445-2 Differential Geometry
Northwestern University
Homework 6
Due February 26 in class.

1) Let (X, ω) be a compact Kähler manifold of complex dimension $n > 1$, and let $u : X \to \mathbb{R}$ be a smooth positive function that satisfies

$$\Delta u \geqslant -Au,$$

where $A \geqslant 1$ is a constant, and Δ is the Laplacian of the metric ω. Using the Moser iteration method prove that

$$\sup_X u \leqslant CA^n \int_X u \, \omega^n,$$

where the constant C depends only on (X, ω).

2) Let (X, ω) be a compact Kähler manifold of complex dimension $n > 1$, $F : X \to \mathbb{R}$ be a smooth function with $\int_X e^F \omega^n = \int_X \omega^n$ and let $\varphi : X \to \mathbb{R}$ be a smooth function with $\omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0$ solving the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F \omega^n.$$

Call g the Hermitian metric defined by ω and $\tilde{g}$ the one of $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$.

Prove directly that $n + \Delta_g \varphi = \mathrm{tr}_g \tilde{g} \geqslant C^{-1}$, and $n - \Delta_{\tilde{g}} \varphi = \mathrm{tr}_{\tilde{g}} g \geqslant C^{-1}$ for some positive constant C that depends only on $\sup_X |F|$ and n.

3) Assume the same setup as problem 2. Using the Moser iteration method we proved in class that

$$\|\varphi\|_{L^\infty(X)} \leqslant C,$$

where C depends only on (X, ω) and $\sup_X e^F$. Recall that φ is normalized by $\int_X \varphi \omega^n = 0$.

Now fix a number $q > n$. Modify this iteration argument to prove the same L^∞ estimate for φ with the constant C depending only on (X, ω), q and $\int_X e^{qF} \omega^n$.