420-1 Partial Differential Equations
Northwestern University
Homework 5
Due November 30 in class.

1. Let u be a positive C^2 function on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, which satisfies
$$\Delta u \geqslant -fu,$$
where f is a function in $L^q(\Omega)$ with $q > \frac{n}{2}$.
Prove that for any $p > 0$ there is a constant C that depends only on p, q, n and $\|f\|_{L^q(\Omega)}$ such that for any ball $B_r(x) \subset \Omega$ we have
$$\sup_{B_{\frac{r}{2}}(x)} u \leqslant \frac{C}{r^{\frac{n}{p}}} \|u\|_{L^p(B_r(x))}$$

2. Let u be a C^4 convex function on $\mathbb{R}^n$ which solves
$$\det(D^2 u) = 1,$$
with $\sup_{\mathbb{R}^n} |D^2 u| < \infty$. Show that all second derivatives of u must be constant, i.e. u is a quadratic polynomial.
(Hint: apply the Evans-Krylov theorem to the rescaled functions $u_\lambda(x) = u(\lambda x)$ and let $\lambda \to \infty$)