420-1 Partial Differential Equations
Northwestern University
Homework 4
Due February 27 in class.

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C^\infty(\Omega)$ with $u = 0$ on $\partial\Omega$. Prove that for every $1 \leqslant p < \frac{n}{n-1}$ there is a constant $C(n,p)$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leqslant C\|\Delta u\|_{L^1(\Omega)}.$$

   You can think of this result as a weak replacement for the elliptic L^1 estimate for Δ, which is false, and which would bound the $W^{2,1}$ norm of u (and therefore also the $W^{1,\frac{n}{n-1}}$ norm by Sobolev) by the L^1 norm of Δu.

2. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain and $u \in W^{2,2}_{\text{loc}}(\Omega)$ be such that $\Delta u \in W^{1,n}_{\text{loc}}(\Omega)$. Show that $D^2 u \in \mathcal{L}^{2,n}_{\text{loc}}(\Omega)$.

3. Show that the function $\log|x|, x \in \mathbb{R}$ lies in $\mathcal{L}^{1,1}(\mathbb{R})$.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in W^{1,2}(\Omega)$ a weak solution of

$$D_i(a^{ij} D_j u) = g + D_i f^i,$$

   where a^{ij} are constant and elliptic, $f^i \in \mathcal{L}^{2,\lambda+2}(\Omega)$ and $g \in L^{2,\lambda}(\Omega)$, with $0 \leqslant \lambda < n$. Show that $Du \in \mathcal{L}^{2,\lambda+2}_{\text{loc}}(\Omega)$, and furthermore for any relatively compact domain $\Omega' \subset \Omega$ we have

$$\|Du\|_{\mathcal{L}^{2,\lambda+2}(\Omega')} \leqslant C(\|u\|_{L^2(\Omega)} + \|g\|_{L^{2,\lambda}(\Omega)} + \|f\|_{\mathcal{L}^{2,\lambda+2}(\Omega)}),$$

   where C depends only on a^{ij}, $d(\Omega', \partial\Omega)$, n, λ.