420-1 Partial Differential Equations
Northwestern University
Homework 1
Due January 18 in class.

1. Consider the minimal surface equation in $\mathbb{R}^n$

$$\sum_{i=1} D_i \left(\frac{D_i u}{(1+|Du|^2)^{\frac{1}{2}}} \right) = 0.$$

Show that it is a quasilinear elliptic equation, and compute the ratio of the largest and lowest eigenvalues of the symbol.

2. Consider the Monge-Ampère equation in $\mathbb{R}^n$

$$\det(D_{ij}u) = f.$$

Find its linearization, and compute the ratio of the largest and lowest eigenvalues of the symbol.

3. Consider a PDE in $\Omega \subset \mathbb{R}^n$ of the form

$$F(D^2u(x), x) = 0,$$

with F smooth in all arguments. Then show that the PDE is uniformly elliptic for all solutions u (in the sense that the eigenvalues of the linearization are bounded away from 0 and infinity) if and only if there exist $\lambda, \Lambda > 0$ such that for every $x \in \Omega$, every symmetric $n \times n$ matrix M and every symmetric $n \times n$ semipositive definite matrix N we have

$$\lambda \|N\| \leqslant F(M+N, x) - F(M, x) \leqslant \Lambda \|N\|,$$

where $\|N\|$ is the maximum of the eigenvalues of N.

4. Prove that every C^2 subharmonic function u in the whole of $\mathbb{R}^2$ which is bounded above must be constant.

(Hint: look at $u(z) - \varepsilon \log |z|$ for $\varepsilon > 0$)

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in L^1(\Omega)$. Let χ_ε be a family of standard mollifiers, and $f_\varepsilon = f * \chi_\varepsilon$. Show that

$$\|f_\varepsilon - f\|_{L^1(\Omega)} \to 0,$$

as ε goes to zero.

(Hint: you can use the fact that $C_c^0(\Omega)$ is dense in $L^p(\Omega)$)