420-1 Partial Differential Equations
Northwestern University
Homework 2
Due January 30 in class.

1. Let u be a harmonic function on a ball $B_{2r}(x)$ in $\mathbb{R}^n$. Prove that

$$\sup_{B_r(x)} u^2 \leqslant \frac{2^n}{|B_{2r}(x)|} \int_{B_{2r}(x)} u^2(x)dx.$$

2. Let u be a harmonic function on a ball $B_{2r}(x)$ in $\mathbb{R}^n$. Prove that

$$\int_{B_r(x)} |Du|^2(y)dy \leqslant Cr \int_{\partial B_r(x)} |Du|^2(y)d\sigma(y),$$

   for a universal constant C.

3. Recall that for any smooth function u we have the Kato inequality

$$|D|Du||^2 \leqslant |D^2u|^2,$$

   valid wherever $|Du| \neq 0$. Prove that if u is harmonic in a domain in $\mathbb{R}^n$ then we can improve this to

$$|D|Du||^2 \leqslant \frac{n-1}{n}|D^2u|^2,$$

   wherever $|Du| \neq 0$.

   (Hint: at any given point you can choose local coordinates so that $D_1u = |Du|$ and $D_ju = 0$ for $j \geqslant 2$)

4. Deduce from the previous problem that if u is harmonic in a domain in $\mathbb{R}^n$ then on the set where $|Du| \neq 0$ we have that the function $|Du|^\alpha$ is subharmonic for all $\alpha \geqslant \frac{n-2}{n-1}$.