445-2 Differential Geometry
Northwestern University
Homework 8
Due March 12 in class.

1) Let (X, ω) be a compact Kähler manifold of complex dimension n with $\mathrm{Ric}(\omega) \leqslant -\varepsilon\omega$ for some $\varepsilon > 0$. Let $\tilde{\omega}$ be another Kähler metric on X with scalar curvature $\tilde{R} \geqslant -A$ for some $A > 0$.

Applying the minimum principle to the quantity $\log \frac{\tilde{\omega}^n}{\omega^n}$, prove that $\tilde{\omega}^n \geqslant C^{-1}\omega^n$, for some constant C that depends only on ε, A, n.

2) Let (X, ω) be a compact Kähler manifold of complex dimension $n \geqslant 2$, and denote by R the scalar curvature of ω. Prove that

$$\int_X R^2 \omega^n \geqslant n^2 \int_X c_1^2(X) \wedge \omega^{n-2},$$

with equality if and only if ω is Kähler-Einstein (so $\mathrm{Ric}(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$).

3) Let X be a $K3$ surface, that is a compact Kähler manifold of complex dimension 2 which is simply connected and with $c_1(X) = 0$. Find the exact universal value β of the L^2 norm of the Riemann curvature tensor of ω, any Ricci-flat Kähler metric on X

$$\int_X |\mathrm{Rm}|^2 \frac{\omega^2}{2!} = \beta.$$

(You can use the fact that every $K3$ surface has Betti number $b_2(X) = 22$). Then prove that for any other Kähler metric ω' on X we have

$$\int_X |\mathrm{Rm}'|^2 \frac{\omega'^2}{2!} \geqslant \beta,$$

with equality if and only if ω' is Ricci-flat. (Note that a similar statement on a complex torus is obvious, with $\beta = 0$).