Donaldson-Thomas type invariants via microlocal geometry

\[ DT(Y) \quad X(X, v) \]

\[ CY_3 \]

1. DT invariants:

\[ Y: CY_3 \text{-fold} \]

\[ \text{(proj n.s scheme/C of dim 3)} \]

\[ w/ \quad \sum y = \omega Y = 0 \]

e.g. quintic hypersurface in \( P^4 \)

\( X \) - a moduli space of coherent sheaves on \( Y \)

with trivialized determinant (\& fixed invariants)

\[ x_0^5 + \ldots + x_5^5 = 0 \]

\[ X = \text{Hilb}^n(Y) = \forall \text{rank 1 torsion free sheaves, i.e. ideal sheaves} \]

\[ \text{s.t. } \dim \ O/I = n/2 \]

\[ X = \{ \ldots O/I = 0 \text{ structure sheaf of } \}

\[ \text{i.e. 1 dim subscheme} \]
$X = \{ \text{stable sheaves of given Chern classes} \}$

$X$-stack

Donaldson-Thomas define $\#_{\text{vir}} X = "\text{virtual \# of points of } X"$

e.g. $X = \text{moduli space of lines on quintic } \mathbb{C}P^4$

$= \text{discrete set of points (conics)}$

$\#_{\text{vir}} X = \# X = 2875 \ (609250)$

In general $\dim X > 0$, highly singular (and non-reduced)

DT invariants defined using deformation theory and intersection theory

$T_X \left[ [E] \right] = \text{Ext}^1(E, E) = H^1(Y, \text{End } E) \quad \text{if } E \text{ bundle}$

$\text{ob} \left[ [E] \right] = \text{Ext}^2(E, E) \quad \text{obstruction space}$

$\mathbb{C} \text{ obstructions to smoothness}$

Suppose $X$ smooth. Then $\dim \text{Ext}^2(E, E)$ is constant as $[E] \in X$ varies.
∃ VB E/X w/ fiber over Ext^2(E, E) and if X is proper:

\[ \#_{\text{vir}}(X) = \sum_{[X]} c_{\text{top}} \]

need \( \text{rk} E = \dim X \)

\[ \Phi_X : \text{Ext}^2(E, E) = \text{Ext}^1(E, E) \]

Serre duality in 3D CY

In fact \( E^* = T_X \Rightarrow E = \Omega_X \)

and

\[ \#_{\text{univ}} = \sum_{[X]} c_{\text{top}} (\Omega_X) = \]

\[ = (-1)^{\dim X} \sum_{[X]} c_{\text{top}} T_X = (-1)^{\dim X} \chi(X) \]

(Grass.-Bouwet)

[Topological Euler char]

Generalize this to singular X.

2. Symmetric obstruction theories

Example (Toy model) \( M: \) smooth scheme/\( \mathbb{C} \)

\( \varphi: M \to \mathbb{C} \) reg \( \text{fn} \)

\( X = \text{Crit}(\varphi) \subset M \) subscheme = \( \mathbb{Z} \cdot df \)
defined: \( T_m \rightarrow \mathcal{O}_m \)

ideal sheaf of \( X \subset M \)

\[
\left[ T_M |_X \xrightarrow{H(f)} \Omega_M |_X \right] = E^* \in \mathcal{D}^{[-1,0]}(\mathcal{O}_X)
\]

\[
\left[ I/I^2 \xrightarrow{d} \Omega_M |_X \right] = \mathcal{E}_{\geq -1} L_X
\]

conormal sheaf

the truncation of the cotangent complex

Get a perfect obstruction theory

\[
E^* \rightarrow \mathcal{E}_{\geq -1} L_X
\]

\[
\begin{align*}
H^0 & \cong H^0 \\
H^{-1} & \rightarrow H^{-1}
\end{align*}
\]

which is symmetric i.e. \( E^* \cong E^*[1] \)

by symmetry of \( H(f) \)
Why an obstruction theory?

\[ \mathcal{T}_M |_X \xrightarrow{H(f)} \Omega_M |_X \rightarrow \Omega_X \]
\[ \text{dualize: } \]
\[ 0 \rightarrow \mathcal{T}_X \rightarrow [\mathcal{T}_M |_X \rightarrow \Omega_M |_X] \rightarrow \text{obstruction} \]
\[ 0 \rightarrow \mathcal{T}_X \rightarrow [\mathcal{T}_M |_X \xrightarrow{\nabla} X/M] \rightarrow \text{cokernel} \]
actual obstructions to smoothness
\[ \text{Spec } (\oplus \frac{I^n}{I^{n+1}}) \rightarrow C_{X/M} \rightarrow \text{CV} \]
Curvilinear obstructions
\[ [X]_{\text{vir}} = \mathcal{O}_{\mathcal{S}_M} [T^*_f] \]
\[ M \xrightarrow{\nabla} \Omega_M \]
\[ A_0(X) \]
Chow group of cycles
modulo rati'1 equiv

If \( X \) cpt:
\#_{\text{vir}} X = \deg [X]_{\text{vir}} = \int_{[X]_{\text{vir}}} \mathbb{E}_{\Omega_M} (O, df)

Def. to normal cone:
\[ t \to f \in C \quad |t| \to \infty \]
get cone inside \( \Omega_M | X \to \Omega_M \)
This is the normal cone \( C_{X/M} \)
embedded into \( \Omega_M \)
by \( df \).

Then [\( X_{\text{vir}} \)] = 0
\[ \Omega_M | X \]
depends only on
\[ E \to LX \]
on the obstruction
\[ \text{thry} \]
Moreover, \( [C_{X/M}] \in \Omega_M \)
is a conic Lagrangian cycle
So it is a characteristic cycle of a constr
\( \mu : X \to \mathbb{Z} \)

which turns out to be:

\[
m(\mathcal{P}) = (-1)^{\dim M} (1 - \chi (\text{Milnor fibre of } \mathcal{P} X))
\]

**Microlocal index theorem (Kashiwara)**

\[
\chi(\text{cpct}) \ast \# \text{vir}(x) = \deg \Omega^1_{\text{ML}X} \Xi(\text{ML}_X)
\]

\[
\chi(M, \mu)
\]

Each critical point gives a contribution in terms of its Milnor fibre.

**Global case** Let \( X \) be a scheme with a symmetric obstruction theory.

\[
E \to \mathcal{T}_X \cong L_X \cong E' \cong [1]
\]

\[
E \to L_X \text{ gives } E \cong \mathcal{O}_X
\]

given by \( E = R\pi_* R\text{Hom}(\mathfrak{e}, \mathfrak{e}) [2] \)
\[ E = R^i_{\tau} R \Hom (\mathcal{E}, \mathcal{E}) \mathbb{F} \]

\[ \mathcal{E} \xrightarrow{\psi} \mathcal{E} \]

\[ X \times Y \xrightarrow{\mathbb{F}} X \]

Pick \( X \subset M \) smooth

Pick \( X \subset M \)

\[
\begin{align*}
\left[ \begin{array}{c}
E^{-1} \\
\varphi^{-1}
\end{array} \right] & \rightarrow \left[ \begin{array}{c}
E^{0} \\
\varphi^{0}
\end{array} \right] = E \\
\varphi & \rightarrow \varphi
\end{align*}
\[
\left[ I/j^2 \rightarrow \Sigma_M \right|_X \right] = \tau_{\geq -1} \mathcal{L}_X
\]

\( \sigma = \mathbb{R}_X \quad (\text{for a symmetric obstruction theory, always so?...}) \)

\[ C \subset C \mathcal{M} \left| X \right. \subset \Sigma \mathcal{M} \]

\[ CV \xrightarrow{\mathbb{F}} \mathbb{R}_X \]

\[ [X]_{\text{vir}} = 0! \quad [C] \]
Fact: locally in $M$ $\exists$ almost closed 1-form $\omega$

$$X = \bar{Z}(\omega)$$

$$\text{d}\omega|_Z \leq \int_M Z(\omega).$$

almost closed:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} (f_1, \ldots, f_n)$$

Fact: deformation to normal cone defines Lagrangian cycle.

$$Z^*_x (X) \xrightarrow{\text{Eu}} \text{Con} (X) \xrightarrow{\text{Char}} L_x (\Omega^*_M) \xrightarrow{\text{d}} [C]$$

$$\overline{u}_x = \sum_{C \in \text{null}(C^*)} (-1)^{\text{dim} \pi (C)} \pi (C)$$

$$A_0 (x)$$
\( X_{\text{cpt}}: \quad \#^{\text{vir}} X = \chi(X, v_X) \)

Now compute DT using stratification of \( X \)...