**445-2 Differential Geometry**
**Northwestern University**
**Homework 4**
**Due February 12 in class.**

1) Let $X$ be a compact complex manifold. Prove that there cannot exist two Kähler metrics $\omega, \omega'$ on $X$ (no assumption on the cohomology classes) with $\mathrm{Ric}(\omega) > 0$ and $\mathrm{Ric}(\omega') \leqslant 0$.

2) Let $(X, g)$ be a compact Kähler manifold, and $V$ a holomorphic vector field on $X$.

(a) Prove that
$$\Delta_g |V|_g^2 = |\nabla V|_g^2 - \mathrm{Rc}(V, \overline{V}).$$

(b) Deduce that if $g$ is Ricci-flat, then every holomorphic vector field on $X$ is either identically zero or never vanishing.

3) Let $M, N$ be (positive-dimensional) compact complex manifolds with $H^1(M, \mathbb{R}) = H^1(N, \mathbb{R}) = 0$ and consider $X = M \times N$. Assume that $X$ admits a Ricci-flat Kähler metric $\omega$. Prove that we must have that

$$\omega = \pi_M^* \omega_M + \pi_N^* \omega_N,$$

where $\omega_M, \omega_N$ are Ricci-flat Kähler metrics on $M, N$ respectively, and $\pi_M, \pi_N$ are the two projections.