445-2 Differential Geometry
Northwestern University
Homework 7
Due March 5 in class.

1) Let (X, ω) be a compact Kähler manifold of complex dimension $n > 1$, $F : X \to \mathbb{R}$ be a smooth function with $\int_X e^F \omega^n = \int_X \omega^n$ and let $\varphi : X \to \mathbb{R}$ be a smooth function with $\omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0$ solving the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F - \lambda\varphi}\omega^n,$$

where λ is $-1, 0$ or 1. Call g the Hermitian metric defined by ω and $\tilde{g}$ the one of $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$. In class we proved an identity for $\Delta_{\tilde{g}}\mathrm{tr}_g\tilde{g}$. Following the same method prove an identity for $\Delta_{\tilde{g}}\mathrm{tr}_{\tilde{g}}g$.

2) Continuing question 1, prove that

$$\Delta_{\tilde{g}} \log \mathrm{tr}_{\tilde{g}}g \geqslant -C\mathrm{tr}_{\tilde{g}}g - C,$$

where C depends only on $(X, \omega), F$ (this holds for any value of $\lambda = -1, 0, 1$). Then proceed as in class and deduce from this that

$$\mathrm{tr}_{\tilde{g}}g \leqslant Ce^{C(\varphi - \inf_X \varphi)},$$

for a (possibly different) constant C that depends on $(X, \omega), F$.