On noncommutative differential forms

Boris Tsygan

Abstract. We review the topic of noncommutative differential forms, following the works of Karoubi, Cuntz-Quillen, Cortiñas, Ginzburg-Schedler, and Waikit Yeung. In particular we give a new proof of the theorem of Ginzburg and Schedler that compares extended noncommutative De Rham cohomology to cyclic homology. This theorem is a stronger version of a theorem of Karoubi. We also describe an algebraic structure, namely a category in DG cocategories, that noncommutative forms and other versions of noncommutative calculus are partial cases of.

1. Introduction

The idea of noncommutative differential geometry is to start with an assocaitive algebra A and produce invariants that, in the case when A is commutative, give classical invariants such as differential forms and the De Rham complex. One way of doing this is to start with invariants from classical homological algebra, such as Hochschild homology. The relation between these invariants and De Rham complexes had been known since the early 60s [**19**], [**32**]. This idea is the basis of cyclic homology theory [**2**], [**3**], [**4**], [**33**].

On the other hand, one can try to define noncommutative differential forms directly from the algebra A. This idea was present in Connes' approach from the very beginning. It was then advanced by Karoubi and by Cuntz-Quilen in [**21**], [**7**]. The relation between the two approaches, one based on Hochschild homology and the other on noncommutative forms, is strong, deep, and subtle. It had been clarified gradually, starting from the works mentioned above and then in the articles by Ginzburg and Schedler [**14**], [**15**].

Let us start with noncommutative forms. They are defined for any unital algebra A over a commutative unital ring k as a differential graded algebra $(\Omega^\bullet(A), d)$ together with a morphism of graded algebras $A \to \Omega^\bullet(A)$ which is universal with such property. The De Rham cohomology is not interesting, not even in characteristic greater than zero. In fact (noncommutative Poincaré lemma), the morphism $k = \Omega^\bullet(k) \to \Omega^\bullet(A)$ is a quasi-isomorphism. However trivial this fact is, its mere generality has some interesting consequences that we will discuss later. For now, note one nontrivial way of defining De Rham cohomology, namely the noncommutative De Rham complex

$$\mathrm{DR}^\bullet(A) = \Omega^\bullet(A)/[\Omega^\bullet(A), \Omega^\bullet(A)] \tag{1.1}$$

2020 *Mathematics Subject Classification.* Primary .

Let us next recall the original definition of cyclic homology (for $\mathbb{Q} \subset k$) [**2**], [**33**] as the homology of the complex

$$A/\mathrm{im}(1-\tau) \xleftarrow{b} A^{\otimes 2}/\mathrm{im}(1-\tau) \xleftarrow{b} \ldots \xleftarrow{b} A^{\otimes(n+1)}/\mathrm{im}(1-\tau) \xleftarrow{b} \tag{1.2}$$

where

$$\tau(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$$

and b is the Hochschild differential. This is a chain complex, as opposed to the noncommutative De Rham complex

$$\mathrm{DR}^0(A) \xrightarrow{d} \mathrm{DR}^1(A) \xrightarrow{d} \ldots \xrightarrow{d} \mathrm{DR}^n(A) \xrightarrow{d}$$

which is a cochain complex. The original observation of Connes was that noncommtative higher traces, i.e. linear functionals

$$\mathrm{DR}^n(A)/d\mathrm{DR}^{n-1}(A) \to k,$$

are cyclic cocycles, i.e. linear functionals

$$A^{\otimes(n+1)}/(\mathrm{im}(1-\tau) + \mathrm{im}(b)) \to k.$$

Let us modify the cyclic complex a little. Put

$$C_n(A) = A \otimes \overline{A}^{\otimes n}$$

where $\overline{A} = A/k \cdot 1$; there are two differentials

$$b : C_n(A) \to C_{n-1}(A);\ B : C_n(A) \to C_{n+1}(A)$$

such that $bB + Bb = 0$. The *reduced* cyclic complex of A (i.e. the quotient of (1.2) by the cyclic complex of k) is quasi-isomorphic to

$$A/([A,A] + k \cdot 1) \xleftarrow{b} C_1(A)/BC_0(A) \xleftarrow{b} \ldots \xleftarrow{b} C_n(A)/BC_{n-1}(A) \xleftarrow{b} \tag{1.3}$$

It turns out that a complex better compatible to the noncommutative De Rham complex is the "dual" cochain complex

$$A/([A,A] + k \cdot 1) \xrightarrow{B} C_1(A)/bC_2(A) \xrightarrow{B} \ldots \xrightarrow{B} C_n(A)/bC_{n+1}(A) \xrightarrow{B} \tag{1.4}$$

Namely: over the ground ring k of characteristic zero, "the original HKR map"

$$\mathrm{HKR}_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \frac{1}{n!} a_0 da_1 \ldots da_n \tag{1.5}$$

is a quasi-isomorphism from (1.4) to the reduced De Rham complex $\mathrm{DR}^\bullet(A)/\mathrm{DR}^\bullet(k)$ (cf. [**21**]).

Now, both complexes (1.3) and (1.4) are quasi-isomorphic to quotients of the periodic cyclic complex

$$\mathrm{CC}^{\mathrm{per}}_\bullet(A) = (C_\bullet(A)((u)), b + uB) \tag{1.6}$$

Those two quotients are described in two diagrams before the beginning of Section 4. (We also take a quotient by the complex of k). This suggests that perhaps the De Rham complex is also a quotient of something larger, and that the De Rham differential d is also part of a pair of differentials, as is the Hochschild differential b.

This is indeed the case, as shown by Ginzburg and Schedler in [**14**], [**15**]. Things being subtle, there are two versions of that. Let us start with the extended De Rham complex (Section 4). It is proven in [**15**] that it is quasi-isomorphic to its cyclic counterpart. Moreover, the two are quasi-isomorphic as filtered complexes (the filtration given by the bicomplex structure). We will present a new proof of this theorem that, we hope, clarifies the situation and puts it in a more general homological algebraic context.

The reduced version of the extended De Rham complex is quasi-isomorphic to the reduced noncommutative De Rham complex. Therefore we recover Karoubi's theorem. Noncommutative reduced De Rham cohomology gets identified with the kernel of the periodicity operator S on reduced cyclic homology as in [**21**].

Another version of a second differential on noncommutative forms is Ginzburg and Schedler's operator ι_Δ. It is actually the first term of the second differential in the extended De Rham complex; it acts on usual, not extended, forms. The pairs (b, B) and (ι_Δ, d) are indeed intertwined by the HKR map Section 4, but only on the image of the Cuntz-Quillen projection. As for the complement of this image: on it, b and d are acyclic whereas $B = \iota_\Delta = 0$. is acyclic in some sense. This allows to conclude that in the periodic cyclic complex (1.6) one can replace $b + uB$ by $\iota_\Delta + ud$. Other versions of cyclic homology may also be computed in terms of $\iota_\Delta + ud$, in a way that is somewhat more involved.

The last part of the paper (section 10) is devoted to a general algebraic structure of which noncommutative forms are an example. Together with other examples they are probably part of a general unified structure. Namely, the structure is of a category in coalgebras (or more generally in cocategories), and a module in cocategories over it. This structure provides a package for various algebraic properties of noncommutative versions of forms and multivectors. One example is the Čech-Alexander complex of an associative algebra, both in the classical commutative version from crystalline cohomology theory [**1**], [**18**] and in the noncommutative version of Cortiñas [**5**]. Note that in the noncommutative case, the Čech-Alexander complex projects directly to the complex of noncommutative forms with the De Rham differential. Another example (subsection 10.3) is a category in cocategories whose objects are algebras, and the cocategory of morphisms between two algebras A and B is constructed from Hochschild cochains of A with values in B. The module in coalgebras puts in correspondence to an algebra A its bar construction.

These examples illuminate how nontrivial algebraic structures appear on Hochschild cochains, noncommutative forms, etc. One starts with a rigid structure with many objects and morphisms; everything is rigid up to homotopy. Then one declares all objects to be the same, and all morphisms to be the identity. Now our homotopies, in general connecting two different things, start to connect the same thing to itself; in other words, they become morphisms or/and cocycles. The first example (coming from the classical Čech-Alexander complex) is the bar differential. Other differentials, e.g. B and ι_Δ above, also can be obtained this way. In the example of subsection 10.3, this is how one produces the Gerstanhaber bracket on Hochschild cochain complexes.

The author is grateful to P. Bressler, A. Gorokhovsky, D. Kaledin, R. Kaufmann, M. Rivera, and R. Nest for fruitful discussions, and to E. Getzler, V. Ginzburg, M. Kontsevich, T. Schedler, and Waikit Yeung for patiently explaining to him various aspects of their work on noncommutative forms and related topics.

2. Noncommutative forms

Let A be an associative unital algebra over a commutative unital ring k.

Let $\Omega^\bullet(A)$ be the graded algebra generated by A and by symbols da, $a \in A$, linear in a and subject to relations

a) $d(ab) = da\, b + adb$;

b) the unit of A is the unit of $\Omega^\bullet(A)$.

The grading $|a| = 0,\ |da| = 1$ makes $\Omega^\bullet(A)$ a graded algebra. We define the differential $d : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ as the unique graded derivation sending a to da and da to zero for all a in A.

We get a DG algebra $(\Omega^\bullet(A), d)$ together with a morphism of graded algebras $i : A \to \Omega^\bullet(A)$. It is universal, in the sense that for any DG algebra $\Omega^\bullet$ together with a morphism of graded algebras $j : A \to \Omega^\bullet$ there exists unique morphism of DG algebras $f : \Omega^\bullet(A) \to \Omega^\bullet$ such that $j = fi$.

2.1. Noncommutative Poincaré lemma.

Lemma 2.1. *The embedding*

$$\Omega^\bullet(k) \to \Omega^\bullet(A)$$

is an isomorphism on the cohomology of the differential d.

Proof. First note that the naive HKR map

$$A \otimes \bar{A}^{\otimes n} \to \Omega^n(A);\ a_0 \otimes \ldots \otimes a_n \mapsto a_0 da_1 \ldots da_n \tag{2.1}$$

is an isomorphism. Indeed, it is clearly surjective, since every product of $a_i' s$ and db_j's may be always transformed into an element of the image of the map above. Then observe that there is an associative product on $A \otimes \bar{A}^\bullet$; the DGA $\Omega^\bullet(A)$ maps to it by universality, and this map inverts (2.1). We get an isomorphism of algebras which intertwines d with

$$a_0 \otimes \ldots \otimes a_n \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_n$$

The cohomology of this differential is clearly k. $\square$

Remark 2.2. One application of the noncommutative Poincaré lemma over $\mathbb{Z}$ is Karoubi's approach to extending De Rham-Sullivan complexes from rational homotopy theory to more general situations [**22**]-[**28**].

3. Noncommutative De Rham complex

Define

$$\mathrm{DR}^\bullet(A) = \Omega^\bullet(A)/[\Omega^\bullet(A), \Omega^\bullet(A)] \tag{3.1}$$

and also

$$\overline{\mathrm{DR}}^\bullet(A) = \mathrm{DR}^\bullet(A)/\mathrm{DR}^\bullet(k) \tag{3.2}$$

4. The extended noncommutative De Rham complex

Let t be a formal variable of degree one. Define

$$\Omega_t^\bullet(A) = \Omega^\bullet(A) * k[t] \tag{4.1}$$

$$\mathrm{DR}_t^\bullet(A) = \Omega_t^\bullet(A)/[\Omega_t^\bullet(A), \Omega_t^\bullet(A)] \tag{4.2}$$

and also

$$\overline{\Omega}_t^\bullet(A) = \Omega^\bullet(A) * k[t]/k[t] \tag{4.3}$$

$$\overline{\mathrm{DR}}_t^\bullet(A) = \overline{\Omega}_t^\bullet(A)/[\overline{\Omega}_t^\bullet(A), \overline{\Omega}_t^\bullet(A)] \tag{4.4}$$

There is the second grading (by powers of T) on $\Omega_t^\bullet$ as well as on all the spaces above. Therefore $\Omega_t^\bullet$ is a bi-graded algebra, and all the above spaces are bi-graded. For the first grading, $|d| = 1$ and $|t| = 0$. For the second, $|d| = 0$ and $|t| = 1$. We denote by $\Omega_t^{p,q}$ the component whose degree is p with respect to thefirst grading and q with respect to the second grading. We get similar decompositions for all the spaces above.

Lemma 4.1.

$$\mathrm{DR}_t^{n,0} = (\Omega/[\Omega,\Omega])^n;\ \mathrm{DR}_t^{n-1,1} \xrightarrow{\sim} \Omega^{n-1}$$

4.1. The derivation ι_t. Let $|\omega|$ be the first grading of ω, *i. e.* $|a| = |t| = 0$ and $|da| = 1$. Define the graded derivation of degree -1 with respect to this grading by

$$\iota_t(a) = \iota_t(t) = 0;\ \iota_t(da) = [t, a]. \tag{4.5}$$

This is a bi-homogeneous map of degree $(-1, 1)$ satisfying

$$\iota_t^2 = 0.$$

We get complexes

$$\mathrm{DR}_t^{n,0} \xrightarrow{\iota_t} \mathrm{DR}_t^{n-1,1} \xrightarrow{\iota_t} \mathrm{DR}_t^{n-2,2} \xrightarrow{\iota_t} \ldots \xrightarrow{\iota_t} \mathrm{DR}_t^{0,n} \tag{4.6}$$

They are the columns of the double complex

$$\begin{tikzcd}
A/[A,A] \arrow[r, "d"] & \mathrm{DR}^1(A) \arrow[r, "d"] \arrow[d, "\iota_t"] & \mathrm{DR}^2(A) \arrow[r, "d"] \arrow[d, "\iota_t"] & \ldots \arrow[r, "d"] & \mathrm{DR}^n(A) \arrow[r, "d"] \arrow[d, "\iota_t"] & {} \\
 & At \arrow[r] & \Omega^1(A)t \arrow[r, "d"] \arrow[d, "\iota_t"] & \ldots \arrow[r, "d"] & \Omega^{n-1}(A)t \arrow[r, "d"] \arrow[d, "\iota_t"] & {} \\
 & & AtAt \arrow[r, "d"] & \ldots \arrow[r, "d"] & \mathrm{DR}_t^{n-2,2}(A) \arrow[r] & {} \\
 & & & \ldots & \ldots \arrow[r] \arrow[d] & {} \\
 & & & & At\ldots At \arrow[r, "d"] & {}
\end{tikzcd}$$

Lemma 4.2. *The projection*

$$\overline{\mathrm{DR}}_t^\bullet(A) \to \overline{\mathrm{DR}}^\bullet(A)$$

is a quasi-isomorphism.

Proof. By Lemma 2.1 and by the Künneth formula, the rows of the reduced version of the double complex are acyclic, and the spectral sequence converges. $\square$

5. Recollection on cyclic homology

Let

$$C_n(A) = A \otimes \bar{A}^{\otimes n} \tag{5.1}$$

where $\bar{A} = A/k \cdot 1$. Define

$$b : C_n(A) \to C_{n-1};\ B : C_n(A) \to C_{n+1}(A)$$

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1} \tag{5.2}$$

and

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{j-1} \tag{5.3}$$

One has

$$b^2 = Bb + bB = B^2 = 0 \tag{5.4}$$

This allows to arrange the terms $C_n(A)$ into a double complex in several different ways. Start with the periodic cyclic complex

$$\mathrm{CC}^{\mathrm{per}}_{\bullet}(A) = (C_{\bullet}(A)((u)), b + uB) \tag{5.5}$$

where u is a formal variable of homological degree -2. Define its subcomplex

$$\mathrm{CC}^{-}_{\bullet}(A) = (C_{\bullet}(A)[[u]], b + uB) \tag{5.6}$$

(the negative cyclic complex) and its quotient complex

$$\mathrm{CC}_{\bullet}(A) = (C_{\bullet}(A)((u))/uC_{\bullet}(A)[[u]], b + uB) \tag{5.7}$$

(the cyclic complex). The latter looks like

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & & & {} \arrow[d, "b"] \\
A \arrow[r, "B"] & C_1 \arrow[r, "B"] \arrow[d, "b"] & \ldots & \ldots \arrow[r, "B"] & C_n \arrow[d, "b"] \\
& A \arrow[r, "B"] & \ldots & \ldots \arrow[r, "B"] & C_{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
& \ldots \arrow[d, "b"] \\
{} \arrow[d, "b"] & {} \\
A \arrow[r, "B"] & C_1 \arrow[d, "b"] \\
& A
\end{tikzcd}$$

The closest to the noncommutative De Rham complex is yet another version that we denote by $\mathrm{CD}_\bullet(A)$:

$$\begin{tikzcd}
A/[A,A] \arrow[r, "B"] & C_1/bC_2 \arrow[r, "B"] \arrow[d, "b"] & C_2/bC_3 \arrow[r, "B"] \arrow[d, "b"] & \dots \arrow[r] & C_n/bC_{n+1} \arrow[r, "B"] \arrow[d, "b"] & {} \\
 & A \arrow[r] & C_1 \arrow[r, "B"] \arrow[d] & \dots \arrow[r, "B"] & C_{n-1} \arrow[r, "B"] \arrow[d, "b"] & {} \\
 & & A \arrow[r, "B"] & \dots \arrow[r, "B"] & C_{n-2} \arrow[r] & {} \\
 & & & \dots & \dots \arrow[r] \arrow[d] & {} \\
 & & & & A \arrow[r] & {}
\end{tikzcd}$$

6. The extended HKR morphism

Define the map

$$\mathrm{HKR}_t : \mathrm{CD}_\bullet(A) \to \mathrm{DR}^\bullet(A) \tag{6.1}$$

as follows. For $k \geq 0$, put

(6.2)

$$(a_0 \otimes \ldots \otimes a_m) \mapsto \frac{1}{(m+k)!} \sum a_0 da_1 \ldots da_{j_1} t da_{j_1+1} \ldots da_{j_2} t \ldots t da_{j_k+1} \ldots da_m$$

where the sum is over all $0 \leq j_1 \leq \ldots \leq j_k \leq m$. The kth row from above in the (b, B) double complex $\mathrm{CD}_{\bullet,\bullet}(A)$ is mapped to the kth row from above in the (ι_t, d) double complex $\mathrm{DR_t}^{\bullet,\bullet}(A)$ by means of $\mathrm{HKR}^{(k)}$.

THEOREM 6.1. *The extended HKR map* (6.2) *is a quasi-isomorphism for every column and for the total complex. In particular, the homology of the complex* (4.6) *at* $\mathrm{DR}_t^{n-j,j}$ *is isomorphic to* $H_j(A, A)$.

As a consequence we recover Karoubi's theorem

THEOREM 6.2. *The nth cohomology of the relative noncommutative De Rham complex* $\overline{\mathrm{DR}}^\bullet(A)$ *is isomorphic to*

$$\mathrm{Ker}(B : \overline{H}C_n(A) \to \overline{H}C_{n+1}(A))$$

7. The extended De Rham complex and Hochschild homology

The extended De Rham complex is related to Hochschild homology in two related ways. First, it is obtained from the tensor algebra of the short bar resolution of the A-bimodule A. We can perform an analogous construction for the *full* bar resolution; we are not sure, however, how to show that the projection to the short version is a quasi-isomorphism. However, if we perform this analogous construction in the dual situation to the DG coalgebra $\mathrm{Bar}(A)$ instead of the algebra A, what we will get is essentially the extended De Rham complex. This is strange because $\mathrm{Bar}(A)$ is larger than A, and the full bar construction is larger than the short one. Such is life in Hilbert's hotel.

It is known that Hochschild and cyclic homology of an algebra can be computed in terms of the dual construction, namely Hochschild and cyclic homology of its bar construction ([**31**], [**11**]). The Hochschild homology that arises here is not only the usual $H_\bullet(A,A)$ but also the higher $H_\bullet(A^{\otimes n}, {}_\alpha A^{\otimes n})$ for all n, where α is the cyclic permutation viewed as an algebra automorphism of $A^{\otimes n}$. But this is isomorphic to $H_\bullet(A,A)$. Similarly, its cyclic version is isomorphic to the cyclic homology [**20**]. This is how we prove Theorem 6.1. We outline the proof in 7.2; full details are in [**30**].

7.1. The extended De Rham complex in terms of the short bar resolution. Recall the standard bar resolution:

$$\mathcal{B}_n(A) = A\otimes \overline{A}^{\otimes n}\otimes A; b' : \mathcal{B}_n(A)\to\mathcal{B}_{n-1}(A)$$

$$b'(a_0\otimes\ldots\otimes a_{n+1}) = \sum_{j=0}^{n}(-1)^j a_0\otimes\ldots\otimes a_j a_{j+1}\otimes\ldots\otimes a_{n+1}$$

If we extend this to $\mathcal{B}_{-1}(A)=A$, we get the bimodule resolution of A.

Define

$$\mathcal{B}_1^{\mathrm{sh}}(A) = \Omega^1_A \xrightarrow{\sim} \mathcal{B}_1(A)/\partial\mathcal{B}_2(A) \tag{7.1}$$

$$\mathcal{B}_\bullet^{\mathrm{sh},(0)}(A) = A;$$

$$\mathcal{B}_\bullet^{\mathrm{sh},(n)}(A) = \mathcal{B}_\bullet^{\mathrm{sh}}(A)\otimes_A\ldots\otimes_A \mathcal{B}_\bullet^{\mathrm{sh}}(A)$$

(n-fold tensor product for $n\geq 1$);

$$\mathcal{B}_\bullet^{\mathrm{sh},(*)}(A) = \bigoplus_{n\geq 0}\mathcal{B}_\bullet^{\mathrm{sh},(n)}(A)$$

Then we have

$$\mathrm{DR}_t^\bullet(A)\xrightarrow{\sim}\mathcal{B}_\bullet^{\mathrm{sh},(*)}(A)/[\mathcal{B}_\bullet^{\mathrm{sh},(*)}(A),\mathcal{B}_\bullet^{\mathrm{sh},(*)}(A)] \tag{7.2}$$

This construction is used in [**35**] where it is denoted by $\Upsilon^{(*)}(A)$.

7.2. The extended De Rham complex and the bar construction. Let $\mathrm{DR}^\bullet_{t,+}(A)$ be the subcomplex of $\mathrm{DR}^\bullet_t(A)$ spanned by elements whose degree with respect to t is positive. This subcomplex can be expressed in the form that we are going to discuss next.

Let us start with any associative unital differential algebra $(\mathcal{A},\partial)$. View $\mathcal{A}$ as a graded algebra. Introduce a new generator ϵ of degree one and square zero. Consider the cross product algebra

$$\widetilde{\mathcal{A}} = k[\epsilon]\ltimes\mathcal{A} \tag{7.3}$$

generated by ϵ and $\mathcal{A}$ subject to a relation $[\epsilon,a]=\partial a$ for all a in $\mathcal{A}$.

In other words, $\widetilde{\mathcal{A}}$ is generated by the algebra $\mathcal{A}$ and by elements $\underline{a}=\epsilon a$, $a\in\mathcal{A}$, of degree $|a|+1$, linear in a and subject to relations

$$\underline{a}\cdot b = \underline{ab};\ a\cdot\underline{b} = (-1)^{|a|}(\underline{ab}-\partial a\cdot b);\ \underline{a}\cdot\underline{b} = (-1)^{|a|-1}\underline{\partial a\cdot b} \tag{7.4}$$

Now one can consider the reduced cyclic homology $\overline{\mathrm{HC}}_\bullet(\widetilde{\mathcal{A}})$ of the graded algebra $\widetilde{\mathcal{A}}$. More precisely, we will compute it using the following specific complex defined for any A:

$$\overline{\mathrm{CC}}'(A) = (\mathrm{Ker}(1-\mathrm{t}),\mathrm{b}'); \mathrm{CC}'(\mathrm{A}) = \overline{\mathrm{CC}}'(\mathrm{A})/\overline{\mathrm{CC}}'(\mathrm{k}) \tag{7.5}$$

where $1-t$ and N are as in the standard $(b, b', 1-t, N)$ double complex. (Recall that

$$(\mathrm{Ker}(1-\mathrm{t}), \mathrm{b}') = (\mathrm{Im}(\mathrm{N}), \mathrm{b}') \xrightarrow{\sim} (\mathrm{C}(\mathrm{A})/\mathrm{Ker}(\mathrm{N}), \mathrm{b}) = (\mathrm{C}(\mathrm{A})/\mathrm{Im}(1-\mathrm{t}), \mathrm{b})$$

and therefore $\overline{\mathrm{CC}}'(A)$ does compute the cyclic homology).

Consider now a dual picture. Let $\mathcal{C}$ be a differential graded counital coalgebra $(\mathcal{C}, \partial)$. For $c \in \mathcal{C}$, let $\underline{c}$ be a formal element of degree $|c|+1$, linear in c. These elements generate the space $\underline{\mathcal{C}}$ which is same as $\mathcal{C}$ but with the grading shifted by one. Let $\widetilde{\mathcal{C}}$ be the graded coalgebra which is a linear direct sum of $\mathcal{C}$ and $\underline{\mathcal{C}}$. The comultiplication is as follows:

$$\Delta c = \sum c^{(1)} \otimes c^{(2)} + \sum (-1)^{|c^{(1)}|} \partial c^{(1)} \otimes \underline{c}^{(2)} \tag{7.6}$$

$$\Delta \underline{c} = \sum \underline{c}^{(1)} \otimes c^{(2)} + (-1)^{|c^{(1)}|} c^{(1)} \otimes \underline{c}^{(2)} + \sum (-1)^{|c^{(1)}|} \underline{\partial c^{(1)}} \otimes \underline{c}^{(2)} \tag{7.7}$$

For any counital DG coalgebra C put

$$\overline{\mathrm{CC}}'(C) = (\mathrm{Coker}(1-\mathrm{t}), \mathrm{b}'); \mathrm{CC}'(\mathrm{C}) = \mathrm{Ker}(\overline{\mathrm{CC}}'(\mathrm{C}) \to \overline{\mathrm{CC}}'(\mathrm{k})) \tag{7.8}$$

Lemma 7.1. *Let $A = k + \overline{A}$ be the algebra obtained from an $\widetilde{\text{algebra}}$ $\overline{A}$ by attaching a unit. Then the complex $\mathrm{DR}^{\bullet}_{t,+}(A)$ is isomorphic to $\mathrm{CC}'(\widetilde{\mathrm{Bar}(\overline{A})})$ where* Bar *stands for the usual bar construction (which is a DG coalgebra).*

Proof. Take a monomial $\omega_1 t \omega_2 t \dots \omega_n t$ in $\mathrm{DR}_{t,+}$. Identify it with $\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_n$ in $\mathrm{CC}'(\widetilde{\mathrm{Bar}(\overline{A})})$ where $\alpha_k = (a_0|a_1|\dots|a_m)$ if $\omega_k = a_0 da_1 \dots da_m$ and $\alpha_k = \underline{(|a_1|\dots|a_m)}$ if $\omega_k = da_1 \dots da_m$. One checks that this gives an isomorphism of complexes. □

7.3. The filtration on the extended De Rham complex. Consider the following filtration on noncommutative forms. We say that a monomial

$$\alpha_1 t \dots t \alpha_N t,\ \alpha_j \in \Omega^{\bullet}(A), \tag{7.9}$$

lies in $\mathcal{F}^p$ if at least p forms α_j are in $d\Omega^{\bullet}$.

We claim that $\mathrm{gr}^*_{\mathcal{F}}(\mathrm{DR}_t{}^{\bullet}(A))$ is dual to (7.2) if one replaces the short bar resolution $\mathcal{B}^{\mathrm{sh}}$ by the full bar resolution $\mathcal{B}$ and the algebra A by the graded coalgebra $T(\overline{A}[1])$, the free coalgebra of $\overline{A}[1]$ where $\overline{A} = A/k\cdot 1$. (This may sound a bit strange since full $\mathcal{B}_{\bullet}$ is larger than $\mathcal{B}^{\mathrm{sh}}_{\bullet}$ and $T(\overline{A}[1])$ is larger than A. Such is life in Hilbert's hotel).

More precisely, for any graded coalgebra C, denote

$$\bar{C}^{\bullet}_{\mathrm{II}}(C)^{(0)} = \overline{\mathrm{CC}}^{\bullet}_{\mathrm{II}}(C) = (\bar{C}^{\otimes(\bullet+1)})_{C_{\bullet+1}} \tag{7.10}$$

with the differential dual to b (or take invariants with the differential dual to b'). For $n > 0$

$$C^{\bullet}_{\mathrm{II}}(C)^{(n)} = (\bar{C}[-1]^{\otimes\bullet} \otimes C \otimes \dots \otimes \bar{C}[-1]^{\otimes\bullet} \otimes C)^{C_n} \tag{7.11}$$

(the tensor product on the right is n-fold); the differential sends a monomial

$$c_1^{(1)} \otimes \dots c_{m_1}^{(1)} \otimes x_1 \otimes \dots \otimes c_1^{(n)} \otimes \dots c_{m_n}^{(n)} \otimes x_n \tag{7.12}$$

to

$$\sum_{i=1}^{n} \sum_{j-1}^{m_i} \pm \dots \otimes \Delta(a_j^{(i)}) \otimes \dots + \sum_{j=1}^{n} (\dots \otimes \Delta'(x_j) \dots + \dots \otimes \Delta''(x_j) \dots)$$

where we use the following notation. First,

$$\Delta' : C \to \bar{C}[-1] \otimes C;\ \Delta'' : C \to C \otimes \bar{C}[-1] \tag{7.13}$$

is the comultiplication followed by the projection. Furthermore, the second half of the $j = n$ term is by definition

$$\pm x_n^{(2)} \otimes c_1^{(1)} \otimes \ldots c_{m_1}^{(1)} \otimes x_1 \otimes \ldots \otimes c_1^{(n)} \otimes \ldots c_{m_n}^{(n)} \otimes x_n^{(1)}$$

where

$$\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$$

To see this, take a monomial

$$\alpha_1^{(1)} t \ldots \alpha_{m_1}^{(1)} t d\beta_1 t \ldots t \alpha_1^{(n)} t \ldots \alpha_{m_n}^{(n)} t d\beta_n \tag{7.14}$$

in $\mathrm{gr}^n(\mathrm{DR}_t^+)$, and associate to it a monomial (7.14) by the following rule: to a form $a_0 da_1 \ldots da_k$, associate an element $(a_0|\ldots|a_k)$ of $T(\bar{A}[1])$. To see that, put

$$(d_+\Omega)^n = d\Omega^{n-1},\ n > 0;\ (d_+\Omega)^0 = k$$

and observe that:

(1)

$$\iota_t(a_0 da_1 \ldots da_n) = \sum_{k=1}^{n} (-1)^{k-1} (a_0 da_1 \ldots da_{k-1}) t (a_k da_{k+1} \ldots da_n)$$

$$\mathrm{mod}(d_+\Omega t\Omega + \Omega t d_+\Omega)$$

(2)

$$\iota_t(da_1 \ldots da_n) = \sum_{k=1}^{n} (-1)^k (a_1 da_2 \ldots da_k) t (da_{k+1} da_{k+2} \ldots da_n) +$$

$$\sum_{k=1}^{n} (-1)^{k-1} (da_1 \ldots da_{k-1}) t (a_k da_{k+1} \ldots da_n) \ \mathrm{mod}(d_+\Omega t d_+\Omega)$$

Now compute the spectral sequence of the filtration $\mathcal{F}$. The first term is the reduced cyclic cohomology of $T(\bar{A}[1])$. Since the coalgebra is cofree,

$$\overline{\mathrm{HC}}^{\bullet}_{\mathrm{II}}(T(\bar{A}[1])) \xrightarrow{\sim} \bigoplus_{n=1}^{\infty} (T^n(\bar{A}[1]))^{C_n} \tag{7.15}$$

By the dual version of [**20**], all homologies of $\mathrm{gr}^n_{\mathcal{F}}$ for $n > 0$ are isomophic to the Hochschild cohomology $\mathrm{HH}^{\bullet}_{\mathrm{II}}(T(\bar{A}[1]))$, which is computed by the short Hochschild complex

$$C^{\bullet}_{\mathrm{sh}}(T(\bar{A}[1])) = (T(\bar{A}[1]) \xrightarrow{b} T(\bar{A}[1]) \otimes \bar{A}[1]) \tag{7.16}$$

7.4. The spectral sequence of the filtration $\mathcal{F}$. We claim that the differential $\mathrm{gr}^k_{\mathcal{F}}\Omega^m_t \to \mathrm{gr}^{k+1}_{\mathcal{F}}\Omega^{m-1}_t$ acts on (7.16) as follows: on $T(\bar{A}[1])$, by b'; on $\bar{A}[1]\otimes T(\bar{A}[1])$, by b. To see that, we have to compare the short complex to the higher full complexes. Namely: we have to compute the compositions

$$C^\bullet_{\mathrm{sh}}(T(\bar{A}[1])) \to C^\bullet_{\mathrm{II}}(T(\bar{A}[1])) \to C^\bullet_{\mathrm{II}}(T(\bar{A}[1]))^{(n)} \tag{7.17}$$

and

$$C_{\mathrm{II}}(T(\bar{A}[1]))^{(n)} \to C^\bullet_{\mathrm{II}}(T(\bar{A}[1])) \to C^\bullet_{\mathrm{sh}}(T(\bar{A}[1])) \tag{7.18}$$

This is done in detail in [**30**].

This argument computes the E^2 term of the spectral sequence for all but two leftmost columns. As soon as we compute it for those two, we will have the theorem proven. Indeed, we will know that the extended HKR map from Proposition **??** is a quasi-isomorphism. Indeed, the truncated Hochschid complex

$$C_n(A)/bC_{n+1}(A) \to C_{n-1}(A) \to \ldots \to C_0(A) \tag{7.19}$$

has its own filtration

$$\mathcal{F}^{n-j}C_j(A) = C_j(A);\ \mathcal{F}^{n-j+1}C_j(A) = 1\otimes \bar{A}^{\otimes j}; \mathcal{F}^{n-j+2}C_j(A) = 0 \tag{7.20}$$

It is straightforward that the extended HKR map preserves the filtration. This finishes the proof of Theorem 6.1 contingent on Theorem 8.4 below.

8. The (ι_Δ, d) double complex

8.1. The differential ι_Δ.

Definition 8.1.

$$\iota_\Delta(a_0da_1\ldots da_n) = \sum_{j=1}^{n}(-1)^{n(j-1)}[a_j, da_{j+1}\ldots da_n a_0 da_1\ldots da_{j-1}]$$

This is just the composition

$$\Omega^n(A) \to \mathrm{DR}^n(A) \xrightarrow{\iota_t} \mathrm{DR}^{n-1,1}(A) \xrightarrow{\sim} \Omega^{n-1}(A)$$

First,

$$\iota_\Delta^2 = [d,\iota_\Delta] = d^2 = 0 \tag{8.1}$$

8.2. The Karoubi operator κ. First look at the component of HKR_t at level one. We get from (6.2)

$$\mathrm{HKR}^{(1)} = \frac{1}{(n+1)!}(1+\kappa+\ldots\kappa^n) \tag{8.2}$$

where

$$\kappa(a_0da_1\ldots da_n) = (-1)^{n-1}da_n\cdot a_0da_1\ldots da_{n-1} \tag{8.3}$$

Lemma 8.2. *On Ω^n one has*

$$\iota_\Delta = (1+\kappa+\ldots+\kappa^{n-1})b$$

This follows from HKR_t being a morphism of double complexes.

8.3. The Cuntz-Quillen projection P.

LEMMA 8.3.
$$(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$$
on Ω^n.

This is proven in [**7**], [**14**].

We see that $(1-\kappa)^2 P(\kappa) = 0$ where $P(x)$ and $1-x$ are coprime polynomials. Therefore we can define P to be the projection on $\mathrm{Ker}(1-\kappa)^2$ along other eigenspaces as a polynomial in κ. This is the Cuntz-Quillen projection. Put also $P^\perp = 1 - P$.

8.4. The main result. Here we summarize the main result from [**15**].

We will use the identification

$$C_\bullet(A) \xrightarrow{\sim} \Omega^\bullet(A); a_0 \otimes \ldots \otimes a_n \mapsto a_0 da_1 \ldots da_n \tag{8.4}$$

(the crude HKR). We get two pairs of commuting differentials on $\Omega^\bullet(A)$: one is (b, B) and the other is (ι_Δ, d). The comparison between the two is given by the following

THEOREM 8.4. (1) *The projections P and $P^\perp$ commute with b, B, ι_Δ, and d.*

(2)
$$\Omega^\bullet(A) = P\Omega^\bullet(A) \oplus P^\perp \Omega^\bullet(A)$$

(3) *On $P\Omega^\bullet(A)$: Let $(\mathcal{N}!)^{-1}$ be the operator whose restriction to $\Omega^n(A)$ is $\frac{1}{n!}\mathrm{Id}$. Then $(\mathcal{N}!)^{-1}$ intertwines b with ι_Δ and B with d.*

(4) *On $P^\perp\Omega^\bullet(A)$: $B = 0$; $\iota_\Delta = 0$; both b and d are contractible.*

Another way to express (3): $P \circ \mathrm{HKR}^{(0)}$ intertwines b with ι_Δ and B with d where

$$\mathrm{HKR}^{(0)}(a_0 \otimes \ldots \otimes a_n) = \frac{1}{n!} a_0 da_1 \ldots da_n \tag{8.5}$$

In particular we get a theorem from [**14**]:

THEOREM 8.5. *The periodic cyclic complex $(C_\bullet(A)((u)), b + uB)$ is quasi-isomorphic to $(\Omega^\bullet(A)((u)), \iota_\Delta + ud)$.*

9. The extended noncommutative De Rham complex and the representation scheme

Here we follow [**14**], [**15**], [**35**]. For an associative algebra A and a natural number n, let $\mathcal{O}_{\mathrm{Rep_n(A)}}$ be the algebra generated by elements $\rho_{jk}(a)$, k-linear in $a \in A$, $1 \le j, k \le n$, subject to relations

$$\rho_{ij}(ab) = \sum_{k=1}^{n} \rho_{ik}(a)\rho_{kj}(b)$$

In other words, points of the scheme $\mathrm{Rep_n(A)} = \mathrm{Spec}(\mathcal{O}_{\mathrm{Rep_n(A)}})$ are n-dimensional representations of A.

Let $(\Omega^\bullet_{\mathrm{Rep_n(A)}}, d)$ be the algebraic De Rham complex of $\mathcal{O}_{\mathrm{Rep_n(A)}}$. The group GL_n and the Lie algebra $\mathfrak{gl}_n$ act on it. Define the Cartan model of the equivariant De Rham complex

$$\Omega^{\bullet, \mathrm{GL}_n}_{\mathrm{Rep_n(A)}} = ((\Omega^\bullet_{\mathrm{Rep_n(A)}}[\mathfrak{g}l_n^*])^{\mathrm{GL}_n}, d + \iota_{\mathfrak{g}l_n}) \tag{9.1}$$

Then there is a morphism of complexes

$$\mathrm{DR}^\bullet_t(A) \to \Omega^{\bullet,\mathrm{GL}_n}_{\mathrm{Rep_n(A)}} \tag{9.2}$$

defined on generators by

$$a \cdot db \cdot c \mapsto \rho_{ij}(a) \cdot d\rho_{jk}(b) \cdot \rho_{ki}(c); \tag{9.3}$$

$$atc \mapsto \sum_{j,k} E^*_{jk} \otimes \rho_{jk}(ca); \tag{9.4}$$

and then extended multiplicatively.

This, and Waikit Yeung's version for multivector fields, provides a bridge between noncommutative geometry of an algebra and classical geometry of its representation scheme.

10. Noncommutative Poincaré lemma and the categorical nature of the differentials

LEMMA 10.1. *Let $f, g : A \to B$ be two morphism of algebras. Then f and g induce homotopic morphisms $(\Omega^\bullet(A), d) \to (\Omega^\bullet(B), d)$.*

PROOF. Follows immediately from the noncommutative Poincaré lemma 2.1 □

The homotopies can be easily constructed explicitly, and in more than one way. Start with

$$\Omega^n(A) \to \oplus_{j=0}^{n-1} \Omega^j(A) t \Omega^{n-1-j}(A) \tag{10.1}$$

be the differential ι_t (4.5) restricted to $\Omega^n(A)$ (i.e. to the homogenous part of degree zero in t of $\Omega^\bullet_t(A)$). We have two morphisms

$$\mu_1, \mu_2 : \Omega^k(A) t \Omega^l(A) \to \Omega^{k+l}(B) \tag{10.2}$$

$$\mu_1(\omega_1 t \omega_2) = f(\omega_1) g(\omega_2);\ \mu_2(\omega_1 t \omega_2) = (-1)^{kl} g(\omega_2) f(\omega_1) \tag{10.3}$$

LEMMA 10.2. *Both μ_1 and μ_2 are homotopies between $f, g : \Omega^\bullet(A) \to \Omega^\bullet(B)$.*

The proof is a direct computation.

Now let $A = B$ and $f = g = \mathrm{id}_A$. It follows from

$$[d, \mu_j] = f - g$$

that in this case

$$[d, \mu_j] = 0.$$

We have

$$\mu_1 = 0;\ \mu_2 = \iota_\Delta.$$

Is there a categorical reason for $\iota_\Delta^2 = 0$? Let us first tighten the structure. Namely, for any $f_1, \ldots, f_m : A \to B$ define

$$\mu_1(f_1, \ldots, f_m) : \Omega^\bullet(A) \to \Omega^{\bullet - m+1}(B) \tag{10.4}$$

as follows. Define

$$\iota_t^{(m)} : \Omega^\bullet(A) \to \Omega^\bullet(A) t \ldots t \Omega^\bullet(A)[1-m] \tag{10.5}$$

$$a_0 da_1 \ldots da_n \mapsto \sum_{1 \le j_1 < \ldots < j_m \le m} (-1)^{j_1 + \ldots + j_m - m} a_0 da_1 \ldots [t, a_{j_1}] \ldots [t, a_{j_m}] \ldots da_n$$

Also put $\mu_1(f) = f$.

LEMMA 10.3.
$$\sum_{j=2}^{m}(-1)^{j-1}\mu_1(f_1,\ldots,\widehat{f_j},\ldots,f_{m+1})+[d,\mu_1(f_1,\ldots,f_{m+1})]=0$$

The proof is by direct verification.

Let $\mathbf{c}(A,B)$ be the category whose objects are morphisms $f:A\to B$ and there is one morphism between any f and g. Lemma 10.3 states that morphisms μ_1 define a pairing

$$\Omega^\bullet(A)\otimes\mathbb{Z}\mathrm{N}(\mathbf{c}(A,B))\to\Omega^\bullet(B) \tag{10.6}$$

Here N stands for the nerve, and $\mathbb{Z}\mathrm{N}$ for the free simplicial Abelian group generated by the nerve and viewed as a chain complex.

There is also the product

$$\mathbb{Z}\mathrm{N}(\mathbf{c}(A,B))\otimes\mathbb{Z}\mathrm{N}(\mathbf{c}(B,C))\to\mathbb{Z}\mathrm{N}(\mathbf{c}(A,C)) \tag{10.7}$$

It is defined as follows. For $f_j:A\to B$ and $g_k:B\to C$,

$$(f_1|\ldots|f_m)(g_1|\ldots|g_n)=\sum\pm(\ldots|g_kf_j|\ldots) \tag{10.8}$$

where the sum is taken over all paths from g_1f_1 to g_nf_m using the following two moves:

$$\text{from } g_kf_j \text{ to } g_{k+1}f_j \text{ or to } g_kf_{j+1}$$

Every permutation of f_j and g_k introduces a factor -1. For example,

$$(f_1|f_2)(g_1|g_2)=(g_1f_1|g_1f_2|g_2f_2)-(g_1f_1|g_2f_1|g_2f_2)$$

LEMMA 10.4. *The product* (10.7) *is associative and compatible with the action* (10.6).

We do not know at the moment if μ_2 (and its partial case ι_Δ) is part of the structure described above. We know this for another related structure that we are going to describe next.

10.1. The Čech-Alexander complex. The noncommutative complex $\Omega^\bullet(A)$ is a quotient of another important complex:

$$A^{(n)}=A^{\otimes(n+1)};\ \check{\partial}:A^{(n)}\to A^{(n+1)};\ \check{\partial}=\sum_{j=0}^{n+1}(-1)^j\partial_j; \tag{10.9}$$

$$\partial_j(a_0\otimes\ldots\otimes a_n)=a_0\otimes\ldots\otimes 1\otimes a_j\otimes\ldots\otimes a_n \tag{10.10}$$

Define

$$\mu(f_1,\ldots,f_m)(a_0\otimes\ldots\otimes a_n)=$$
$$\sum\pm f_1(a_0)\otimes\ldots\otimes f_1(a_{j_1})f_2(a_{j_1+1})\otimes\ldots\otimes f_{m-1}(a_{j_{m-1}})f_m(a_{j_{m-1}+1})\otimes\ldots\otimes f_m(a_n)$$

The sum is taken over all $0\le j_1<\ldots<j_m\le n-1$. We get the pairing

$$A^{(\bullet)}\otimes\mathbb{Z}\mathrm{N}(\mathbf{c}(A,B))\to B^{(\bullet)} \tag{10.11}$$

LEMMA 10.5. *The product* (10.7) *is compatible with the action* (10.11).

There is an extra piece of structure here. Namely, the deconcatenation operation turns both $\mathbb{Z}\mathrm{N}(\mathbf{c}(A,B)$ and

$$\bar{A}^{(\bullet)}[1]=(A^{(\bullet)}/k^{(\bullet)})[1] \tag{10.12}$$

into differential graded coalgebras.

PROPOSITION 10.6. The product (10.7) and the pairing

$$\bar{A}^{(\bullet)}[1] \otimes \mathbb{Z}\mathrm{N}(\mathbf{c}(A,B)) \to \bar{B}^{(\bullet)}[1]$$

(induced by (10.11)) are morphisms of differential graded coalgebras.

When $A = B$ and $f = g = \mathrm{id}_A$ then we recover the bar differential

$$\mu(\mathrm{id},\mathrm{id})(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n \tag{10.13}$$

The identity $\mu(\mathrm{id},\mathrm{id})^2 = 0$ follows from (10.5) though of course it can be checked directly.

We have constructed a category in differential graded coalgebras; Čech-Alexander complexes form "a module in coalgebras" over it. This provides a strong form of the statement that all morphisms define the same morphism on Čech-Alexander cohomology.

REMARK 10.7. Of course the last sentence above is obviously true because the embedding $k^{(\bullet)} \to A^{(\bullet)}$ is a quasi-isomorphism. (In fact, for any section $s : A \to k$ of the embedding $k \to A$, the map

$$a_0 \otimes \ldots \otimes a_n \mapsto s(a_0)a_1 \otimes \ldots \otimes a_n \tag{10.14}$$

is a homotopy between that embedding and the identity). There are cases, however, when we can modify the complex and then its cohomology becomes interesting. For example, let A be a finitely generated commutative algebra. Let $\pi : P \to A$ be an epimorphism where P is a polynomial algebra in finitely many variables. Let I be the kernel of π. Denote by $\widehat{P}^{(n)}$ the $I^{(n)}$-adic completion of $P^{\otimes(n+1)}$ where

$$I^{(n)} = \mathrm{Ker}(P^{\otimes(n+1)} \to A)$$

In characteristic p one can also take the completed divided powers envelope. Then the differential $\check{\partial}$ and the maps $\mu(f_1, \ldots, f_m)$ extend to $P^{(\bullet)}$ while the homotopy (10.14) does not. For every $\pi : P \to A$ and $\rho : Q \to B$ and for every $f : A \to B$ there is a morphism $\widetilde{f} : P \to Q$ such that $\rho\widetilde{f} = f\pi$. Therefore the complex $P^{(\bullet)}$ depends, essentially, only on A. Lemma 10.5 and Proposition 10.6 also provide a recipe for gluing the complexes $P^{(\bullet)}$ for a sheaf of commutative algebras. This gives a definition of crystalline cohomology [**1**], [**18**].

10.2. Noncommutative Čech-Alexander complex. Now let R be an associative algebra. Let $R^{(n)}$ be the $(n+1)$-fold *free* product of R with itself. (This is the $(n+1)$-fold coproduct in the category of associative rather than commutative algebras). One defines the coface maps

$$\partial_j : R^{(n-1)} \to R^{(n)},\ 0 \le j \le n \tag{10.15}$$

as follows. We denote by R_k the kth copy of R in the free product. Then ∂_j is the morphism of algebras that sends R_k identically to R_k for $k < j$ and to R_{k+1} for $k \le j$.

Now let R and S be two associative algebras. Let $f_1, \ldots, f_m : R \to S$ be morphisms of algebras. The maps

$$\mu(f_1, \ldots, f_m) : R^{(n)} \to S^{(n-m+1)}$$

are defined analogously to the above. Namely, for any nondecreasing surjection $\sigma : \{0, \ldots, n\} \to \{0, \ldots, n-m+1\}$ let

$$h(\sigma) : R^{(n)} \to S^{(n-m+1)}$$

be the algebra morphism that sends R_k to $S_{\sigma(k)}$ by the morphism $f_{k-\sigma(k)+1}$. Then

$$\mu(f_1, \ldots, f_m) = \sum (-1)^{p(\sigma)} h(\sigma)$$

where

$$p(\sigma) = \sum_{l=0}^{n-m+1} l \mathrm{Card}(\sigma^{-1}(\{l\}))$$

The coalgebra structure on the bar construction also lifts from the commutative case. Define for $0 \leq j < n$

$$\Delta_{j,n} : R^{(n)} \to R^{(j)} \otimes R^{(n-1-j)}$$

to be the algebra morphism that sends R_k identically to $R_k \otimes 1$ for $k \leq j$ and to $1 \otimes R_{k-j-1}$ for $k > j$. Then

$$\Delta | R^{(n)} = \sum_{j=0}^{n-1} \Delta_{j,n} \tag{10.16}$$

defines a DG coalgebra structure on $R^{(\bullet)}[1]$.

Proposition 10.8. The product (10.7) and the pairing

$$\bar{R}^{(\bullet)}[1] \otimes \mathbb{Z}\mathrm{N}(\mathbf{c}(R, S)) \to \bar{S}^{(\bullet)}[1]$$

(induced by (10.11)) are morphisms of differential graded coalgebras. They satisfy the associativity condition when three algebras A, B, C are given.

10.2.1. *Noncommutative crystalline cohomology in characteristic zero.* Here we follow Cortiñas [**5**]. Let A be an associative algebra. Let $R \to A$ be an epimorphism where R is a free algebra. Let $J^{(n)} = \mathrm{Ker}(R^{(n)} \to A)$. Let $R^{(\bullet)}/[R^{(\bullet)}, R^{(\bullet)}]^{\frown}$ be the completion of $R^{(\bullet)}/[R^{(\bullet)}, R^{(\bullet)}]$ with respect to the filtration induced by powers of $J^{(\bullet)}$.

Theorem 10.9. [**5**] *The complex*

$$R_\natural{}^{(\bullet)\frown} = R^{(\bullet)}/[R^{(\bullet)}, R^{(\bullet)}]^{\frown}, \check{\partial}) \tag{10.17}$$

computes the periodic cyclic homology of A.

All the morphisms $\mu(f_1, \ldots, f_m)$ extend to the completion. The coalgebra structure descends to the quotient by commutators. Also, denote

$$\bar{R}_\natural^{(\bullet)\frown} = R_\natural{}^{(\bullet)\frown}/k_\natural{}^{(\bullet)\frown}$$

We get a stronger form of the statement [**5**] that the cohomology is independent of the choice of R :

Proposition 10.10. The product (10.7) and the pairing

$$\bar{R}_\natural^{(\bullet)\frown} \otimes \mathbb{Z}\mathrm{N}(\mathbf{c}(R, S)) \to \bar{S}_\natural^{(\bullet)\frown}$$

(induced by (10.11)) are morphisms of differential graded coalgebras that satisfy the associativity condition.

Let us say a few words about the proof of Theorem 10.17. In the commutative case, the classical proof starts with the Čech-Alexander-De Rham double complex and shows that its total complex is quasi-isomorphic to both Čech-Alexander and De Rham complexes. In the noncommutative case, the Čech-Alexander complex already *is* a version of the De Rham complex. This is shown by expressing the algebras $R^{(\bullet)}$ in terms of noncommutative forms (generalizing the case $\bullet = 1$ due to Cuntz and Quillen). After that, the theorem follows from Karoubi's Theorem 6.2 and Goodwillie's theorem [**17**].

10.3. Bar construction and Hochschild cochains. There is another, possibly related, example of the structure discussed above. Namely, replace the additive category $\mathbb{Z}\mathbf{c}(A, B)$ by the differential graded category $\mathbf{C}^\bullet(A, B)$ as follows. Its objects are morphisms $f : A \to B$. The complex of morphisms is

$$\mathbf{C}^\bullet(A, B)(f, g) = C^\bullet(A, {}_fB_g) = \mathrm{Hom}_k(A^{\otimes \bullet}, B), \tag{10.18}$$

the Hochschild cochain complex of A with coefficients in B viewed as an A-bimodule with the action $a_1 \cdot b \cdot a_2 = f(a_1)bg(a_2)$. The composition is given by the cup product.

There are morphisms of DG coalgebras

$$\mathrm{Bar}(\mathbf{C}^\bullet(A, B)) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(B, C)) \to \mathrm{Bar}(\mathbf{C}^\bullet(A, B)) \tag{10.19}$$

$$\mathrm{Bar}(A) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(B, C)) \to \mathrm{Bar}(B) \tag{10.20}$$

The first one is associative and the two agree with each other. The morphisms are a categorized version of the morphisms based on brace operations [**12**], [**13**]. The details can be found in [**34**] and [**30**].

This one way of saying that algebras form a two-category, i.e. a category in DG categories: the above says that they form a (strict) category in DG cocategories; one can apply the bar construction and define a version of a 2-category up to homotopy.

Remark 10.11. One gets a feeling that the algebraic structures described in (10.6), Propositions 10.6 and 10.10, (10.19), and (10.20) are parts of the same structure.

References

1. P. Berthelot *Cohomologie Cristalline des Schémas de Caractéristique* $p > 0$, Lecture Notes in Math. **407**, Springer-Verlag, New York (1974).
2. A. Connes, *Non Commutative Differential Geometry, Part I, The Chern Character in K-Homology,* Preprint I.H.E.S. (1982).
3. A. Connes, *Non Commutative Differential Geometry, Part II, De Rham Homology and Non Commutative Algebra*, Preprint I.H.E.S. (1983).
4. A. Connes, *Non-commutative differential geometry*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques volume **62**, pages41–144 (1985).
5. G. Cortiñas, *Periodic cyclic homology as sheaf cohomology*, Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part II. K-Theory **20** (2000), no. 2, 175–200.
6. G. Cortiñas, *Infinitesimal K-theory*, J. Reine Angew. Math. **503** (1998), 129–160.
7. J. Cuntz, D. Quillen, *Operators on noncommutative differential forms and cyclic homology*, Geometry, topology, and physics, 77–111, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.
8. W. G. Dwyer, D. M. Kan, *Three homotopy theories for cyclic modules*, Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), J. Pure Appl. Algebra **44** (1987), no. 1-3, 165–175.
9. W. G. Dwyer, D. M. Kan, *The homotopy theory of cyclic sets*,Trans. Amer. Math. Soc. **291** (1985), no. 1, 281–289.

10. W. G. Dwyer, D. M. Kan, *Normalizing the cyclic modules of Connes*, Comment. Math. Helv. **60** (1985), no. 4, 582–600.
11. B. Feigin, B. Tsygan, *Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math. **1289**, Springer, Berlin, 210–239, (1987).
12. M. Gerstenhaber, A. Voronov, *Homotopy G-algebras and moduli space operad*, Internat. Math. Res. Notices, no. 3 (1995), 141–153.
13. E. Getzler, J. D. S. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, hep-th/9305013, 1–70 (1993).
14. V. Ginzburg, T. Schedler, *Free products, cyclic homology, and the Gauss-Manin connection* (Appendix by B. Tsygan), Adv. Math. **231** (2012), no. 3–4, 2352–2389.
15. V. Ginzburg, T. Schedler, *A new construction of cyclic homology*, Proc. Lond. Math. Soc. (3) **112** (2016), no. 3, 549–587.
16. A. Goncharov, M. Kontsevich, *Spectral description of non-commutative local systems on surfaces and non-commutative cluster varieties*, arXiv:2108.04168.
17. T. Goodwillie, *Cyclic homology, derivations, and the free loopspace*, Topology, **24** (2) (1985), pp. 187-215.
18. A. Grothendieck, *Crystals and the de Rham cohomology of schemes*, Adv. Stud. in pure Math., 306–358, North-Holland (1968).
19. G. Hochschild, B. Kostant, A. Rosenberg, *Differential forms on regular affine algebras*, Transactions AMS **102** (1962), No.3, 383–408.
20. D. Kaledin, *Cartier isomorphism for unital associative algebras*, Proc. Steklov Inst. Math., **290** (2015), 35–51.
21. M. Karoubi, *Homologie ciclique et K-théorie*, Astérisque **149** (1987), 1–147.
22. M. Karoubi, *Corréspondance de Dold-Kan et formes différentielles*, J. Algebra **198** (1997), no. 2, 618–626.
23. M. Karoubi, *Formes différentielles non commutatives et opérations de Steenrod*, Topology **34** (1995), no. 3, 699–715.
24. M. Karoubi, *Formes différentielles non commutatives et cohomologie à coefficients arbitraires*, Trans. Amer. Math. Soc. **347** (1995), no. 11, 4277–4299.
25. M. Karoubi, *Formes topologiques non commutatives*, (French), Ann. Sci. École Norm. Sup. (4) **28** (1995), no. 4, 477–492.
26. M. Karoubi, *Méthodes quantiques en topologie algébrique*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 9, 755–758.
27. M. Karoubi, *Quantum methods in algebraic topology*, Topology, geometry, and algebra: interactions and new directions (Stanford, CA, 1999), 177–193, Contemp. Math., **279**, Amer. Math. Soc., Providence, RI, 2001.
28. M. Karoubi, *Braiding of differential forms and homotopy types*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 10, 757–762.
29. J. L. Loday, Cyclic homology, Gründlehren der Mathematischen Wissenschaften, vol. **301**, Springer, Berlin, 1992.
30. R. Nest, B. Tsygan, *Cyclic Homology*, book in progress, available at:
31. D. Quillen, *Algebra cochains and cyclic cohomology*, Inst. Hautes Études Sci. Publ. Math. **68** (1988), 139–174 (1989). https://sites.math.northwestern.edu/~tsygan/Part1.pdf
32. 10. G. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222.
33. B. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology* (Russian), Uspekhi Mat. Nauk **38** (1983), no. 2(230), 217–218 https://sites.math.northwestern.edu/~tsygan/Part1.pdf
34. B. Tsygan, *Noncommutative calculus and operads*,Topics in noncommutative geometry, Clay Math. Proc., **16**, 19–66. Amer. Math. Soc., Providence, RI (2012).
35. Wai-Kit Yeung, *Pre-Calabi-Yau structures and moduli of representations*, arXiv:1802.05398.

Department of Mathematics, Northwestern University, Evanston IL 60208 US
Email address: `b-tsygan@northwestern.edu`