Bezrukavnikov

X smooth

$D(X)$-mod = q-coh + flat connection

$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}} \Omega^1$

deformation (q-n) of $\mathcal{O}(T^*X)$

$QCoh(T^*X)$ = Higgs sheaves

$\mathcal{F} : \bar{\nabla} \to \mathcal{F} \otimes_{\mathcal{O}} \Omega^1$

$M \in D$-mod; the choice of a good filtration gives a <u>degeneration</u>...

$M \mapsto \bar{M} \in QCoh(T^*X)$ gives a map

$$K^0(D_X\text{-mod}) \to K^0(Coh\, T^*X)$$

$$M \rightsquigarrow \bar{M}$$

<u>Sometimes</u> $\bar{M}$ (or smth like it) is canonical.

1) For X as before/$\mathbb{C}$ there is a category of MHM (Mixed Hodge Mods)

$$\begin{array}{l} \mathrm{MHM} \\ \quad \Psi \\ \mathcal{M} \longmapsto \mathrm{Forg}(\mathcal{M}) \in D\text{-mod} \\ \quad \searrow \overline{\mathrm{Forg}}(\mathcal{M}) \in \mathrm{Coh}(T^*X) \end{array}$$

MHM = holonomic D-mod (w/ reg sing) with a fixed good filtration and satisfying some properties.

Given $f: X \to Y$ we have pull-back/push-fwd functors between $\mathrm{MHM}(X)$, $\mathrm{MHM}(Y)$. If is proper, Forg, $\overline{\mathrm{Forg}}$ are compatible with f_+

(need to do $\Omega^{1/2}$-twist here).

Here f_+ for Higgs sheaves comes from the diagram

$$\begin{array}{ccc} T^*Y \times_Y X & \to & T^*X \\ \downarrow & & \\ T^*Y & & \end{array}$$

(as in D-modules)

E.g. $\mathcal{E}$ - variation of Hodge structures
$\mathcal{E}$ ~~has~~ is a bundle with a filtration

$$\mathrm{gr}\,\mathcal{E} \xrightarrow{\mathrm{gr}\nabla} \mathrm{gr}\,\mathcal{E} \otimes \Omega^1$$

is a sheaf on T^*X set-theoretically supported on the zero section.

assume X projective.

Thm (Simpson) then $\mathcal{E}_1, \mathcal{E}_2$ - 2 variations of HS

$$\operatorname{Ext}_{Sh(X)}(\mathcal{E}_1, \mathcal{E}_2) \simeq$$

$$\simeq \operatorname{Ext}_{Coh(T^*X)}(gr\,\mathcal{E}_1, gr\,\mathcal{E}_2)$$

Example $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{O}$: get

$$H^*(X) \underline{\simeq} \oplus H^i(\Omega^j)$$

In other words: for the Ext calculation, the quasi-classical approximation is EXACT.

The thm drastically fails for δ_0.

L.H.S. = $\mathbb{C}$; R.H.S. = $\mathcal{O}_{T_0^*\mathbb{A}^1}$ (?)

Nevertheless there is something.

Example of a similar statement

G - semisimple algebraic group;

$$Coh\, T^*({}_N\backslash G/B) = Coh^N(\mu^{-1}(n^\perp))$$

$X = N \backslash G / B$

$$T^*(G/B) \xrightarrow{\mu} \mathfrak{g}^*$$

Thm (R.B., S. Riche):

there is a full exact embedding

$$D^b(D\text{-mod}_N(G/B)) \longrightarrow D^b(\mathrm{Coh}^N(\mu^{-1}(\mathfrak{n}^\perp)))$$

fitting commutative triangle

$$\begin{tikzcd} D^b(D\text{-mod}_N(G/B)) \arrow[rr] & & D^b(\mathrm{Coh}^N(\mu^{-1}(\mathfrak{n}^\perp))) \\ & MHM_N(G/B) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Thm in progress (R.B., Vilonen)

The same holds for ${}_N\backslash G/_K$

$$K = G^\theta$$

In particular $j_w : (G/B)_w \hookrightarrow G/B$

$j_{w*}(\mathcal{O})$ an object in MHM

$\overline{\mathrm{Forg}}(j_{w*}(\mathcal{O})) = \mathcal{O}(Z_w)$ Z_w - explicit subscheme in $T^*(G/B)$

Rmk Here we get a map

$$K^0(D\text{-mod}(X)) \longrightarrow K^0(\mathrm{Coh}^{\mathbb{C}^*}(T^*X))$$

and the image generates the R.H.S. under the action of $\mathrm{Pic}(X)$.

Char p X/k of char p

$$Z(\mathcal{D}(X)) = \mathcal{O}(T^*X^{(1)})$$

Ex $X = \mathbb{A}^1$ $\quad \mathcal{D}(X) = k\langle x, \partial_x \rangle / \partial x - x\partial = 1$

$$x^p, \partial_x^p \in Z(\mathcal{D}(X))$$

$$\mathcal{D}(X) / (x^p - \alpha, \partial_x^p - \beta) \simeq \mathrm{Mat}_p(k) \text{ if } \alpha, \beta \in k^p$$

$$\mathcal{D}(X) = \text{Azumaya algebra} \mid T^*X^{(1)}$$

So any $M \in D(X)$-module can be thought of as a sheaf on T^*X with an action of an Azumaya algebra.

If A.A. splits on the support of M, we get a coherent sheaf on $T^*X^{(1)}$.

Ex $M = \mathcal{O}$ $\quad \nabla = d + df$; $\mathrm{supp}(M) = \mathrm{graph}(df)$

Can check (Ogus-Vologodsky):

$$\mathcal{D}\text{-mod} \longrightarrow \text{"}q\mathrm{Coh}\text{"}\, T^*X'$$

(gerbe twist...)

Commutes with pullback, pushforward. f_*, f^*

$f : X \to Y$

In the example $X = {}_N\backslash G/B$

$$M = j_{w*}(\mathcal{O}) \rightsquigarrow \mathcal{O}(Z_w)$$

same Z_w as in char 0.

$T^*(G/B) \longrightarrow \mathcal{N}$ is an example of a symplectic resolution.

$X \to Y$ given, say, a fin-dim module M over the quantization of Y:

$\mathcal{O}_{quant}(Y)$

can reduce to char p; often can prove an equivalence

$$\mathcal{D}^b(\mathcal{O}_{quant}(X)\text{-mod}) \xrightarrow[R\Gamma]{\sim} \mathcal{D}^b(\mathcal{O}_{quant}(Y))$$

$\mathcal{O}_{quant}(X)$ – A.A. over $X^{(1)}$

$\mathrm{Supp}(\mathcal{F}) \subset \pi^{-1}(0)$

A.A. splits on the formal nbhd of $\pi^{-1}(0)$; $\mathcal{F}$ defines a coherent sheaf on X;

idea / general expectation:
same coherent sheaf could be obtained by Hodge construction.