Operations on Hochschild and cyclic complexes

Tight structure $\rightsquigarrow$ rich structure

$\lim_{f \to id}$

Ex. 1 From Čech-Alexander to homological algebra.

D – an algebra / k. Čech-Al.:

$$D \xrightarrow{\check{\partial}} D^{\otimes 2} \xrightarrow{\check{\partial}} D^{\otimes 3} \longrightarrow \dots$$

$$\check{\partial}(a_0 \otimes \dots \otimes a_n) = \sum_{j=0}^{n+1} (-1)^j a_0 \otimes \dots \otimes 1 \otimes a_j \otimes \dots$$

$a_0 \mapsto 1 \otimes a_0 - a_0 \otimes 1 \quad \dots$

$\Big(X \leftleftarrows X \times_S X \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \dots \qquad X = \operatorname{Spec} D \ \dots \Big)$

This is <u>not</u> an interesting cohomology (knows nothing about the product; is acyclic: $D^{(\cdot)}(k) \xrightarrow{\sim} D^{(\cdot)}(A)$

$$\begin{matrix} \wr\!\wr \\ k \end{matrix}$$

if $k \hookrightarrow A$ has a section.

contracting homotopy: using this section...

Or more geometrically:

$$a_0 \otimes \ldots \otimes a_n \longmapsto a_0(x_0) \cdot a_1 \otimes \ldots \otimes a_n$$

$$x_0 \in X.$$

<u>In particular</u>: $D \underset{g}{\overset{f}{\rightrightarrows}} E$

two morphisms; $f_* \sim g_* : D^{(\cdot)} \to E^{(\cdot)}$

In fact: can choose a GOOD homotopy.

$$h(f,g)(a_0 \otimes \dots \otimes a_n)$$

$$\|$$

$$\sum_{j=0}^{n-1} (-1)^j f(a_0) \otimes \dots \otimes \underbrace{f(a_j)\, g(a_{j+1})}_{} \otimes$$

$$\otimes \dots \otimes g(a_n)$$

$$[\check{\partial}, h(f,g)] = f_* - g_*$$

<u>In particular:</u> $\underline{f = id}$:

$$[\check{\partial}, h(id, id)] = 0$$

$$\|$$

bar differential

<u>Q1</u> why $\partial_{bar}^2 = 0$?

<u>Know</u>: $[\check{\partial}, \partial_{bar}^2] = 0 \Rightarrow \partial_{bar} = [\check{\partial}, ?_1]$

(Why is $?_1$ zero, or even good? Not sure). Continuing, we get

$$\left(\delta + \partial_{\text{ber}} + ?_1 + ?_2 + \ldots\right)^2 = 0$$

Seems best we can predict from general "principle".

All this becomes interesting when:

$$\begin{array}{ll} J_A \hookrightarrow D & \text{– polynomial ring} \\ \quad\;\;\downarrow & \\ \quad\;\; A & \text{– comm ring} \end{array}$$

$$J_A^{(n)} = \ker\left(D^{\otimes n} \longrightarrow A\right)$$

$\hat{D}^{(n)} = J_A^{(n)}$-adic completion of $D^{\otimes n+1}$ $(n \geq 0)$

The homotopy does not extend to completions; cohomology nontrivial.

OTO:

$$\begin{tikzcd} D_1 \arrow[r, shift left, "\exists f"] \arrow[dr, two heads] & D_2 \arrow[l, dashed, shift left, "\exists g"] \arrow[d, two heads] \\ & A \end{tikzcd}$$

Fact:

$$\begin{tikzcd} D \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & E \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

$h(f,g)$ <u>does</u> extend to

$$\hat{D}^{(\bullet)} \xrightarrow{h(f,g)} \hat{E}^{(\bullet-1)}$$

So: Čech-Alexander cohomology of A does not depend on $D \twoheadrightarrow A$.

<u>Variant</u>: A is over $\mathbb{F}_p$; D polynomial <u>over $\mathbb{Z}_p$</u>; and we take <u>divided power envelope</u> (completed) of $J_A^{(n)}$. (crystalline cohomology).

But is the structure <u>really</u> tight?

$$\begin{tikzcd} D \arrow[r, bend left=60, "f_1"] \arrow[r, "f_2"] \arrow[r, bend right=60, "f_3"'] & E \end{tikzcd}$$

$$h(f_1, f_3) - h(f_1, f_2) - h(f_2, f_3) - h(f_1, f_2) \circ h(f_2, f_3) = [\check{\partial}, h(f_1, f_2, f_3)]$$

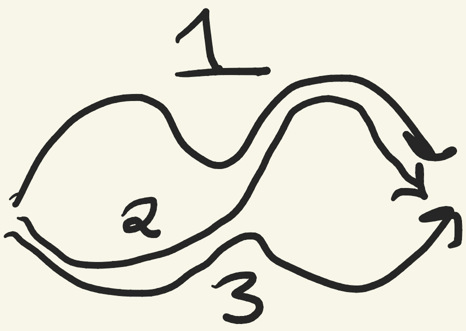
etc.

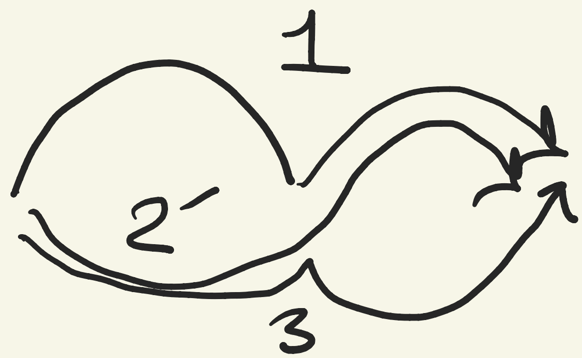
...

<u>Also</u>: agree with compositions $D \xrightarrow{f} E \xrightarrow{g} F$

$$\begin{tikzcd} D \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & E \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & F \end{tikzcd}$$

g_1





$$[\check{\partial}, h(1,2,3)] = h(1,2) + h(2,3) - \cancel{h(1,3)}$$

$-$

$$[\check{\partial}, h(1,2',3)] = h(1,2') + h(2',3) - \cancel{h(1,3)}$$

$$[\check{\partial}, h(g_1,g_2)\circ h(f_1,f_2)] =$$

$$= (g_{1*} - g_{2*})\, h(f_1,f_2) - h(g_1,g_2)(f_{1*} - f_{2*})$$

$$h(g_1,g_2)\circ h(f_1,f_2) = h(1,2,3) - h(1,2',3)$$

(no loose ends).

Ex 2) (DR to Ginzburg-Schedler's realisation of Hochschild and cyclic).

$\Omega^{\bullet, NC}_{A/k}$ = dga : gen. $\underset{0}{a}, \underset{1}{da}$ (lin. in $a \in A$)

rel.: $d(ab) = da.b + a.db$

$d : \Omega^{\cdot} \to \Omega^{\cdot +1}$ graded der; $\quad d : a \mapsto da \mapsto 0$

$$k \simeq \Omega^{\bullet}_{k/k} \xrightarrow{\sim} \Omega^{\bullet}_{A/k}$$

Therefore $\forall\ A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B$:

$$f_{1*} = f_{2*} : \Omega^{\cdot}_{A/k} \longrightarrow \Omega^{\cdot}_{B/k}$$

Actually : good homotopy exists:

1) $\Omega^{\cdot} \xrightarrow[\deg -1]{} \Omega^{\cdot}_A \otimes \Omega^{\cdot}_A$

$$a_0 da_1 \dots da_n$$

$$\downarrow$$

$$\sum_{j=1}^{n} (-1)^{j-1} a_0 da_1 \dots da_{j-1} \cdot (a_j \otimes 1 - 1 \otimes a_j) \cdot da_{j+1} \dots da_n$$

2) $\Omega_A^{\cdot} \otimes \Omega_A^{\cdot} \longrightarrow \Omega_B^{\cdot}$

$$\omega_1 \otimes \omega_2 \longmapsto \pm f_{2*}(\omega_2) f_{1*}(\omega_1)$$

e.g. $a_0 \cdot da_1 \longmapsto a_0 a_1 \otimes 1 - a_0 \otimes a_1$

$$\downarrow$$

$$f_1(a_0 a_1) - f_2(a_1) f_1(a_0)$$

$$\parallel$$

$$f_2(a_1) \cdot f_1(a_0) - f_1(a_0) \cdot f_1(a_1)$$

$$[d, \tau(f_1, f_2)] = f_{1*} - f_{2*}$$

$$[d, \tau(id, id)] = 0$$

$$\iota_\Delta := \iota(\mathrm{id}, \mathrm{id})$$

the Ginzburg-Schedler differential.

$\iota(f_1, \dots, f_k)$:

$$\Delta(a) = a \otimes 1 - 1 \otimes a$$

$$a_0 da_1 \dots da_n$$
$$\downarrow$$
$$\sum \pm a_0 \dots \Delta(a_{j_1}) da_{j_1+1} \dots \Delta(a_{j_2}) \dots \Delta(a_{j_k}) \dots$$

$$\underset{\deg -k+1}{\Omega^\bullet_A} \to \Omega^\bullet_A \otimes \dots \otimes \Omega^\bullet_A \qquad \omega_1 \otimes \dots \otimes \omega_k$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\Omega^\bullet_B \qquad\qquad \pm f_k(\omega_k) f_1(\omega_1) \dots f_{k-1}(\omega_{k-1}) \dots$$

$da_1 \cdots da_n$

$$\sum_{k=1}^{n} f_1(da_1 \cdots da_{k-1}) \cdot \left(f_1(da_k) \otimes 1 - 1 \otimes f_2(da_k)\right) \cdot f_2(da_{k+1} \cdots da_n)$$

$$\sum_{k=1}^{n} f_1(da_1 \cdots da_{k-1}) f_2(da_k \cdots da_n) - \sum_{k=1}^{n} f_1(da_1 \cdots da_k) f_2($$

$(\mathcal{D}^{(\cdot)}, \check{\partial})$ $\quad a_0 \otimes \_ \otimes a_n \mapsto \sum (a_0 \otimes \_ \otimes a_j) \otimes (a_{j+1} \otimes \_)$

$\tilde{\mathcal{D}}^{(\cdot)} = \mathcal{D}^{(\cdot)}/k$ dg coalgebra

$\mathbb{Z}N(D, E)$ $\quad \_ \cdot \_$

$$\left.\begin{array}{l} \tilde{\mathcal{D}}^{(\cdot)} \otimes \mathbb{Z}N(D,E) \to \tilde{\mathcal{E}}^{(\cdot)} \\ \mathbb{Z}N(C,D) \otimes \mathbb{Z}N(D,E) \to \mathbb{Z}N(C,E) \end{array}\right\} \text{ mms of dg coalgs}$$

$A \overset{id}{\underset{id}{\rightrightarrows}} A$ acts by $\partial_{bar} : A^{\otimes \cdot} \circlearrowleft$

well-definedness of crys cohom can be packaged as a <u>category in cocategories</u> acting upon a <u>module in cocategories</u>

$A \rightsquigarrow \Omega_A^{\cdot, nc}$ $\qquad \Delta : a_0 da_1 \_ da_n \mapsto \sum a_0 da_1 \_ da_j \otimes \_$

$$\Omega_A^{\cdot, nc} \otimes \mathbb{Z}N(A,B) \longrightarrow \Omega_B^{\cdot, nc} \Big\} \text{ up to higher homotopies ...}$$

$A \overset{id}{\underset{id}{\rightrightarrows}} A$ by $\overline{G\text{-}S}$ --- 1_Δ.

NC forms <u>will</u> reappear; definitely part of a bigger structure.

$\underline{\text{Ex. } 2\tfrac{1}{2}}$ Saw in $\underline{\text{Ex. } 2}$:

$$\begin{array}{ccc} A\otimes A & & a_0\otimes a_1 \\ \downarrow & & \downarrow \\ B & & f_1(a_0)\cdot f_1(a_1) - f_2(a_1)\cdot f_1(a_0) \end{array}$$

Say, $A = B$, $f_1 = \mathrm{id}$, $f_2 = f$:

$$\begin{array}{c} a_0\otimes a_1 \\ \downarrow \\ a_0a_1 - f(a_1)a_0 \end{array}$$

$$\cdots \longrightarrow \bar{A}^{\otimes n}\otimes A \longrightarrow \cdots$$

We $\underline{\text{know}}$ one:

$$b_f(a_0\otimes\cdots\otimes a_n) = f(a_n)a_0\otimes \underline{\quad\quad} + \sum \pm\, a_0\otimes\cdots\otimes a_ja_{j+1}\otimes \underline{\quad\quad}$$

$$C_\bullet(A, {}_fA) = A \overset{L}{\underset{A\otimes A^{op}}{\otimes}} {}_fA$$

${}_fA$ = graph(f)

Whatever structure we have on it, it is TIGHT (believe so).

So: $\quad id_*, f_* : C_\bullet(A, {}_f A) \circlearrowleft$

$$a_0 \otimes - \longmapsto fa_0 \otimes - \otimes fa_n$$

Should be homotopic.

$$[b_f, B_f] = id - f_*$$

Look for that, find:

$$B_f(a_0 \otimes \dots \otimes a_n) = 1_f \otimes \sum (-1)^{nj} fa_j \otimes \dots \otimes fa_n \otimes$$
$$\otimes a_0 \otimes \dots \otimes a_{j-1}$$

$f = id$:

$$[b, B] = 0.$$

$$B = \lim_{f \to id} \frac{f_* - id}{\text{"}b_f\text{"}}$$

And, yes. B is NC DR differential

In lecture 3: will discuss some pict… we uniting (b, B) and (ι_Δ, d). Now:

HKR.

$$\begin{array}{ccc} C_\bullet(A,A) & \longrightarrow & \Omega^{nc,\bullet}_A \\ b\ ,B & & \iota_\Delta, d \end{array}$$

$$\begin{tikzcd}
a_0 \otimes \dots \otimes a_n \arrow[r, mapsto] \arrow[dr, "\mathrm{HKR}"'] & \frac{1}{(n+1)!} \sum_{j=0}^{n} (-1)^{nj} \, da_j \cdot \ldots \cdot da_n \cdot a_0 \cdot da_1 \cdot \ldots \cdot da_{j-1} \arrow[d, "A \text{ commutative}"] \\
& \Omega^\bullet_{A/k}
\end{tikzcd}$$

(Is there any "animation" of this?)

Ex. 3 Hochschild cochains

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & B \end{tikzcd} \qquad C^\bullet(A, {}_{f_1}B_{f_2})$$

$$\wr$$

$$\mathbb{R}\mathrm{Hom}_{A\otimes B^{op}}({}_{f_2}B, {}_{f_1}B)$$

They form a dg category with objects $f_1, f_2, \ldots$ (Yoneda / $\cup$ product)

Explicitly:

$$C^\bullet(A, {}_{f_1}B_{f_2}) = \mathrm{Hom}_k(\bar{A}^{\otimes \bullet}, B)$$

$$(\varphi \smile \psi)(a_1, \ldots, a_{m+n}) = \pm \varphi(a_1, \ldots, a_m)\psi(a_{m+1}, \ldots, a_{m+n})$$

$\varphi \in C^\bullet(A, {}_{f_1}B_{f_2})$

$\psi \in C^\bullet(A, {}_{f_2}B_{f_3})$

$\varphi \smile \psi \in C^\bullet(A, {}_{f_1}B_{f_3})$

Another operation:

$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, bend left=60, "f_1"{name=U}] \arrow[r, bend right=60, "f_2"'{name=L}] \arrow[from=U, to=L, phantom, "\Downarrow\varphi"] & C \arrow[r, "g"] & D \end{tikzcd}$$

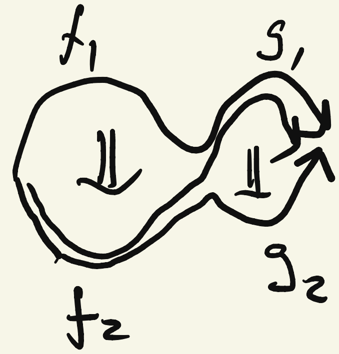
$\varphi \in C^\bullet(B, {}_{f_1}C_{f_2})$

$\rightsquigarrow$

$g_* h^* \varphi \in C^\bullet(A, {}_{gf_1h}D_{gf_2h})$

Tight structure^2 $\rightsquigarrow$ new structure
$f, g \mapsto 1$

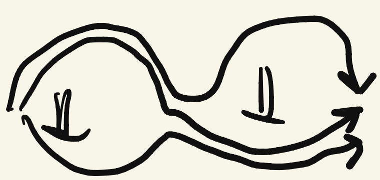
$$\begin{tikzcd} A \arrow[r, bend left=60, ""{name=U1}] \arrow[r, bend right=60, ""'{name=L1}] \arrow[from=U1, to=L1, phantom, "\varphi\Downarrow"] & B \arrow[r, bend left=60, ""{name=U2}] \arrow[r, bend right=60, ""'{name=L2}] \arrow[from=U2, to=L2, phantom, "\psi\Downarrow"] & C \end{tikzcd} \rightsquigarrow$$



$\rightsquigarrow g_{1*}\varphi \cup f_2^*\psi$

homotopic

$\rightsquigarrow$



$\rightsquigarrow f_1^*\psi \cup g_{2*}\varphi$

When $A = B = C$, $f_1 = \dots = g_2 = \mathrm{id}$:

two homotopies: $\psi\{\varphi\}$ and $\varphi\{\psi\}$.

$[\varphi, \psi] = \varphi\{\psi\} \mp \psi\{\varphi\}$: OOO

Recall:

dg Cat $\mathcal{C}$ (small) $\rightsquigarrow$ dg CoCat $\mathrm{Bar}(\mathcal{C})$

Same objects;

$$\mathrm{Bar}(\mathcal{C})(x,y) = \bigoplus_{x_1,\dots,x_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(x,x_1) \otimes \dots \otimes \mathcal{C}(x_n,x)[n+1]$$

Coproduct = cutting word in two.
Differential = ∂_{bar}.

Thm

$$\mathrm{Bar}\, C^{\bullet}(A,B) \otimes \mathrm{Bar}\, C^{\bullet}(B,C) \to \mathrm{Bar}\, C^{\bullet}(A,C)$$

$$\mathrm{Bar}(A) \otimes \mathrm{Bar}\, C^{\bullet}(A,B) \to \mathrm{Bar}(B)$$

morphisms of dg cocategories; associative, compatible.

In other words: Algebras form a category in dg cats, plus a module in dg cats.

Lect. 3

Cat in dgcocats:

$\forall A, B \rightsquigarrow$ dg cocat $\mathcal{B}(A,B)$

$$\mathcal{B}(A,B) = \mathrm{Bar}\, C^\bullet(A,B)$$

(1) $\quad \mathrm{Bar}(A) \otimes \mathcal{B}(A,B) \to \mathcal{B}(B)$

$\forall A, B, C$:

(2) $\quad \mathcal{B}(A,B) \otimes \mathcal{B}(B,C) \to \mathcal{B}(A,C)$

morphisms of dg cocats

all associative, compatible.

<u>Rmk</u> (1) commutes also with $\check{\partial}$ on $\mathrm{Bar}(A)$, $\mathrm{Bar}(B)$. By any chance, are Ex. 1, Ex. 3 parts of the same structure?

Now: include Hochschild chains

$f : A \to A \qquad C_\bullet(A, {}_f A) = A \overset{\mathbb{L}}{\underset{A \otimes A^{op}}{\otimes}} {}_f A$

These form a dg mod$/ C^\bullet(A,A)$ ✓ dg cat as above

TR_A – its bar constr. = dg comod$/ \mathcal{B}(A,A)$

$$TR_A(f) = \bigoplus_{f_1, \dots, f_n} C^\bullet(f, f_1)[1] \otimes \dots \otimes C^\bullet(f_{n-1}, f_n)[1] \otimes C_\bullet(A, {}_{f_n} A)$$

Now have:

$\forall A, B$ – dgcocat $\mathcal{B}(A,B)$

$m_{ABC} : \mathcal{B}(A,B) \otimes \mathcal{B}(B,C) \to \mathcal{B}(A,C)$

assoc

$\forall A$: dg comod $TR(A)$ over $\mathcal{B}(A,A)$

$$\forall A, B: \quad \begin{array}{ccc} & \mathcal{B}(A,B) \otimes \mathcal{B}(B,A) & \\ \swarrow^{m_{ABA}} & & \searrow^{m_{BAB}} \\ \mathcal{B}(A,A) & & \mathcal{B}(B,B) \end{array}$$

$$m^*_{ABA}\,TR(A) \xrightarrow[\tau_{AB}]{\sim} m^*_{BAB}\,TR(B)$$

<u>$\forall A, B, C$:</u>

$$\begin{tikzcd}[column sep=small]
 & \mathcal{B}(A,B)\otimes\mathcal{B}(B,C)\otimes\mathcal{B}(C,A) \arrow[dl, "m_{ABCA}"'] \arrow[d, "m_{B_B}"'] \arrow[dr, "m_{C_C}"] & \\
\mathcal{B}(A,A) & \mathcal{B}(B,B) & \mathcal{B}(C,C)
\end{tikzcd}$$

$$\begin{tikzcd}
m^*_{ABCA}\,TR(A) \arrow[r] & m^*_{B_B}\,TR(B) \arrow[r] & m^*_{C_C}\,TR \arrow[ll, bend right]
\end{tikzcd}$$

$$\boxed{\tau^3}$$

get an automorphism of the dg comodule/ $\mathcal{B}(A,B)\otimes\_$

$$id \sim \tau^3$$

<u>So</u>: Algebras (dg cats) form a category in dg cocats with a trace functor. (shadow).

Remarks

1). How surprising is all this?

Not much.

Another model of $\mathcal{C}(A,B)$:

dg modules / $A \otimes B^{op}$; perfect as
(cofibrant) $\quad$ B-modules

$$\mathrm{Hom}_{A\otimes B^{op}}(-,-)$$

Composition: $\otimes_B$

So: we get (strict) cat in

dg cats with more objects.

I guess we can get below by transfer of structure somehow...

Our approach got us:

dg cat in dgcocats

To pass to cat in dgcats:

Take Cobar.

$$\mathcal{C}(A,B) = \mathrm{Cobar}\, \mathcal{B}(A,B) =$$

$$= \mathrm{Cobar}\, \mathrm{Bar}\, C^\bullet(A,B)$$

Cobar, though, is only laxly monoidal:

$$\mathrm{Cobar}(\mathcal{B}_1 \otimes \mathcal{B}_2) \to \mathrm{Cobar}\, \mathcal{B}_1 \otimes \mathrm{Cobar}\, \mathcal{B}_2$$

coassociative

Solution:

$$\mathcal{C}(A_1, \dots, A_{n+1}) = \mathrm{Cobar}\big(B(A_1, A_2) \otimes \dots \otimes B(A_n, A_{n+1})\big)$$

Two types of morphisms:

I $\mathcal{C}(A_1, \dots, A_{n+1}) \to \mathcal{C}(A_1, A_{i_1}, \dots, A_{i_k}, A_{n+1})$

induced by
$$\begin{array}{c} B(-) \otimes B(-) \\ \downarrow \\ B(-) \end{array}$$

II $\mathcal{C}(A_1, \dots, A_{n+1}) \xrightarrow[\sim]{\text{w.e.}} \mathcal{C}(A_1, \dots, A_j) \otimes \mathcal{C}(A_j, \dots, A_{n+1})$

agreeing w/ each other.

Very much Segal-like; due to T. Leinster.

Some version of dg nerve prob. makes this a Segal $(\infty, 2)$ category.

<u>Also</u>: $M \mapsto M \otimes_{A \otimes A^{op}} A$ is an <u>actual</u> trace functor, so again some transfer of structure should produce a homotopy version on Hochschild chain/cochains.

<u>OTOH</u>:

<u>calculus flavor</u>:

<u>Recall</u> $C^\bullet(A,A)$ nc multivectors

$C_\bullet(A,A)$ nc forms

With this in mind: what geometric intuition is involved in the above operations?

	Higher analogues of
$\mathcal{B}(A,B)$ dg cocat:	$\wedge$ on $\wedge^\bullet T_X$
$\mathcal{B}(A,B)\otimes\mathcal{B}(B,C)\to\mathcal{B}(A,C)$	$[\,,\,]_{Sch}$ on $\wedge^{\bullet+1}T_X$
$TR(A)$ dg comod	$\iota_a\omega$, $a\in\wedge^\bullet T_X$, $\omega\in\Omega^\bullet$
$\tau_{AB}: m^*_{ABA}TR(A) \cong m^*_{BAB}TR(B)$	$L_a\omega$, $a\in\wedge^{\bullet+1}T_X$, $\omega\in\Omega^\bullet$
$\mathrm{id} \sim \tau^3$	d_{DR}

And nothing more

<u>Other remark</u>: what about $C_\bullet(A, {}_fB_g)$?

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$

${}_f B$ right dualizable A–B bimod

$({}_g B)^\vee = B_g$ (left dualisable B–A bimod)

$${}_f B_g = {}_f B \otimes_B B_g$$

$$A \overset{\mathbb{L}}{\underset{A \otimes A^{op}}{\otimes}} ({}_f B \otimes_B B_g) = C_\bullet(A, {}_f B \otimes_B {}_g B^\vee)$$

Actually, get some interesting structure in coalgebras.

1. $\forall A, B$: dg cobimodule $\mathcal{M}(A,B)$ over $\mathcal{B}(A,B)$
2. Some nontrivial variation on

$$\mathcal{B}(A,B) \otimes \mathcal{M}(B,C) \otimes \mathcal{B}(C,D) \to \mathcal{M}(A,D)$$

In particular, for $A = B = \ldots = A$, $f = g = \ldots = \mathrm{id}_A$: some twisted version of Shoikhet's tetramodule.

Application: nc crystalline complex. $p>2$

Objects: (free) $\mathbb{Z}_p$-modules A with a product associative mod p.

Morphisms: $f: A \to B$ is morphism of algebras mod p.

Thm

Back to nc forms

(+ Hochschild (co)chains)

1. A common home for forms and chains (Ginzburg–Schedler):

$$\begin{tikzcd} \Omega^{1,nc} \arrow[r] \arrow[d, equal] & A \otimes A \arrow[d, equal] \\ B_1^{sh}(A) \arrow[r] & B_0^{sh}(A) \end{tikzcd}$$

$$a_0\, da_1\, a_2 \longmapsto a_0 (a_1 \otimes 1 - 1 \otimes a_1) a_2$$

Notation:

a) a short resolution of A as $A \otimes A^{op}$-mod; (semi)free if A is.

b) $A \otimes A = A t_* A$; then $da \mapsto [a, t_*]$

NC forms (a version):

$$\mathbb{Y}^{(*)}(A) = \bigoplus_{n\geq 0} \mathbb{Y}^{(n)}(A)$$

$$\mathbb{Y}^{(n)}(A) = \underbrace{B_1^{sh} \otimes_A \cdots \otimes_A B_1^{sh}}_{n \text{ times}}$$

$$= \Omega_A^{\bullet, nc}\langle t_* \rangle$$

Two differentials:

$$\iota_\Delta : da \longmapsto [a, t_*]$$
$$a, t_* \longmapsto 0$$

$$d : a \longmapsto da \, ; \; da, t_* \longmapsto 0$$
$$[d, \iota_\Delta] = [t_*, -]$$

$$Y^{(*)}(A) = \mathcal{Y}^{(\cdot)} / [\mathcal{Y}^{(\cdot)}, \mathcal{Y}^{(\cdot)}]$$

2) Add nc multivectors:

$$B^{sh,\vee} = \mathrm{Hom}_k(B^{sh}_\bullet, \underbrace{A \otimes A}_{\substack{\text{inner} \\ \text{bimod structure}}})$$

$$A \otimes A \longrightarrow \mathrm{Der}(A, \underbrace{A \otimes A}_{\text{in}})$$

$$\Vert$$

$$A t^* A$$

$$t^* \longmapsto \left(a \overset{\Delta}{\longmapsto} a \otimes 1 - 1 \otimes a\right)$$

$$\mathcal{X}^{(k)} = B^{sh\vee}_\bullet \underset{A}{\otimes} \cdots \underset{A}{\otimes} B^{sh\vee}_\bullet$$

[or perhaps $\mathrm{Hom}_{A \otimes \cdots \otimes A}(B^{sh}_\bullet \otimes \cdots \otimes B^{sh}_\bullet, A \otimes \cdots)$]

$$\mathcal{X}^{(*)} = \mathcal{X}^{(*)} / [\mathcal{X}^{(*)}, \mathcal{X}^{(*)}]$$

$$\hat{\mathfrak{X}}^{(*)} = \left(\mathbb{X} / [\mathbb{X}, \mathbb{X}] \right)^{\wedge \text{ appropriately.}}$$

definitely $\prod_{k \geq 0} (\ldots)^{(k)}$

(Kontsevich–Vlassopoulos ; Waikit Yeung).

<u>Prop.</u> Let A be quasi-free.

<u>Then</u>:

$\Upsilon^{(n)}(A)$ computes

α = cyclic shift

$$HH_\bullet(A^{\otimes n}, {}_\alpha A^{\otimes n})_{C_n}$$

char = 0

$$\wr$$

$$HH_\bullet(A, A)$$

$HH^\bullet(A, A)$

$\not\cong$

$\hat{\mathfrak{X}}^{(n)}$ computes $HH^\bullet(A^{\otimes n}, {}_\alpha A^{\otimes n})$

Operations on $\hat{\mathfrak{X}}, \Upsilon$:

1) The Schouten-style bracket on $\hat{\mathfrak{X}}$ (K.-V., Waikit Y.)

2) $\iota_a \omega \in \Upsilon$, $\quad a \in \hat{\mathfrak{X}}, \omega \in \Upsilon$

3) $\iota_\omega a \in \mathfrak{X}$, $\quad a \in \mathfrak{X}, \omega \in \Upsilon$

<u>Used for:</u>

Left CY structure on A

$\rightsquigarrow$

nc version of a shifted structure on A (using $\mathfrak{X}, \Upsilon$ calculus)

$\rightsquigarrow$

a shifted symplectic structure on (derived) $\mathrm{Rep}(A)/GL$.

a right CY structure

generality?

$\wr$
$\downarrow$

nc analog of a shifted Poisson structure.

$\uparrow$
$\wr$

Those, i.e. MC elements of the dgla $\hat{\mathfrak{X}}^{(*)}$, are called pre-CY structures on A.

Question: how to "animate" this picture?

One suggestion: consider the tight structure on the following:

$$\begin{tikzcd}
 & A_3 \arrow[r, "M_{33}"] \arrow[dl, "M_{31}"'] & B_3 & \\
B_1 & & & A_2 \arrow[ul, "M_{23}"'] \arrow[dl, "M_{22}"] \\
 & A_1 \arrow[ul, "M_{11}"] \arrow[r, "M_{12}"'] & B_2 &
\end{tikzcd}$$

Hochschild cochains:

$$\mathrm{Hom}_{\otimes A_i - \otimes B_i}\left(\bigotimes M_{ii}, \bigotimes M_{i,i+1}\right)$$

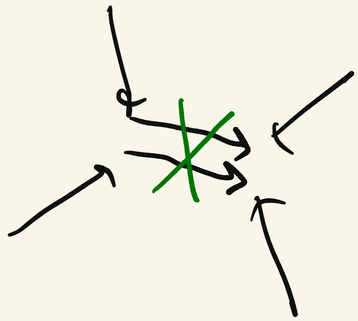
Hochschild chains:

$$\bigotimes M_{ii}^{\vee} \otimes_{\otimes A_i - \otimes B_i} \bigotimes M_{i,i+1}$$

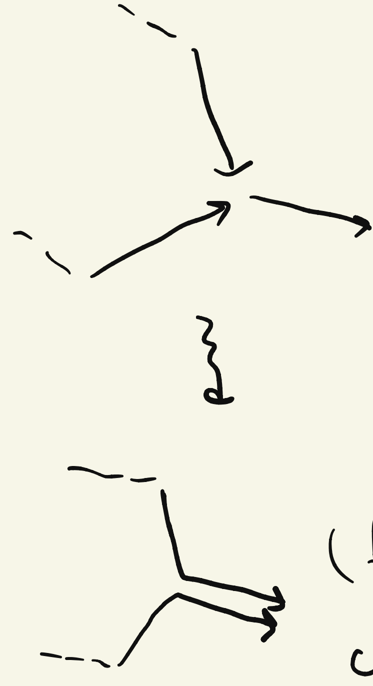
And a mixed chain-cochain version ...

Operations of the form:

1)



2)



(for cochains)

and ... (for chains)

<u>Question</u> what "rich structure" will we get on

$$C^\bullet(A^{\otimes n}, {}_\alpha A^{\otimes n}) \quad \text{and} \quad \begin{matrix} C_\bullet(A^{\otimes n}, {}_\alpha A^{\otimes n}) \\ \cong \\ C_\bullet(A, A) \end{matrix} \; ?$$

Another piece of this structure should be:

nc Frobenius

Example

$$\begin{tikzcd}
A_1 \arrow[r, "M_{11}"] \arrow[d, "N_{12}"'] & B_1 \\
B_2 & A_2 \arrow[u, "N_{21}"'] \arrow[l, shift right, "M_{22}"'] \arrow[l, shift left, "M_{22}"] \arrow[d, "M_{23}"] \\
A_3 \arrow[u, "M_{32}"] \arrow[r, "M_{33}"'] & B_3
\end{tikzcd}$$

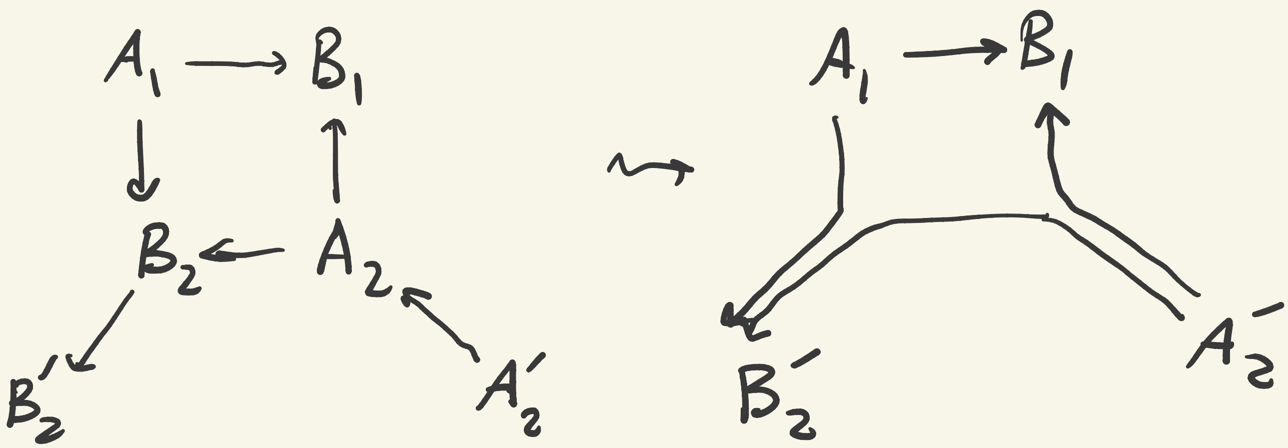
$$\mathrm{RHom}_{A_1 \otimes A_2 - B_1 \otimes B_2}\left(M_{11} \otimes M_{22},\ N_{12} \otimes N_{21}\right)$$

$$\mathrm{RHom}_{A_2 \otimes A_3 - B_2 \otimes B_3}\left(M_{32} \otimes M_{23},\ M_{22} \otimes M_{23}\right)$$

$$\downarrow$$

$$\mathrm{RHom}_{A_1 \otimes A_2 \otimes A_3 - B_1 \otimes B_2 \otimes B_3}\left(M_{11} \otimes M_{32} \otimes M_{23},\ N_{12} \otimes N_{21} \otimes M_{33}\right)$$

As well as:



Having this, there are two ways to merge pictures at vertices:

$$
\begin{tikzcd}
 & C_2 & B_3 \arrow[l] \arrow[d] \\
A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[u] & C_1 \\
B_2 & A_2 \arrow[l] \arrow[u] &
\end{tikzcd}
\quad \longrightarrow \quad
\begin{tikzcd}
 & C_2 & B_3 \arrow[l] \arrow[d] \\
A_1 \arrow[ur] \arrow[rr] \arrow[d] & & C_1 \\
B_2 & A_2 \arrow[l] \arrow[ur] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & C_2 & B_3 \arrow[l] \arrow[d] \\
A_1 \arrow[ur] \arrow[d] & & C_1 \\
B_2 & A_2 \arrow[l] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

In a tight structure, they are homotopic. Homotopy is of degree -1.

Summing over all vertices, get the KV bracket.

[when $A_i = B_i = A$
$M_{ij} = N_{ij} = A_{diag}$

2. Obj: k (one obj)

Mor: $C(k,k) = k\text{-mod}$ $\otimes_k$

Sets

Ab

$\forall$ $TR_k : C(k,k) \to Ab$

$\forall$ k-algebra A:

$$A^{\natural}_{TR} = TR_k(A \otimes \dots \otimes A)_{\bullet+1}$$

$\bullet \geq 0$

Action of $\Lambda^{op} \Rightarrow A^{\natural}_{TR}$ is a cyclic object in Ab

Sets

$$\boxed{W : \mathbb{F}_p\text{-mods} \longrightarrow \mathbb{Z}_p\text{-mods}}$$

(Kaledin's nc Witt vectors)