445-2 Differential Geometry
Northwestern University
Homework 1
Due January 22 in class.

1) Let p, q be two positive integers and consider the $\mathbb{C}$-action on $(\mathbb{C}^p \backslash \{0\}) \times (\mathbb{C}^q \backslash \{0\})$ given by

$$t \cdot (z_1, \dots, z_p, w_1 \dots, w_q) = (e^t z_1, \dots, e^t z_p, e^{it} w_1, \dots, e^{it} w_q),$$

where $t \in \mathbb{C}$, and $(z_1, \dots, z_p) \in \mathbb{C}^p \backslash \{0\}, (w_1, \dots, w_q) \in \mathbb{C}^q \backslash \{0\}$.

(a) Show that this action is free.

Call then $X_{p,q}$ the quotient complex manifold.

(b) Show that $X_{p,q}$ is diffeomorphic to $S^{2p-1} \times S^{2q-1}$, a product of odd-dimensional spheres.

2) Let X be a compact complex manifold of complex dimension n and assume that $f : \mathbb{CP}^n \to X$ is a holomorphic map which is also a finite sheeted covering. Prove that X must be biholomorphic to $\mathbb{CP}^n$. Then find a compact real manifold Y of dimension $2n$ and a nontrivial finite sheeted smooth covering map $f : \mathbb{CP}^n \to Y$ for any n odd.

3) Let $X = \mathbb{C}^2/\Lambda$ be the complex torus given by the square lattice Λ generated by $(1, 0), (i, 0), (0, 1), (0, i)$ over $\mathbb{Z}$. Consider the holomorphic map $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\sigma(z, w) = \left(z + \frac{1}{2}, \ -w \right).$$

(a) Show that σ induces a holomorphic involution $\sigma : X \to X$ that has no fixed points.

It follows that the quotient $Y = X/\sigma$ is a compact complex manifold with complex dimension 2.

(b) Show that there is no nonzero holomorphic 2-form on Y.

(c) Show that there is a never-vanishing smooth $(2, 0)$-form on Y.