The Geometric Weil Representation

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Joint work with:

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(0) Motivation - GEOMETRIZING FOURIER SYMMETRIES

- Discrete Fourier Transform

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- ℓ-adic sheaves (Grothendieck 60s)

$$D_c^b(\mathbb{A}^1) \quad \rightsquigarrow \quad \mathbb{C}(\mathbb{F}_q).$$

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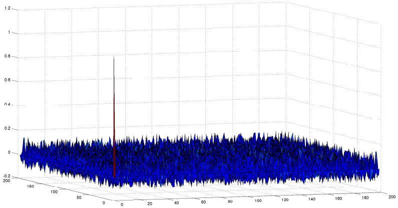
- Application: For $SL_2(\mathbb{F}_q) \supset T$, $\varphi_\chi \in \mathcal{H}_\chi$,

$$\left| \left\langle \varphi_\chi \, , \; \pi(\tau,\omega)\varphi_\chi \right\rangle \right| \leq \frac{2}{\sqrt{q}}, \quad (\tau,\omega) \in \mathbb{F}_q \times \mathbb{F}_q - (0,0).$$



$q = 199$

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Theorem (Stone–von Neumann)

There exists a unique (up to $\simeq$) irreducible representation $\underbrace{\pi : H \to GL(\mathcal{H})}_{\text{Heisenberg rep'n}}$ *s.t.* $\pi(z) = \psi(z) \cdot Id_{\mathcal{H}}$, $z \in Z(H) = k$.

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$$\pi \overset{\rho(g)}{\widetilde{\longrightarrow}} \pi^g(v, z) = \pi(gv, z),$$

  i.e.,
$$\rho(g)\pi(h)\rho(g)^{-1} = \pi(g[h]), \quad \text{for every } h \in H. \qquad (1)$$

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Problem

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3. **Formula** (G–Hadani): On $U = \{g \in Sp; \det(g - I) \neq 0\}$

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- *Proof.* Geometric Weil Rep'n.

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Theorem (Pseudo-randomness, G–Hadani)

For $v \neq 0$ we have

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Theorem (Deligne, Weil Conjectures II)

$\mathbf{X}/\mathbb{F}_q$, $\dim \mathbf{X} = 1$, $(\mathcal{F}, \bigtriangledown)$ *(Hermitian) line bundle with flat connection. Then*

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- In our case $\mathcal{F}$ is explicit. Can compute $c = 2$.

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- (3) *Topology.*

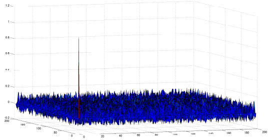
Theorem (Deligne, Weil Conjectures II)

$\mathbf{X}/\mathbb{F}_q$, $\dim \mathbf{X} = 1$, $(\mathcal{F}, \nabla)$ *(Hermitian) line bundle with flat connection. Then*

$$\left| \sum_{x \in \mathbf{X}} f^{\mathcal{F}}(x) \right| \leq c \cdot \sqrt{q},$$

iff $\mathcal{F}$ has non-trivial monodromy.

  - In our case $\mathcal{F}$ is explicit. Can compute $c = 2$.
- Done!



$$\left\langle \pi(v)\varphi_\chi \, , \, \varphi_\chi \right\rangle, \; q = 199$$

THANK YOU

Ronny