420-1 Partial Differential Equations
Northwestern University
Homework 3
Due February 13 in class.

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class C^1. Show that a function u belongs to $W^{1,\infty}(\Omega)$ iff u is Lipschitz continuous on Ω.

2. Give an example of a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(\Omega)$ which is not Lipschitz continuous on Ω.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ of class C^2. Show that the boundary value problem for the biharmonic equation
$$\Delta\Delta u = f \text{ in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$
with $f \in L^2(\Omega)$ has a unique weak solution $u \in W_0^{2,2}(\Omega)$, where weak solution means that
$$\int_\Omega \Delta u \Delta v = \int_\Omega f v,$$
for all $v \in C_c^\infty(\Omega)$.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that a function $u \in L^2_{\text{loc}}(\Omega)$ satisfies
$$\int_\Omega u \Delta v = \int_\Omega f v,$$
for all $v \in C_c^\infty(\Omega)$, where $f \in C^\infty(\Omega)$. Prove that $u \in C^\infty(\Omega)$ and $\Delta u = f$ holds on Ω.

5. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ which satisfies weakly
$$D_i(a^{ij}(x) D_j u) = 0,$$
with (a^{ij}) bounded and uniformly elliptic. Prove that if $\int_{\mathbb{R}^n} u^2 < \infty$ then u is a constant.