Categorical Weil Representation & Sign Problem

Shamgar Gurevich

Madison

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Joint work with:

- **Ronny Hadani (Math, Austin)**

Theorem (Canonical vector space, G–Hadani '04)

There exists a natural functor

$$\mathcal{H} : \underset{over\ k=\mathbb{F}_q}{Symp} \longrightarrow \underbrace{Vect}_{over\ \mathbb{C}}.$$

(0) Motivation - CANONICAL CATEGORY

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- For $V \in Symp$ we have

$$\rho_V : Sp(V) \to GL(\mathcal{H}(V)) \quad \text{— Weil representation.}$$

- **Want:** lax 2-functor

$$\underbrace{\mathbf{Symp}}_{\text{In } Var \text{ over } k} \ni \mathbf{V} \mapsto \mathcal{C}(\mathbf{V}) \text{ — canonical category of } \ell\text{-adic sheaves.}$$

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- For $\mathbf{V} \in \mathbf{Symp}$ get

$$\rho_{\mathbf{V}} : Sp(\mathbf{V}) \to Aut(\mathcal{C}(\mathbf{V})) \text{ — categorical Weil representation.}$$

(I) Canonical Vector Space - CONSTRUCTION

- Heisenberg group

$$H = V \times k, \quad (v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\omega(v, v')).$$

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- Irreducible rep'n of H with central character ψ

$$\mathcal{H}_{L^\circ} = \{f : H \to \mathbb{C}; f(l \cdot z \cdot h) = \psi(z) f(h) \ \text{ for } l \in L, z \in Z, h \in H\}.$$

Canonical Vector Space - CONSTRUCTION

- Vector bundle

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Theorem (Strong S-vN, G–Hadani '04)

We have a natural Sp-equivariant trivialization: $\{T_{M^\circ,L^\circ} : \mathcal{H}_{L^\circ} \to \mathcal{H}_{M^\circ}\}$ *with*

$$T_{N^\circ,M^\circ} \circ T_{M^\circ,L^\circ} = T_{N^\circ,L^\circ}, \quad \textit{for every } N^\circ, M^\circ, L^\circ.$$

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- **Canonical vector space**

$$\mathcal{H}(V) = \{ \ (f_{L^\circ} \in \mathcal{H}_{L^\circ}, \ L^\circ \in OLag) \text{ with } T_{M^\circ,L^\circ}(f_{L^\circ}) = f_{M^\circ} \ \}.$$

Canonical Vector Space - KERNELS

- Kernels

$$\begin{cases} \mathbb{C}(M\backslash H/L,\psi)\widetilde{\rightarrow}Hom_H(\mathcal{H}_{L^\circ},\mathcal{H}_{M^\circ}), \\ \qquad\qquad K_{M^\circ,L^\circ} \longmapsto T_{M^\circ,L^\circ}. \end{cases}$$

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- Function of kernels

$$\begin{cases} K \in \mathbb{C}(OLag^2 \times H), \\ K * K = K. \end{cases}$$

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- Canonical vector space

$$\mathcal{H}(V) = \{\ f \in \mathbb{C}(OLag \times H) \text{ with } K * f = f\}.$$

(II) Geometric Kernels - DEFINITION

Theorem (Geometrization, G-Hadani '06))

There exists a geometrically irreducible, perverse, ℓ-adic Weil sheaf

$\underbrace{\mathcal{K}}_{\text{sheaf of kernels}}$ *on* $\mathbf{OLag}^2 \times \mathbf{H}$ *with*

1. *Convolution.* *Canonical isomorphism* $\theta : \mathcal{K} * \mathcal{K} \widetilde{\rightarrow} \mathcal{K}$.
2. *Function. We have* $\underbrace{f^{\mathcal{K}}}_{\text{sheaf-to-function}} = K$.

Geometric Kernels - SIGN PROBLEM

- Consider the commutative diagram with scalar morphism $C = c \cdot Id$

$$\begin{array}{ccc}
(\mathcal{K} * \mathcal{K}) * \mathcal{K} & \xrightarrow{\alpha} & \mathcal{K} * (\mathcal{K} * \mathcal{K}) \\
\downarrow \theta * id & & \downarrow id * \theta \\
\mathcal{K} * \mathcal{K} & & \mathcal{K} * \mathcal{K} \\
\downarrow \theta & & \downarrow \theta \\
\mathcal{K} & \xrightarrow{C} & \mathcal{K}
\end{array}$$

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$$\begin{tikzcd}
(\mathcal{K} * \mathcal{K}) * \mathcal{K} \arrow[r, "\alpha"] \arrow[d, "\theta * id"] & \mathcal{K} * (\mathcal{K} * \mathcal{K}) \arrow[d, "id * \theta"] \\
\mathcal{K} * \mathcal{K} \arrow[d, "\theta"] & \mathcal{K} * \mathcal{K} \arrow[d, "\theta"] \\
\mathcal{K} \arrow[r, "C"] & \mathcal{K}
\end{tikzcd}$$

Problem (The sign problem, Bernstein–Deligne)

Compute the scalar $c = ?$.

Sign Problem - SOLUTION

Theorem (G–Hadani '11, with Gabber)

We have $c = 1$.

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Proof.

$$\begin{tikzcd}
((\mathcal{K} * \mathcal{K}) * \mathcal{K}) * \mathcal{K} \arrow[r, "\alpha"] \arrow[d, "\alpha * id"] & (\mathcal{K} * \mathcal{K}) * (\mathcal{K} * \mathcal{K}) \arrow[d, "\alpha"] \\
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- By successive application of θ, each term is identified with $\mathcal{K}$, and by naturality of α, the arrows become C.

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- Hence $C^3 = C^2$, so $C = 1$.

(III) Canonical Category - DEFINITION

Definition

We define

$$\mathcal{C}(\mathbf{V}) = \left\{ \begin{array}{c} (\mathcal{F}, \eta), \\ \mathcal{F} \in D^b(\mathbf{OLag} \times \mathbf{H}), \\ \eta : \mathcal{K} * \mathcal{F} \widetilde{\rightarrow} \mathcal{F}, \end{array} \right.$$

such that η is compatible with α and θ, i.e., the following diagram is commutative

$$\begin{tikzcd}
(\mathcal{K} * \mathcal{K}) * \mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\theta * id"] & \mathcal{K} * (\mathcal{K} * \mathcal{F}) \arrow[d, "id * \eta"] \\
\mathcal{K} * \mathcal{F} \arrow[d, "\eta"] & \mathcal{K} * \mathcal{F} \arrow[d, "\eta"] \\
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\end{tikzcd}$$

We call $\mathcal{C}(\mathbf{V})$ the **canonical category** associated with $\mathbf{V} \in \mathbf{Symp}$.

THANK YOU

Ronny