{\alpha-[\alpha]}$ as $\alpha \to \frac{d-3}{2}$ when $d = 2m+1$. There is no singularity in even dimensions. In odd dimensions the real part is $\cos \pi\lambda x_-^\lambda + x_+^\lambda$ and we always seem to have a pole in each term!

But in any dimension, the imaginary part is well-defined and we have

$$\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}(x,y) = C_o \mathrm{sgn}\,(t) \sum_{j=0}^\infty (-1)^j w_j(x,y) \frac{(r^2-t^2)_-^{j-\frac{d-3}{2}-1}}{4^j \Gamma(j-\frac{d-3}{2})} \mod C^\infty. \tag{6.8}$$

By taking the time derivative we also have,

$$\cos t\sqrt{\Delta}(x,y) = C_o|t| \sum_{j=0}^\infty (-1)^j w_j(x,y) \frac{(r^2-t^2)_-^{j-\frac{d-3}{2}-2}}{4^j \Gamma(j-\frac{d-3}{2}-1)} \mod C^\infty. \tag{6.9}$$

where C_o is a universal constant and where $W_j = \tilde{C}_o e^{-ij\frac{\pi}{2}} 4^{-j} w_j(x,y),$.

6.1. **Proof of the Hadamard-Riesz parametrix.** We try to construct the kernel as a homogeneous oscillatory integral

$$E(t,x,y) = \int_0^\infty e^{i\theta(r^2-t^2)} A(t,x,y,\theta) d\theta, \tag{6.10}$$

where A is a polyhomogeneous symbol in θ,

$$A(t,x,y,\theta) \sim \sum_{j=0}^\infty W_j(t,x,y) \theta_+^{\frac{n-1}{2}-j} d\theta \mod C^\infty \tag{6.11}$$

Here, θ_+^s is the homogeneous distribution with singularity at $\theta = 0$ regularized as in [HoI, Chapter 3.2] or in [Be]. The leading term $\theta_+^{\frac{n-1}{2}}$ of the amplitude has the correct power for $\cos t\sqrt{\Delta}$. It should be $\theta_+^{\frac{n-3}{2}}$ for $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$.

Applying the Fourier transform formula for $\mathcal{F}\theta_+^s$ [HoI, p. 167] gives,

$$\int_0^\infty e^{i\theta(r^2-t^2)} \theta_+^{\frac{n-3}{2}-j} d\theta = i e^{i(\frac{n-1}{2}-j)\pi/2} \Gamma(\frac{n-3}{2} - j + 1)(r^2 - t^2 + i0)^{j-\frac{n-3}{2}-2}.$$

When n is odd, $\Gamma(\frac{n-3}{2} - j + 1)$ has poles at the negative integers. Thus, this parametrix does not quite work on odd dimensional spaces (= even dimensional spacetimes). But the correct formulae may be obtained by analytic continuation (cf [**?**, Be]). Riesz defined a holomorphic family of Riesz kernels $(t^2-r^2)_+^\alpha$ and used analytic continuation to define the value when α is a negative integer. He only studied the imaginary part, where there is no pole. Hadamard used a different regularization procedure (discussed below). In the end,

$$\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}(x,y) = C_o \mathrm{sgn(t)} \sum_{\mathrm{j}=0}^{\infty} (-1)^{\mathrm{j}} \mathrm{w_j(x,y)} \frac{\mathrm{(r^2-t^2)_-^{j-\frac{n-3}{2}-1}}}{\mathrm{4^j \Gamma(j-\frac{n-3}{2})}} \mod \mathrm{C}^\infty \tag{6.12}$$

Here, $\mathrm{sgn(x)} = \frac{\mathrm{x}}{|\mathrm{x}|}$ for $x \neq 0$ and $= 0$ for $x = 0$.

By taking the time derivative we also have,

$$\cos t\sqrt{\Delta}(x,y) = C_o |t| \sum_{j=0}^{\infty} (-1)^j w_j(x,y) \frac{(r^2-t^2)_-^{j-\frac{n-3}{2}-2}}{4^j \Gamma(j-\frac{n-3}{2}-1)} \mod C^\infty \tag{6.13}$$

where C_o is a universal constant and where the Hadamard-Riesz coefficients $w_j(x,y)$ solve certain transport equations.

The coefficients W_j are determined inductively by the transport equations

$$\begin{aligned} &\tfrac{\Theta'}{2\Theta} W_0 + \tfrac{\partial W_0}{\partial r} = 0 \\ &4ir(x,y)\{(\tfrac{k+1}{r(x,y)} + \tfrac{\Theta'}{2\Theta}) W_{k+1} + \tfrac{\partial W_{k+1}}{\partial r}\} = \Delta_y W_k. \end{aligned} \tag{6.14}$$

The solutions are given by (6.6), i.e.

$$\begin{aligned} &W_0(x,y) = \Theta^{-\frac{1}{2}}(x,y) \\ &W_{j+1}(x,y) = \Theta^{-\frac{1}{2}}(x,y) \int_0^1 s^k \Theta(x,x_s)^{\frac{1}{2}} \Delta_2 W_j(x,x_s) ds \end{aligned} \tag{6.15}$$

where x_s is the geodesic from x to y parametrized proportionately to arc-length and where Δ_2 operates in the second variable.

For $U(t) = \exp it\sqrt{\Delta}$ one may apply $\sqrt{\Delta}$ to the parametrix for $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$, resulting in an oscillatory with the same phase and a different amplitude. One may then use Duhamel's formula to construct the exact solution as a Volterra series,

$$U(t,x,y) = U_N(t,x,y) + \int_0^t U_N(t-s)(\partial_t^2 - \Delta) U_N(t-s) ds + \cdots,$$

where U_N is an approximate solution obtained by using N terms of a series above.

6.2. **Sketch of proof of Hadamard's construction.** Let $\Theta = \sqrt{\det(g_{jk})}$ be the volume density in normal coordinates based at y, $dV = \Theta(y,x)dx$. That is,

$$\Theta(x,y) = \left| \det D_{\exp_x^{-1}(y)} \exp_x \right|.$$

Fix $x \in M$ and endow $B_\varepsilon(x)$ with geodesic polar coordinates r, θ. That is, use the chart $\exp_x^{-1} : B_r(x) \to B^*_{x,r}M$ combined with polar coordinates on $T^*_x M$. Then $g^{11} = 1, g^{1j} = 0$ for $j = 2, \ldots, n$. Also, $dV = \Theta(x,y)dy =$

$\Theta(x,r,\theta)r^{n-1}drd\theta$. So the volume density J relative to Lebesgue measure $drd\theta$ in polar coordinates is given by $J = r^{n-1}\Theta$.

In these coordinates,

$$\Delta = \frac{1}{J}\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}\left(Jg^{jk}\frac{\partial}{\partial x_k}.\right) = \frac{\partial^2}{\partial r^2} + \frac{J'}{J}\frac{\partial}{\partial r} + L,$$

where L involves no $\frac{\partial}{\partial r}$ derivatives. Equivalently,

$$\Delta = \frac{\partial^2}{\partial r^2} + (\frac{\Theta'}{\Theta} + \frac{n-1}{r})\frac{\partial}{\partial r} + L,$$

The first step in the parametrix construction is to find the phase function. Hadamard chooses to use Γ. In the Lortenzian metric, Γ satisfies

$$\nabla\Gamma\cdot\nabla\Gamma = 4\Gamma. \tag{6.16}$$

This is not the standard Eikonal equation $\sigma_{\Box}(d\varphi) = 0$ of geometric optics, but rather has the form

$$\sigma_{\Box}(d\Gamma) = 4\Gamma.$$

But Γ is a good phase, since the Lagrangian submanifold

$$\{(t, d_t\Gamma, x, d_x\Gamma, y, -d_y\Gamma)\}$$

is the graph of the bichafacteristic flow. This is because the $d_x r(x,y)$ is the unit vector pointing along the geodesic joining x to y and $d_y r(x,y)$ is the unit vector pointing along the geodesic pointing from y to x.

To proceed, we introduce the simplifying notation

$$M = \Box\Gamma = -4 - 2r\frac{(n-1)}{r} - 2r\frac{\Theta_r}{\Theta} = 2m + 2r\frac{\Theta_r}{\Theta}$$

where $m = n+1$. We then have,

$$\begin{aligned}\Box\left[f(\Gamma)U_j\right] &= \Box\left[f(\Gamma)\right]U_j + 2\nabla\left[f(\Gamma)\right]\nabla U_j + f(\Gamma)\Box U_j\\ &= \big(f''(\Gamma)\nabla(\Gamma)\cdot\nabla(\Gamma) + f'(\Gamma)\Box(\Gamma)\big)U_j + 2f'(\Gamma)\nabla\Gamma\cdot\nabla U_j + f(\Gamma)\Box U_j.\end{aligned}$$

In addition to (6.16), we further have

$$\begin{cases}\Box\Gamma = 4 + \dfrac{J_r}{J}2r\\ \nabla\Gamma\cdot\nabla = \nabla(t^2-r^2)\cdot\nabla = 2(t\dfrac{\partial}{\partial t} + r\dfrac{\partial}{\partial r}) = 2s\dfrac{d}{ds},\end{cases}$$

where we recall that that we are using the Lorentz metric of signature $+-{}--$. Here $s^2 = \Gamma$, and the notation $s\frac{d}{ds}$ refers to differentiation along a spacetime geodesic.

We then have

$$\Box\left[f(\Gamma)U_j\right] = \left(f''(\Gamma)(4\Gamma) + f'(\Gamma)(4 + \frac{J_r}{J}2r)\right)U_j + 2f'(\Gamma)(-2s\frac{d}{ds}U_j) + f(\Gamma)\Box U_j.$$

We now apply this equation with $f = x^{\frac{2-m}{2}+j}$ (and later to $f = \log x$), in which case

$$f' = (\frac{2-m}{2} + j)x^{\frac{2-m}{2}+j-1}, \quad f'' = (\frac{2-m}{2} + j)(\frac{2-m}{2} + j - 1)x^{\frac{2-m}{2}+j-2}.$$

We then attempt to solve

$$\Box \left(\Gamma^{\frac{2-m}{2}} \sum_{j=0}^{\infty} U_j \Gamma^j \right) = 0 \tag{6.17}$$

away from the characteristic conoid by setting the coefficient of each power $\Gamma^{\frac{2-m}{2}+j-1}$ of Γ equal to zero. The resulting 'transport equation' is

$$0 = \left\{ -4 \left((\frac{2-m}{2} + j)(\frac{2-m}{2} + j - 1) \ + (\frac{2-m}{2} + j)(-4 - \frac{J_r}{J} 2r) \right) + 2(\frac{2-m}{2} + j)(-2s\frac{d}{ds}) \right\} U_j + \Box U_{j-1}.$$

They are impossible to solve for all j when m is even because the common factor $(\frac{2-m}{2} + j)$ vanishes when $j = \frac{m-2}{2}$. We thus first assume that m is odd so that it is non-zero for all j We then recursively solve Hadamard's transport equations in even space dimensions,

$$4s\frac{dU_k}{ds} + (M - 2m + 2r\frac{J_r}{J})U_k = -\Box U_{k-1}.$$

When $k = 0$ we get

$$2s\frac{dU_0}{ds} + 2s\frac{\Theta_s}{\Theta} = 0,$$

which is solved by

$$U_0 = \Theta^{-\frac{1}{2}}.$$

The solution of the ℓth transport equation is then,

$$U_\ell = -\frac{U_0}{4\ell s^{m+\ell}} \int_0^s U_0^{-1} s^{\ell+m-1} \Box U_{\ell-1} ds.$$

Hence we have a formal solution with the singularity of the Green's function in the elliptic case, and by comparison with the Euclidean case we see that it solves $\Box E = \delta_0$.

We now consider the necessary modifications in the case of even dimensional spacetimes. In this case, $\Gamma^{\frac{2-m}{2}}\Gamma^j$ is always an integer power. If we could solve the transport equation for $j = \frac{m-2}{2}$, the resulting term would be regular with power Γ^0. The problem is that Γ^0 should actually be a term with a logarithmic singularity $\log \Gamma$.

Thus the parametrix (6.17) is inadequate in even spacetime dimensions. Hadamard therefore introduced a logarithmic term $V \log(\Gamma)$. By a similar calculation to the above,

$$\Box \left[(\log \Gamma) V \right] = \left(-\Gamma^{-2}(4\Gamma) + \Gamma^{-1}(-4 - \tfrac{J_r}{J} 2r) \right) V + 2\Gamma^{-1}(-2s\tfrac{d}{ds} V) + \log \Gamma \Box V.$$

Due to (6.16), all terms except the logarithmic term have the same singularity Γ^{-1}. On the other hand, the only way to eliminate the logarithmic term is to insist that $\Box V = 0$. We further assume that

$$V = \sum_{j=0}^{\infty} V_j \Gamma^j.$$

We then return to the unsolvable transport equations for U_j for $j \geq \frac{m-2}{2}$, which now acquires the new V_0 term to become:

$$0 = \Big\{ -4\left((\frac{2-m}{2} + j)(\frac{2-m}{2} + j - 1) \; + (\frac{2-m}{2} + j)(-4 - \frac{J_r}{J} 2r) \right) \\ + 2(\frac{2-m}{2} + j)(-2s\frac{d}{ds}) \Big\} U_j + \Box U_{j-1} \\ + \Gamma^{-1}\left(4 + (-4 - \frac{J_r}{J} 2r) + 2\frac{d}{ds}\right) V_0.$$

When $j = \frac{m-2}{2}$, everything cancels in the Γ^{-1} term except $\Box U_{m-1}$. Hence, we drop the U_j for $j \geq \frac{m-2}{2}$ and assume the non logarithmic part is just the finite sum $\sum_{j=0}^{m-1} U_j \Gamma^j$. But adding in the V_0 term we get the transport equation,

$$-4s\frac{dV_0}{ds} - 2r\frac{J_r}{J} V_0 = -\Box U_{m-1}.$$

Here, U_{m-1} is known and we solve for V_0 to get,

$$V_0 = -\frac{U_0}{4s^m} \int_0^s U_0^{-1} s^{m-1} \Box U_{m-1} ds.$$

The condition $\Box V = 0$ imposed above then determines the rest of the coefficients V_j,

$$V_\ell = -\frac{U_0}{4\ell s^{m+\ell}} \int_0^s U_0^{-1} s^{\ell+m-1} \Box V_{\ell-1} \, ds.$$

We now have two equations: the original $\Box(U\Gamma^{\frac{2-m}{2}} U + V \log \Gamma) = 0$ and the new $\Box V = 0$. By solving the transport equations for $U_0, \ldots, U_{m-1}, V_0, V_j (j \geq 1)$ we obtain a solution of an inhomogeneous equation of the form,

$$\Box(U\Gamma^{\frac{2-m}{2}} + V \log \Gamma) = \sum_{j=0} w_j \Gamma^j,$$

where the right side is regular. To complete the construction, we add a new term of the form $W = \sum_{\ell=1}^{\infty} W_\ell (r^2 - t^2)^\ell$ in order to ensure that

$$\Box\Big(\sum_{j=0}^{m-1} U_j (r^2 - t^2)^{-m+j} + V \log(r^2 - t^2) + W \Big) = 0$$

away from the characteristic conoid. It then suffices to find W_j so that

$$\Box \sum_{j=1}^{\infty} W_j \Gamma^j = \sum_{j=0} w_j \Gamma^j.$$

This leads to more transport equations which are always solvable (by the Cauchy-Kowalevskaya theorem). This concludes the sketch of the construction of the Hadamard parametrix.

7. Convergence in the real analytic case

The above parametrix construction was formal. However, when the metric is real analytic, Hadamard proved that the formal series converges for $|t|$ and $|\Gamma|$ sufficiently small. The convergence proof based on the method of majorants.

Theorem **14.** [H] *(see also* [Gar]*) Assume that* (M, g) *is real analytic. Then there exists* $K > 0$ *so that the Hadamard parametrix converges for any* (t, y) *such that* $t \neq 0$, $r(x, y) < \varepsilon = \mathrm{inj}(x_0)$ *and*

$$|t^2 - r^2| \leq \frac{\left((1 - \frac{\|y\|}{\varepsilon}\right)^2}{\left(1 + \frac{m_1}{\varepsilon} + \frac{m_1^2}{\varepsilon^2}\right) K}, \qquad (m_1 = \frac{m-2}{2}). \tag{7.1}$$

It follows that the Hadamard fundamental solutions holomorphically extend to a neighborhood of $\mathcal{C}_{\mathbb{C}}$ as branched meromorphic functions with $\mathcal{C}_{\mathbb{C}}$ as branch locus. To obtain single valued distributions, one then needs to restrict the kernels to regions where a unique branch can be defined.

8. Hadamard parametrix on a quotient manifold without conjugate points

The wave kernels $\cos(t\sqrt{\Delta})(x, y)$ and $\frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}}$ can be constructed globally in time on a Riemannian manifold (M, g) without conjugate points, such as a non-positively curved manifold. We refer to §**??** for the geometric notions and notations. We denote the universal Riemannian cover of (M, g) by $(\tilde{M}, \tilde{g})$. By definition, there is a unique geodesic (unit speed) between any two points (x, y) of $\tilde{M}$ and the geodesic distance function (squared) is a global smooth function $r^2(x, y)$.

On $\tilde{M}$, the wave operator $\tilde{E}$ can be globally constructed (modulo $C^\infty(\mathbb{R} \times M \times M)$) by the Hadamard-Riesz parametrix method ([Be]). That is, the wave kernel $\tilde{E}(t, x, y) = \cos(t\sqrt{-\Delta})$ is given modulo smooth kernels by the Hadamard parametrix,

$$\tilde{E}(t, x, y) \equiv \int_0^\infty e^{i\theta(r^2 - t^2)} \sum_{j=0}^{\infty} W_j(x, y) \theta^{\frac{n-1}{2} - j} \chi(\theta)\, d\theta \tag{8.1}$$

where χ is (as above) a smooth cutoff near 0 and where the W_j are given recursively by the formulae in (6.15). Note that r^2 and $\Theta^{-\frac{1}{2}}$ are smooth for a metric WCP.

The wave kernel $E(t, x, y)$ on M is obtained by projecting this kernel from $\tilde{M}$, i.e. by summing over the deck transformation group:

$$E(t, x, y) = \sum_{\gamma \in \Gamma} \tilde{E}(t, x, \gamma \cdot y) \equiv \sum_{\gamma \in \Gamma} \int_0^\infty e^{i\theta(r^2(x,\gamma y) - t^2)} \sum_{j=0}^\infty W_j(x, \gamma y) \theta^{\frac{n-1}{2} - j} \chi(\theta) \, d\theta.$$

9. Dimension 3

In dimension 3, the Hadamard-Riesz parametrix is relatively elementary, and its relation to spherical means is simpler. The Hadamard parametrix is constructed for the cosine propagator in dimension in [Don] and we follow its exposition in the section. The sine-propagator $S(t)$ is one degree smoother but the calculations are equivalent since $S'(t) = C(t)$.

Let $C(t, x, q)$ be the cosine propagator. For each k we construct a parametrix in the first sense so that

$$\begin{cases} \Box C^k \in C^{k-1}(\mathbb{R} \times M), \\ \\ C^k(0, x, q) - \delta_q \in C^{k-1}(M). \end{cases}$$

We then do the same for the sine propagator $S(t, x, q)$.

We follow [Don] and start by relating the notation there to the one in the Hadamard parametrix method. Donnelly writes $g = \det(g_{ij})$ in normal coordinates based at x. We denote the same quantity by $\Theta(x, y)$ so that $dV(y) = \Theta(x, y) dy = \sqrt{g(y)} dy$. We then change to geodesic polar coordinates (ρ, ω) so that $dy = \rho^{n-1} d\rho d\omega$. We write $\Theta(x, \rho, \omega)$ for $\Theta(x, \cdot)$ in polar coordinates. Thus, $dV(y) = \Theta \rho^{n-1} d\rho d\omega$.

Let $\Theta = \sqrt{\det(g_{jk})}$ be the volume density in normal coordinates based at y, $dV = \Theta(y, x) dx$. That is,

$$\Theta(x, y) = \left| \det D_{\exp_x^{-1}(y)} \exp_x \right|.$$

Fix $x \in M$ and endow $B_\varepsilon(x)$ with geodesic polar coordinates r, θ. That is, use the chart $\exp_x^{-1} : B_r(x) \to B^*_{x,r} M$ combined with polar coordinates on $T^*_x M$. Then $g^{11} = 1, g^{1j} = 0$ for $j = 2, \ldots, n$. Also, $dV = \Theta(x, y) dy = \Theta(x, r, \theta) r^{n-1} dr d\theta$. So the volume density J relative to Lebesgue measure $dr d\theta$ in polar coordinates is given by $J = r^{n-1} \Theta$.

In these coordinates,

$$\Delta = \frac{1}{J} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(J g^{jk} \frac{\partial}{\partial x_k} . \right) = \frac{\partial^2}{\partial r^2} + \frac{J'}{J} \frac{\partial}{\partial r} + L, \tag{9.1}$$

where L involves no $\frac{\partial}{\partial r}$ derivatives. Equivalently,

$$\Delta = \frac{\partial^2}{\partial r^2} + (\frac{\Theta'}{\Theta} + \frac{n-1}{r}) \frac{\partial}{\partial r} + L,$$

PROPOSITION **15.** *For general metrics on Riemannian manifolds of dimension 3, for any k and for $|t| < \text{inj}(\mathrm{M}, \mathrm{g})$, there exists a kth order parametrix of the form,*

$$\begin{aligned} C^k(t,x,q) &= a_{-2}(x,q)\delta'(\rho-t) + a_{-1}(x,q)\delta(\rho-t) + a_0(x,q)H^0(\rho-t) \\ & \quad + \cdots + a_k(x,q)H^k(\rho-t). \end{aligned} \tag{9.2}$$

Here, $H^k(s) = \frac{1}{j!}s_+^j$ *and the* a_j *are constructed so that as* $\rho \to 0$*,*

$$\begin{cases} a_0 = O(1), \\ a_0 + \rho a_1 = O(\rho), \\ \cdots \\ a_0 + \rho a_1 + \cdots + a_k\rho^k = O(\rho^k). \end{cases}$$

Remark: On $\mathbb{R}^3$ the sine propagator was $\frac{\delta(\rho-t)}{\rho}$ and its t-derivative is the cosine propagator $\frac{\delta'(\rho-t)}{\rho}$. The factor of $\frac{1}{\rho}$ will be absorbed into the amplitudes a_j.

Proof. Suppose that $f(\rho)$ is a function depending only on ρ. In view of (9.1), we have for $n = 3$,

$$\Delta(f(\rho)\alpha) = \left(f''(\rho) + \frac{J_\rho}{J}f'(\rho)\right)\alpha + 2f'(\rho)\frac{\partial\alpha}{\partial\rho} + f(\rho)\Delta\alpha. \tag{9.3}$$

Define the transport operator

$$\mathcal{R} := J^{-\frac{1}{2}}\frac{\partial}{\partial\rho}J^{\frac{1}{2}}.$$

A straightforward caclulation based on (9.3) gives

$$\begin{aligned} (\tfrac{\partial^2}{\partial t^2} - \Delta)C^k(t,x,q) &= (-2\mathcal{R}a_{-2})\,\delta''(\rho-t) \\ &+ (-2\mathcal{R}a_{-1} - \Delta a_{-2})\,\delta'(\rho-t) \\ &\quad + \cdots + (-\mathcal{R}a_{j-1})\,H^{j-1}(\rho-t) \\ &+ \cdots(-\Delta a_k)H^k(\rho-t). \end{aligned}$$

Note that the coefficient of δ''' cancels due to the argument $\rho - t$. To make the coefficient of δ'' zero, we need to solve

$$\mathcal{R}\,a_{-2} = 0 \iff \frac{\partial}{\partial\rho}J^{\frac{1}{2}}a_{-2} = 0 \iff J^{\frac{1}{2}}a_{-2} = \text{Const}.$$

It follows that

$$a_{-2} = C\frac{\Theta^{\frac{1}{2}}}{\rho}.$$

To obtain the desired initial condition, one makes the constant the same as in the Euclidean case, i.e. $C = 4\pi$. Thus, a_{-2} has a similar form to the Euclidean case where $\Theta \equiv 1$.

The coefficient of $\delta(\rho - t)$ equals

$$-2\mathcal{R}a_{-1} + \Delta a_{-2},$$

and to make it zero we need to define a_{-1} so that

$$\mathcal{R}a_{-1} = -\Delta a_{-2}.$$

Then,

$$a_{-1} = CJ^{-\frac{1}{2}} - \frac{1}{2}J^{-\frac{1}{2}}\int_0^\rho J^{\frac{1}{2}}\Delta J^{-\frac{1}{2}}ds.$$

One must have $C = 0$ if a_{-1} is smooth. Adjusting for constants,

$$a_{-1} = \frac{1}{8\pi}\frac{1}{J^{\frac{1}{2}}}\int_0^\rho J^{\frac{1}{2}}\Delta J^{-\frac{1}{2}}ds.$$

This may be simplified using the special identity in dimension 3,

$$J^{\frac{1}{2}}\Delta J^{-\frac{1}{2}} = \Theta^{-1}\Delta\Theta^{-\frac{1}{2}},$$

to give,

$$a_{-1} = J^{-\frac{1}{2}}\int_0^\rho \Theta^{\frac{1}{2}}\Delta\Theta^{-\frac{1}{2}}ds.$$

In general one has transport equations

$$\mathcal{R}a_j = -\tfrac{1}{2}\Delta a_{j-1},$$

which are solved iteratively as in the case $j = -1$. If we solve the first k transport equations we get

$$\Box C^k = -(\Delta a_k)H^k \in C^{k-1}(\mathbb{R}\times M).$$

This uniquely determines the a_k and implies that

$$C^k(0, x, q) - \delta_q \in C^{k-1}(M).$$

As in [Don, Theorem 2.4], we claim:

THEOREM **16.** $C(t, x, q) - C^k(t, x, q) \in C^{k-1}(\mathbb{R}\times M)$.

Proof. One has,

$$\begin{cases} \Box(C - C^k) \in C^{k-1}(\mathbb{R}\times M), \\ C(0, x, q) - C^k(0, x, q) \in C^{k-1}(\mathbb{R}\times M), \\ \frac{\partial}{\partial t}(C(t, x, q) - C^k(t, x, q))|_{t=0} = 0. \end{cases}$$

The last equation holds if we extend the solution to be even in t from $t > 0$. Further, by the recursive procedure,

$$C^{k+1} - C^k \in C^k(\mathbb{R} \times M).$$

To prove that $C - C^k \in C^{k-1}$ we use a form of Duhamel's principal for second order equations and the fact that the wave propagator is unitary on Sobolev spaces. It is sufficient to cite [HoIII, Lemma 15.5.4]. Let $E(t) = \int_M (|v(x,t)|^2 + |\nabla v(x,t)|^2) dV$.

Lemma **17.** *Let $v \in C^\infty([0,T] \times M)$ be the solution of the inhomogeneous initial value problem on $[0,T] \times M$:*

$$\begin{cases} \Box v = h, \\ \\ v(x,0) = v_t(x,0) = 0. \end{cases}$$

Then,

$$E(t) \leq C(\int_0^t ||h(s,x)|| ds.$$

First,

$$\langle \Delta \dot{v}, v \rangle_{L^2} = \langle \dot{v}, \Delta v \rangle_{L^2}.$$

Hence,

$$2\langle h, \dot{v} \rangle_{L^2(M)} = \frac{\partial}{\partial t} E(t).$$

Let

$$M^2 := \sup_{0 \leq s \leq t} ||\dot{v}(s\cdot)||^2_{L^2} + ||\nabla v(s,\cdot)||^2_{L^2}/C_1 = \sup_{0 \leq s \leq t.} E(s).$$

Therefore,

$$E(t) \leq 2M \int_0^t ||h(s,\cdot)|| ds.$$

Thus,

$$M^2 \leq 2M \int_0^t ||h(s,\cdot)|| ds,$$

proving the Lemma.

□

One can iterate the argument to obtain estimates on higher derivatives of v in terms of higher derivatives of h. The full estimate is [HoIII, (17.5.11)]:

$$\sum_{j=0}^{k+1} ||D_t^{K+1-j} v(t,\cdot)||_{(j)} \leq C_k \left(\int_0^t ||D_s^k h(s,\cdot)|| ds + \sum_{j=0}^{k-1} ||D_t^{k-1-j} h(t,,\cdot)||_{(j)} \right).$$

It is proved inductively using $\ddot{v} = h - \Delta v$ which equals 0 when $t = 0$.

This concludes the proof of Proposition 15.

□

9.1. **Sine kernel.** The same parametrix construction works for the sine propagator $S(t)$. In Euclidean space it equals $\frac{\delta(\rho-t)}{\rho}$. The factor $\frac{1}{\rho} = a_{-2}$ and $\Theta = 1$.

Proposition **18.** *For general metrics on Riemannian manifolds of dimension 3, there exists a parametrix of the form,*

$$S^k(t,x,q) = a_{-2}(x,q)\delta(\rho-t)+a_{-1}(x,q)H^0(\rho-t)+a_0(x,q)H^1(\rho-t)+\cdots+a_k(x,q)H^{k-1}(\rho-t). \tag{9.4}$$

Here, $H^k(s) = \frac{1}{j!}s_+^j$.

One can go through the same steps or simply observe that $S(t) = \int_0^t C(s)ds$. Since the a_j are independent of t one simply integrates up the $H^k(s)$. Note that $\frac{d}{ds}H^k(s) = H^{k-1}(s)$.

10. Spherical means and ball means as Fourier integral operators

The spherical means operator is an averaging operator over the Riemannian sphere $S_r(x)$ of radius r centered at x and the ball means operator averages over the ball $B_r(x)$ centered at x of radius r. There are two natural definitions according to the measure one puts on the spheres, resp. balls. The tangential spherical means operator is defined by

$$L_r^0 f(x) = \int_{S_x^*M} f(\exp_x r\xi)d\mu_x(\xi), \tag{10.1}$$

where $d\mu_x$ is the surface measure on S_x^*M induced by the metric g. The tangential ball means is defined by

$$M_r^0 f(x) = \int_{rB_x^*M} f(\exp_x r\xi)d\mathcal{L}(r\xi),$$

where $d\mathcal{L}$ is Lebesgue measure in the tangent space In both cases one uses Euclidean surface area, resp. volume, forms on the tangent space at x.

The (ordinary) spherical means operator is defined by

$$L_r f(x) = \frac{1}{|S_r(x)|}\int_{S_r(x)} f dA_x,$$

where $|S_r(x)|$ is the Riemannian surface area of $S_r(x)$, and the ball means operator is

$$M_r f(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} f dV_g$$

where $|B_r(x)|$ is the volume of the ball.

In Euclidean $\mathbb{R}^n$ the two types of spherical means (resp. ball means) operators agree, but they differ in general on curved Riemannian manifolds, where the surface measure on a geodesic sphere is not the pushforward under the exponential map of the Euclidean surface measure in the tangent space.

We follow Tsujishita [T]. The (tangential) spherical means operator L_t is defined by the diagram

$$\begin{array}{ccc} C(M) & \xrightarrow{L_t} & C(M) \\ \pi^* \downarrow & & \downarrow \pi_* \\ C(S^*M) & \xrightarrow{(G^t)^*} & C(S^*M). \end{array}$$

That is,

$$L_t = \pi_*(G^t)^*\pi^*.$$

Proposition **19.** *For fixed $0 \leq t \leq inj(M,g)$, L_t is a Fourier integral operator on $M \to M$ of order $-\frac{1}{2}(n-1)$ whose canonical relation Λ_t is the conormal bundle of the characteristic conoid*

$$C_t := \{(x,y) : d(x,y) = t\} \subset M \times M,$$

i.e.

$$\Lambda_t = N^*C_t \subset T^*(M \times M).$$

Remark: It is possible to view $L : [0,\varepsilon] \times M \to M$ as a Fourier integral operator, i.e. with the time variable included.

11. Hormander parametrix

We would like to construct a parametrix of the form

$$\int_{T_x^*M} e^{i\langle \exp_y^{-1} x, \eta\rangle} e^{it|\eta|_y} A(t,x,y,\eta) d\eta.$$

This is a homogeneous Fourier integral operator kernel (see §**??**).

Hörmander actually constructs one of the form

$$\int_{T_x^*M} e^{i\psi(x,y,\eta)} e^{it|\eta|} A(t,x,y,\eta) d\eta,$$

where ψ solves the Hamilton Jacobi Cauchy problem,

$$\begin{cases} q(x, d_x\psi(x,y,\eta)) = q(y,\eta), \\ \psi(x,y,\eta) = 0 \iff \langle x-y, \eta\rangle = 0, \\ d_x\psi(x,y,\eta) = \eta, \quad (\text{for } x = y \end{cases}$$

The question is whether $\langle \exp_y^{-1} x, \eta\rangle$ solves the equations for ψ. Only the first one is unclear. We need to understand $\nabla_x\langle \exp_y^{-1} x, \eta\rangle$. We are only interested in the norm of the gradient at x but it is useful to consider the entire expression. If we write $\eta = \rho\omega$ with $|\omega|_y = 1$, then ρ can be eliminated from the equation by homogeneity. We fix $(y,\eta) \in S_y^*M$ and consider $\exp_y : T_yM \to M$. We wish to vary $\exp_y^{-1} x(t)$ along a curve. Now the level sets of $\langle \exp_y^{-1} x, \eta\rangle$ define a notion of local 'plane waves' of (M,g) near y. They are actual hyperplanes normal to ω in flat $\mathbb{R}^n$ and in any case are far different from distance spheres. Having fixed (y,η), $\nabla_x \langle \exp_y^{-1} x, \omega\rangle$ are normal to the plane waves defined by (y,η). To determine the length we need to see how $\nabla_x \langle \exp_y^{-1} x, \omega\rangle$ changes in directions normal to plane waves.

The level sets of $\langle \exp_y^{-1} x, \eta\rangle$ are images under $\exp_y$ of level sets of $\langle \xi, \eta\rangle = C$ in T_yM. These are parallel hyperplanes normal to η. The radial geodesic in the direction η is of course normal to the exponential image of the hyperplanes. Hence, this radial geodesic is parallel to $\langle \exp_y^{-1} x, \eta\rangle$ when $\exp_y t\eta = x$. It follows that $|\nabla_x \langle \exp_y^{-1} x, \eta\rangle$ at this point equals $\frac{\partial}{\partial t} \langle \exp_y^{-1} \exp_y t \frac{\eta}{|\eta|}, \eta\rangle = t|\eta|_y$. Hence $|\nabla_x \langle \exp_y^{-1} x, \eta\rangle|_x = 1$ at such points.

11.1. **Relation to the phase $\langle \exp_y^{-1}(x), \xi\rangle$.** One may construct a short time parametrix for the wave kernel with phase $\psi(t,x,y) = \langle \exp_y^{-1}(x), \xi\rangle - t|\xi|_y$. Let us consider the operator

$$K_0(t,x,y) = \chi(x,y) \int_{T_y^* M} e^{i(\langle \exp_y^{-1}(x), \xi\rangle - t|\xi|_y)} d\xi$$

where χ is a cutoff to the diagonal. If we introduce polar coordinates on T_y^*M (not on M!) we get

$$K(t,x,y) = \chi(x,y) \int_0^\infty \int_{S_y^* M} e^{i\rho(\langle \exp_y^{-1}(x), \omega\rangle - t)} \rho^{n-1} d\rho d\omega.$$

We then consider the inner integral

$$K(t,\rho,x,y) = \int_{S_y^* M} e^{i\rho(\langle \exp_y^{-1}(x), \omega\rangle - t)} d\omega.$$

The singularity of $K(t,x,y)$ at $t = d_g(x,y)$ is determined by the asymptotic expansion of $K(t,\rho,x,y)$ as $\rho \to \infty$. We may therefore evaluate it by the stationary phase method. As a function of ω is it a classical problem that first arose in lattice point counting problems and we know that

$$K(t,\rho,x,y) \simeq \rho^{-(n-1)/2} \sum_{\omega = \pm \exp_y^{-1}(x)} e^{i\rho(\pm d_g(x,y) - t)} e^{\pm i\pi/2} + \cdots .$$

The Hessian is well known to be the second fundamental form of the tangent sphere and since it is a Euclidean sphere its second fundamental form is the identity.

Plugging back in we find that

$$K(t,x,y) = \chi(x,y) \sum_{\pm} \int_0^\infty \rho^{-(n-1)/2} e^{i\rho(\pm d_g(x,y) - t)} e^{\pm i\pi/2} + \cdots) \rho^{n-1} d\rho.$$

There is no singularity from the $-$ sign so

$$K(t,x,y) \simeq C_0 \chi(x,y) \int_0^\infty e^{i\rho(d_g(x,y) - t)} + \cdots) \rho^{\frac{n-1}{2}} d\rho = \chi(x,y) D_t^{\frac{n-1}{2}} \delta(d(x,y) - t).$$

12. Appendix on Homogeneous distributions

Let $D_t\varphi(x) = \varphi_t(x) = t\varphi(tx)$. A distribution on $\mathbb{R}\backslash\{0\}$ is called homogeneous of degree a if $\langle E, \varphi_t\rangle = t^{-a}\langle E, \varphi\rangle$. One uses the same definition in $\mathbb{R}^n\backslash\{0\}$ with $\varphi_t(x) = t^n\varphi(tx)$. If it extends to $\mathbb{R}^n$ as a distribution with the same property it is called homogeneous of degree a on $\mathbb{R}^n$. The following is from [HoI].

12.1. x_+^a**.** For $\operatorname{Re} a > -1$ define

$$x_+^a = \begin{cases} x^a, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

We want to extend the definition to all $a \in \mathbb{C}$ so that

$$\frac{d}{dx}x_+^a = a x_+^{a-1}, \quad x x_+^{a-1} = x_+^a.$$

There is a problem already at $a = 0$ since $\frac{d}{dx}x_+^0 = \delta_0(x)$.

We define

$$I_a(\varphi) = \int_0^\infty x^a \varphi(x)dx,$$

so that

$$I_a(\varphi') = -aI_{a-1}(\varphi), \quad \operatorname{Re} a > 0.$$

Then for $\operatorname{Re} a > -1$ and $k \in \mathbb{Z}_+$,

$$I_a(\varphi) = \frac{(-1)^k}{(a+k)\cdots(a+1)} I_{a+k}(\varphi^{(k)}).$$

This defines I_a as an analytic family of distributions for $\operatorname{Re} a > -k-1$ except for poles at $a = -1, \ldots, -k$. At $a = -k$ the residue is

$$\lim_{a\to -k}(a+k)I_a(\varphi) = \frac{(-1)^k}{(-1)\cdots(-k+1)} I_0(\varphi^{(k)}) = \frac{\varphi^{(k-1)}}{(k-1)!}.$$

Thus,

$$\lim_{a\to -k}(a+k)x_+^a = (-1)^k \frac{\delta_0^{(k-1)}}{(k-1)!}.$$

Thus one defines

$$x_+^{-k}(\varphi) = \int_0^\infty (\log x)\varphi^{(k)}(x)dx/(k-1)! + \varphi^{(k-1)}(0)(\sum_{j=1}^k 1/j)/(k-1)!.$$

12.2. x_-^a**.** For $\operatorname{Re} a > -1$ define

$$x_+^a = \begin{cases} 0, & x \geq 0 \\ |x|^a, & x < 0. \end{cases}$$

It is the reflection of x_+^a through the origin.

12.3. χ_+^a. Also, define
$$\chi_+^\alpha = \frac{x_+^\alpha}{\Gamma(\alpha+1)}.$$
This is a holomorphic family of homogeneous distributions and
$$\chi_+^{-k} = \delta_0^{(k-1)}.$$

12.4. $(x+i0)^a$. Define the function z^a on $\mathbb{C}\backslash\mathbb{R}$ defined by $e^{a\log z}$ where $\log z \in \mathbb{R}$, for $z \in \mathbb{R}_+$. Its boundary values are denoted $(x \pm i0)^a$. For $\operatorname{Re} a > 0$ one has
$$(x \pm i0)^a = x_+^a + e^{\pm i\pi a} x_-^a.$$

We also need that $(x+i0)^\lambda$ is entire and
$$(x+i0)^\lambda = \begin{cases} e^{i\pi\lambda}|x|^\lambda, & x<0 \\ \\ x_+^\lambda, & x>0. \end{cases}$$

One has
$$(x+\pi i0)^{-k} = x_+^{-k} + (-1)^k x_-^{-k} \pm i\pi(-1)^k \delta_0^{(k-1)}/(k-1)!.$$

12.5. x^{-n}. Here and above t^{-n} is the distribution defined by $t^{-n} = Re(t + i0)^{-n}$ (see [Be], [G.Sh., p.52,60].)

12.6. **Fourier transforms of homogeneous distributions.** According to [GeSh, p. 171],
$$\int_0^\infty e^{i\theta\sigma}\theta_+^\lambda d\lambda = ie^{i\lambda\pi/2}\Gamma(\lambda+1)(\sigma+i0)^{-\lambda-1}.$$
Also, $(t+i0)^{-n} = e^{-i\pi\frac{n}{2}} \frac{1}{\Gamma(n)} \int_0^\infty e^{itx} x^{n-1} dx$.

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