445-2 Differential Geometry
Northwestern University
Homework 3
Due February 5 in class.

1) Let (X^n, g) be a Kähler manifold of complex dimension $n > 1$, and let f be a smooth positive nonconstant real function on X. Prove that the conformally rescaled Hermitian metric $f \cdot g$ is never Kähler.

2) Let $X = (\mathbb{C}^2 \backslash \{(0,0)\})/\mathbb{Z}$, be the Hopf surface, where the $\mathbb{Z}$-action is generated by the dilation $\lambda : (z, w) \mapsto (2z, 2w)$. By direct calculation, prove that

$$H^{1,0}_{\bar\partial}(X) = 0, \quad H^{2,0}_{\bar\partial}(X) = 0.$$

Then use Homework 2, problem 3, to show that $H^{0,1}_{\bar\partial}(X) \neq 0$. This gives an example of a compact complex surface with $H^{1,0}_{\bar\partial}(X) \ncong H^{0,1}_{\bar\partial}(X)$.

3) Let (X^n, g) be a Hermitian manifold, and as usual let ω be the fundamental $(1,1)$-form of g. Let β be a real $(1,1)$-form on X such that $\omega^{n-1} \wedge \beta = 0$. Prove that

$$*\beta = -\frac{1}{(n-2)!}\, \beta \wedge \omega^{n-2},$$

where $*$ is the Hodge star operator of g (which is determined by $\eta \wedge *\psi = \langle \eta, \psi \rangle_g \frac{\omega^n}{n!}$ for all $(1,1)$-forms η, ψ on X, where $\langle \cdot, \cdot \rangle_g$ is the pointwise tensor inner product defined by g).