The asymptotic behaviour of doubly periodic instantons and Stokes structure

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Introduction

Let T be an elliptic curve over $\mathbb{C}$. For $\lambda \in \mathbb{C}$, let $\mathscr{M}^\lambda$ denote the moduli space of line bundles of degree 0 with a flat λ-connection.

$$\mathscr{M}^\lambda := \big\{(L, \mathbb{D}^\lambda) \,\big|\, L \in \mathrm{Pic}_0(T),\ \mathbb{D}^\lambda\text{:}\lambda\text{-connection of } L\big\}/\sim$$

A flat λ-connection is a differential operator $\mathbb{D}^\lambda : L \longrightarrow L \otimes \Omega^1_T$ such that (i) $\mathbb{D}^\lambda(fs) = f\mathbb{D}^\lambda s + (\lambda\partial_T + \overline{\partial}_T)f \cdot s$ for $f \in C^\infty(T), s \in C^\infty(T, L)$, (ii) $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$.

The space $\mathscr{M}^\lambda$ is affine space bundle over $T^\vee := \mathrm{Pic}_0(T)$

Goal

1. The behaviour of holomorphic vector bundles on $\mathscr{M}^\lambda$ around ∞. (Hukuhara-Turrittin type theorem, Stokes structure....)
2. Application to instantons on $T^\vee \times \mathbb{C}$.

Vector bundles on $\mathscr{M}^\lambda$

Let $T = \mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$. Then $T^\vee := \mathrm{Pic}_0(T) \simeq \mathbb{C}/\Lambda^\vee$, where

$$\Lambda^\vee := \big\{z \in \mathbb{C} \,\big|\, \mathrm{Im}(z\overline{\zeta}) \in \pi\mathbb{Z}\ \forall \zeta \in \Lambda\big\}.$$

The identification is induced by $\mathbb{C} \ni z \longmapsto (\underline{\mathbb{C}}, \overline{\partial}_T + z\, d\overline{\zeta})$ ($\underline{\mathbb{C}} = T \times \mathbb{C}$.)

$\mathscr{M}^\lambda$ is described as the quotient $\mathscr{M}^\lambda = \big\{(\xi, \eta) \in \mathbb{C}^2\big\}/\sim$

$$(\xi, \eta) \sim (\xi + \chi, \eta - \lambda\overline{\chi}) \quad \exists \chi \in \Lambda^\vee$$

The identification is induced by $(\xi, \eta) \longmapsto (\underline{\mathbb{C}}, \overline{\partial}_T + \lambda\partial_T + \xi\, d\overline{\zeta} + \eta\, d\zeta)$.

$$(\underline{\mathbb{C}}, \overline{\partial}_T + \lambda\partial_T + \xi\, d\overline{\zeta} + \eta\, d\zeta) \simeq (\underline{\mathbb{C}}, \overline{\partial}_T + \lambda\partial_T + (\xi + \chi)\, d\overline{\zeta} + (\eta - \lambda\overline{\chi})d\zeta) \iff \chi \in \Lambda^\vee$$

The isomorphism is induced by $\rho_\chi(\zeta) = \exp(2\sqrt{-1}\,\mathrm{Im}(\chi\overline{\zeta})) = \exp(\chi\overline{\zeta} - \overline{\chi}\zeta)$ on T

$$\rho_\chi(\zeta)^{-1} \circ (\overline{\partial}_T + \lambda\partial_T) \circ \rho_\chi(\zeta) = (\overline{\partial}_T + \lambda\partial_T) + \chi\, d\overline{\zeta} - \lambda\overline{\chi}\, d\zeta$$

The fibration $\mathscr{M}^\lambda \longrightarrow T^\vee$ is given by $(\xi, \eta) \longmapsto \xi$.

$\mathscr{M}^0 = T^\vee \times \mathbb{C}$. We use the natural coordinate (z, w). We have a natural diffeomorphism $\mathscr{M}^0 \simeq \mathscr{M}^\lambda$ given by

$$(\xi, \eta) = (z + \lambda\overline{w}, -\lambda\overline{z} + w).$$

$$\overline{\partial}_\xi = \frac{1}{1 + |\lambda|^2}(\overline{\partial}_z + \lambda\partial_w), \quad \overline{\partial}_\eta = \frac{1}{1 + |\lambda|^2}(-\lambda\partial_z + \overline{\partial}_w)$$

We have the projection $\Psi_0 : \mathscr{M}^0 \longrightarrow \mathbb{C}$ given by $(z, w) \longmapsto w$. The induced C^∞-map $\mathscr{M}^\lambda \longrightarrow \mathbb{C}$ is also denoted by Ψ_0.

A holomorphic vector bundle on $\mathscr{M}^\lambda$ is equipped with commutative actions of $\overline{\partial}_\xi$ and $\overline{\partial}_\eta$. We regard them as a generalization of a flat λ-connection on $\mathbb{C}$ through Ψ_0.

λ-flat bundle

A λ-flat bundle on a complex manifold X is a C^∞-bundle V with a differential operator $\mathbb{D}^\lambda : V \longrightarrow V \otimes \Omega^1_X$ such that

$$\mathbb{D}^\lambda(fs) = f\mathbb{D}^\lambda s + (\lambda\partial_X + \overline{\partial}_X)f \cdot s \quad f \in C^\infty(X),\ s \in C^\infty(X, V)$$
$$\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$$

- If $X \subset \mathbb{C}$, a flat λ-connection is given by commutative actions $\mathbb{D}^\lambda_w$ and $\mathbb{D}^\lambda_{\overline{w}}$, satisfying

$$\mathbb{D}^\lambda_w(fs) = f\mathbb{D}^\lambda_w s + \lambda\partial_w f \cdot s \qquad \mathbb{D}^\lambda_{\overline{w}}(fs) = f\mathbb{D}^\lambda_{\overline{w}} s + \overline{\partial}_w f \cdot s.$$

Hitchin transform

Let $X \subset \mathbb{C}$ be an open subset. We obtain an open subset $\Psi_0^{-1}(X) \subset \mathscr{M}^\lambda$.

λ-flat bundle on X $\Longrightarrow$ holomorphic bundle on $\Psi_0^{-1}(X)$

Let $(V, \mathbb{D}^\lambda)$ be a λ-flat bundle on X. Because $\Psi_0 : \mathscr{M}^0 \longrightarrow \mathbb{C}$ is holomorphic, $\Psi_0^{-1}(V)$ is equipped with a flat λ-connection $\mathbb{D}^\lambda$ on $\mathscr{M}^0$. We set

$$\overline{\partial}_\xi := \frac{1}{1 + |\lambda|^2}(\mathbb{D}^\lambda_{\overline{z}} + \mathbb{D}^\lambda_w), \quad \overline{\partial}_\eta := \frac{1}{1 + |\lambda|^2}(-\mathbb{D}^\lambda_z + \mathbb{D}^\lambda_{\overline{w}})$$

We obtain a holomorphic vector bundle $\Psi_0^*(V, \mathbb{D}^\lambda) := (\Psi_0^{-1}(V), \overline{\partial}_\xi, \overline{\partial}_\eta)$ on $\Psi_0^{-1}(X) \subset \mathscr{M}^\lambda$.

Proposition We have the following equivalence:

$$\begin{pmatrix} T^\vee\text{-equivariant} \\ \text{holomorphic vector bundle} \\ \text{on } \Psi_0^{-1}(X) \end{pmatrix} \longleftrightarrow \begin{pmatrix} \lambda\text{-flat bundle} \\ \text{on } X \end{pmatrix}$$

Push-forward

Let $\mathscr{C}_X^\infty$ be the sheaf of C^∞-functions on X.

Holomorphic vector bundle on $\Psi_0^{-1}(X) \Longrightarrow$ flat λ-connection on X

From a holomorphic vector bundle E, we obtain a $\mathscr{C}_X^\infty$-module $\Psi_{0*}(E)$ on X:

$$\Psi_{0*}(E)(U) = \{C^\infty\text{-sections of } E \text{ on } \Psi_0^{-1}(U)\} \quad (U \subset X \text{ open})$$

It is equipped with the actions of $(1+|\lambda|^2)\overline{\partial}_\xi$ and $(1+|\lambda|^2)\overline{\partial}_\eta$. They give a flat λ-connection of $\Psi_{0*}(E)$.

It can be regarded as "a λ-flat bundle of infinite rank".

We have a natural inclusion

$$(V,\mathbb{D}^\lambda) \subset \Psi_{0*}\Psi_0^*(V,\mathbb{D}^\lambda) \simeq (V,\mathbb{D}^\lambda)\otimes\Psi_{0*}(\mathscr{O}_{\mathscr{M}^\lambda}).$$

The analogy of holomorphic vector bundles on $\mathscr{M}^\lambda$ and λ-flat bundles on $\mathbb{C}$ can be more acute around ∞.

Recall $\mathscr{M}^\lambda \longrightarrow T^\vee$ is affine space bundle given by $(\xi,\eta)\longmapsto\xi$.
We obtain the natural projective completion $\overline{\mathscr{M}}^\lambda$, by adding $\eta=\infty$.

Let T_∞^λ denote $\{\eta=\infty\}$, which is naturally isomorphic to $T^\vee$.

$$\overline{\mathscr{M}}^\lambda = \mathscr{M}^\lambda \sqcup T_\infty^\lambda \quad \text{(set theoretically)}$$

- We will consider vector bundles E on a neighbourhood of T_∞^λ such that $E_{|T_\infty^\lambda}$ is semistable of degree 0.
- E has a kind of Stokes structure, if $\lambda \neq 0$. (The case $\lambda = 0$ is simpler.)

Semistable bundle of degree 0 on an elliptic curve

Holomorphic vector bundles on an elliptic curve was studied by Atiyah in 50's, and then by many people. In particular, semistable vector bundle of degree 0 is very easy to understand.

Let E be a holomorphic vector bundle on an elliptic curve C.

$$\deg(E) := \int_C c_1(E) \quad \text{(degree)}$$

$$\mu(E) := \frac{\deg(E)}{\operatorname{rank} E} \quad \text{(slope)}$$

E **semistable** $\overset{\text{def}}{\Longleftrightarrow}$ $\mu(F)\leq\mu(E)$ holds for any subbundle $F\subset E$.

Example

Let $C\simeq\mathbb{C}/\Lambda$. We use the standard coordinate z of $\mathbb{C}$.

A finite dimensional $\mathbb{C}$-vector space V induces a C^∞-bundle $\underline{V} := V\times C$ over C. It has a natural holomorphic structure

$$\overline{\partial}_0 : C^\infty(C,\underline{V}) \longrightarrow C^\infty(C,\underline{V}\otimes\Omega_C^{0,1})$$

$f\in\operatorname{End}(V)$ gives a holomorphic structure $\overline{\partial}_0 + f\,d\overline{z}$ of $\underline{V}$.

Lemma $(\underline{V},\overline{\partial}_0+f\,d\overline{z})$ is semistable of degree 0.
Conversely, any semistable vector bundle of degree 0 can be expressed as above (not uniquely).

Ambiguity of descriptions

Let $\Lambda^\vee$ be defined by

$$\Lambda^\vee := \{\zeta\in\mathbb{C} \,|\, \operatorname{Im}(\zeta\overline{z})\in\pi\mathbb{Z}\ \forall z\in\Lambda\}$$

$\chi\in\Lambda^\vee$ gives the function ρ_χ on C:

$$\rho_\chi(z) := \exp\bigl(2\sqrt{-1}\operatorname{Im}(\chi\overline{z})\bigr)$$

The multiplication of ρ_χ induces an isomorphism

$$(\underline{V},\overline{\partial}_0+f\,d\overline{z}) \simeq (\underline{V},\overline{\partial}_0+(f+\chi\,\mathrm{id}_V)\,d\overline{z})$$

Essentially, all the ambiguity is given in this way.

Let E_0 be a semistable bundle of degree 0 on C. We have the Fourier-Mukai transform $\mathrm{FM}(E_0)$ on $C^\vee = \mathrm{Pic}_0(C)$.

Fourier-Mukai transform (the simplest case)

We have the universal line bundle $\mathscr{L}$ (Poincaré bundle) on $C\times C^\vee$.
Let $C \xleftarrow{p_1} C\times C^\vee \xrightarrow{p_2} C^\vee$ be the projections.
For an $\mathscr{O}_C$-module M, we obtain $\mathrm{FM}(M) := p_{2*}(p_1^*M\otimes\mathscr{L})[1]$ in $D^b(\mathscr{O}_{C^\vee})$.
If M is a semistable bundle of degree 0, $\mathrm{FM}(M)$ is a torsion $\mathscr{O}_{C^\vee}$-module.

Let $\iota : C^\vee \longrightarrow C^\vee$ be given by $\iota(\zeta) = -\zeta$. We set

$$\mathfrak{s}(E_0) := \text{the support of } \iota^*\mathrm{FM}(E_0)$$

If $E_0 = (\underline{V},\overline{\partial}_0+f\,d\overline{z})$, $\mathfrak{s}(E_0) = \{\text{the eigenvalue of } f \text{ modulo } \Lambda^\vee\}$.
(V,f) is unique up to isomorphisms, once we fix a lift of $\mathfrak{s}(E_0)$ to $\mathbb{C}$.

An equivalence

Let $\widetilde{\mathfrak{s}} \subset \mathbb{C}$ be a finite set such that $\widetilde{\mathfrak{s}} \longrightarrow \mathbb{C} \longrightarrow C^{\vee}$ is injective. The image is denoted by $\mathfrak{s}$.

$VB_0^{ss}(C,\mathfrak{s})$: Semistable bundles E_0 of degree 0 on C such that $\mathfrak{s}(E_0) \subset \mathfrak{s}$.

$VS^*(\widetilde{\mathfrak{s}})$: Vector spaces with an endomorphism (V,f) such that the eigenvalue of $f \in \widetilde{\mathfrak{s}}$

The construction $(V,f) \longmapsto (\underline{V}, \overline{\partial}_0 + f\, d\overline{z})$ gives an equivalence of categories

$$VS^*(\widetilde{\mathfrak{s}}) \simeq \mathrm{VB}_0^{ss}(C,\mathfrak{s})$$

This equivalence will be enhanced later.

Vector bundles on $\overline{\mathscr{M}}^{\lambda}$

Analogy around infinity

$$\overline{\mathscr{M}}^{\lambda} = \mathscr{M}^{\lambda} \sqcup T_{\infty}^{\lambda}$$

The map $\Psi_0 : \mathscr{M}^{\lambda} \longrightarrow \mathbb{C}$ is extended to a C^{∞}-map $\Psi_0 : \overline{\mathscr{M}}^{\lambda} \longrightarrow \mathbb{P}^1$. Let $\overline{X}$ be a neighbourhood of ∞ in $\mathbb{P}^1$.

We would like to explain the analogy between

- holomorphic vector bundles E on $\Psi_0^{-1}(\overline{X})$ such that $E_{|T_{\infty}^{\lambda}}$ are semistable of degree 0.
- vector bundles V on $\overline{X}$ with a meromorphic λ-connection $\mathbb{D}^{\lambda}$ such that $\mathbb{D}^{\lambda}(V) \subset V \otimes dw$. (Note that dw has pole of order 2 at ∞.)

Construction Ψ_1^*

It is convenient to consider the C^{∞}-maps $\Psi_1 : \mathscr{M}^{\lambda} \longrightarrow \mathbb{C}$ or $\Psi_1 : \overline{\mathscr{M}}^{\lambda} \longrightarrow \mathbb{P}^1$ given by $\Psi_1(\xi,\eta) = (1+|\lambda|^2)\Psi_0(\xi,\eta) = \eta + \lambda\overline{\xi}$.

For $(\tau, y) = (\xi, \eta + \lambda\overline{\xi})$, we have

$$\overline{\partial}_{\xi} = \overline{\partial}_{\tau} + \lambda\partial_y, \quad \overline{\partial}_{\eta} = \overline{\partial}_y$$

Let $X \subset \mathbb{C}$ be open.

λ-flat bundle on $X \Longrightarrow$ holomorphic bundle on $\Psi_1^{-1}(X)$

Let $(V, \mathbb{D}^{\lambda})$ be a λ-flat bundle on X. A C^{∞}-bundle $\Psi_1^{-1}(V)$ on $\Psi_1^{-1}(X)$ is equipped with an induced flat λ-connection $\mathbb{D}^{\lambda}$ (with respect to (τ, y)). Then,

$$\overline{\partial}_{\xi} = \mathbb{D}_{\overline{\tau}}^{\lambda} + \mathbb{D}_y^{\lambda}, \quad \overline{\partial}_{\eta} = \mathbb{D}_{\overline{y}}^{\lambda}$$

gives a holomorphic structure on $\Psi_1^{-1}(V)$. The holomorphic bundle is denoted by $\Psi_1^*(V, \mathbb{D}^{\lambda})$.

Comparison of Ψ_0^* and Ψ_1^*

Ψ_0^* and Ψ_1^* are essentially the same construction. (They are the same in the casse $\lambda = 0$.)

Let $X_0 := \{|w| > R\}$ and $X_1 := \{|w| > (1+|\lambda|^2)R\}$.

- We have $\Psi_0^{-1}(X_0) = \Psi_1^{-1}(X_1)$.
- a λ-flat bundle on $X_0 \longleftrightarrow$ a λ-flat bundle on X_1.

Let $(V, \mathbb{D}^{\lambda})$ on X_0. By the parallel transport of the flat λ-connection along the segment connecting w and $(1+|\lambda|^2)w$, we obtain an isomorphism

$$V_{|w} \simeq V_{|(1+|\lambda|^2)w}.$$

It induces a C^{∞}-isomorphism

$$\Psi_0^*(V) \simeq \Psi_1^*(V).$$

We can check that it is holomorphic by an easy computation.

Extension at ∞.

Let $\overline{X} := \{y \in \mathbb{C} \,\big|\, |y| \geq R\} \cup \{\infty\}$.

Meromorphic λ-connection on $\overline{X} \Longrightarrow$ holomorphic vector bundle on $\Psi_1^{-1}(\overline{X})$

Let V be a holomorphic vector bundle on $\overline{X}$ with a meromorphic flat λ-connection $\mathbb{D}^{\lambda}$ such that $\mathbb{D}^{\lambda}(V) \subset V \otimes dy$. The construction Ψ_1^* gives a holomorphic bundle $\Psi_1^*(V, \mathbb{D}^{\lambda})$ on $\Psi_1^{-1}(\overline{X})$.

Let $v_1, \ldots, v_r$ be a holomorphic frame of V. Let A be determined by $\mathbb{D}_y^{\lambda}(v_1, \ldots, v_r) = (v_1, \ldots, v_r)A(y^{-1})$, which is holomorphic in y^{-1}. We set $\widetilde{v}_i := \Psi_1^{-1}(v_i)$. Then,

$$\overline{\partial}_{\eta}(\widetilde{v}_1, \ldots, \widetilde{v}_r) = 0 \quad \overline{\partial}_{\xi}(\widetilde{v}_1, \ldots, \widetilde{v}_r) = (\widetilde{v}_1, \ldots, \widetilde{v}_r)A(y^{-1}).$$

Remark Ψ_0^* is not naturally extended on $\overline{X}$. We use Ψ_0^* in relation with instantons.

Let $\widetilde{\mathfrak{s}} \subset \mathbb{C}$ be a finite subset such that $\widetilde{\mathfrak{s}} \longrightarrow T$ is injective. The image is denoted by $\mathfrak{s}$.

$\mathrm{VB}_0^{ss}(\overline{\mathscr{X}}^{\lambda}, \mathfrak{s})$: Holomorphic vector bundles E on $\overline{\mathscr{X}}^{\lambda} := \Psi_1^{-1}(\overline{X})$ s.t. $E_{|T_{\infty}^{\lambda}} \in \mathrm{VB}_0^{ss}(T_{\infty}^{\lambda}, \mathfrak{s})$.

$\mathrm{Conn}^{\lambda}(\overline{X}, \widetilde{\mathfrak{s}})$: Vector bundles V on $\overline{X}$ with a meromorphic flat λ-connection $\mathbb{D}^{\lambda}$ such that
(i) $\mathbb{D}^{\lambda}(V) \subset V \otimes dy$,
(ii) the eigenvalues of $\mathrm{Top}(\mathbb{D}^{\lambda})$ is contained in $\widetilde{\mathfrak{s}}$, i.e., $(V_{|\infty}, \mathrm{Top}(\mathbb{D}^{\lambda})) \in VS^*(\widetilde{\mathfrak{s}})$.

If $\mathbb{D}^{\lambda}(V) \subset V \otimes dy$, we have the induced endomorphism $\mathrm{Top}(\mathbb{D}^{\lambda})$ of $V_{|\infty}$.

We have the functor $\Psi_1^* : \mathrm{Conn}^{\lambda}(\overline{X}, \widetilde{\mathfrak{s}}) \longrightarrow \mathrm{VB}_0^{ss}(\overline{\mathscr{X}}^{\lambda}, \mathfrak{s})$.

Formal case

Let $\widehat{X}$ denote the formal completion of $\mathbb{P}^1_y$ at ∞. Let $\widehat{\mathscr{X}^\lambda}$ denote the formal completion of $\overline{\mathscr{X}}^\lambda$ along T^λ_∞. We have the formal version of the functor Ψ_1^*.

Theorem $\Psi_1^* : \mathrm{Conn}^\lambda(\widehat{X}, \widetilde{\mathfrak{s}}) \longrightarrow \mathrm{VB}_0^{ss}(\widehat{\mathscr{X}^\lambda}, \mathfrak{s})$ is an equivalence.

It might be useful to describe the behaviour of a holomorphic vector bundle on $\overline{\mathscr{M}}^\lambda$ around T^λ_∞.

Classical Hukuhara-Levelt-Turittin decomposition

Let $K = \mathbb{C}((z))$ be the field of Laurent power series. Let V be a differential K-vector space. If we take an appropriate extension $K \subset K' = \mathbb{C}((z^{1/e}))$, we have a formal isomorphism

$$V \otimes K' \simeq \bigoplus_{\mathfrak{a} \in z^{-1/e}\mathbb{C}[z^{-1/e}]} L_\mathfrak{a} \otimes R_\mathfrak{a}$$

where $R_\mathfrak{a}$ are regular singular, and $L_\mathfrak{a} = \mathbb{C}((z^{1/e}))v_\mathfrak{a}$ such that $\partial_z v_\mathfrak{a} = v_\mathfrak{a}\partial_z\mathfrak{a}$.

The set $\{\mathfrak{a} \mid R_\mathfrak{a} \neq 0\}$ and the formal monodromy of $R_\mathfrak{a}$ are the important invariants for the differential module V.

By the equivalence $\mathrm{Conn}^\lambda(\widehat{X}, \widetilde{\mathfrak{s}}) \simeq \mathrm{VB}_0^{ss}(\widehat{\mathscr{X}^\lambda}, \mathfrak{s})$, these invariants are transferred to objects in $\mathrm{VB}^{ss}(\widehat{\mathscr{X}^\lambda}, \mathfrak{s})$.

"Local Fourier transform and Stationary phase formula" (Interlude)

Recall the simplest version of the generalized Fourier-Mukai transform due to Laumon-Rothstein.

Over $T \times \mathscr{M}^\lambda$, we have a universal family of line bundles $\mathscr{L}$ with a family of flat λ-connections $\mathbb{D}^\lambda : \mathscr{L} \longrightarrow \mathscr{L} \otimes \Omega^1_{T \times \mathscr{M}^\lambda / \mathscr{M}^\lambda}$.
Let $T \xleftarrow{p_1} T \times \mathscr{M}^\lambda \xrightarrow{p_2} \mathscr{M}^\lambda$ be the projections.
For a meromorphic λ-flat bundle $(M, \mathbb{D}^\lambda)$ on T, we obtain

$$\mathrm{FM}^{LR}(M) := p_{2+}\big(p_1^*(M, \mathbb{D}^\lambda) \otimes (\mathscr{L}, \mathbb{D}^\lambda)\big)[1] \in D^b_{\mathrm{coh}}(\mathscr{O}_{\mathscr{M}^\lambda})$$

If M is simple with $\mathrm{rank}\, M \neq 1$, $\mathrm{FM}^{LR}(M)$ is an algebraic vector bundle on $\mathscr{M}^\lambda$. Hence, it naturally gives a locally free $\mathscr{O}_{\overline{\mathscr{M}}^\lambda}(*T^\lambda_\infty)$-module.

Classical Fourier transform

We have a line bundle with a flat connection $\big(\mathscr{O}_{\mathbb{C}_z \times \mathbb{C}_\zeta}, d + d(z\zeta)\big)$ on $\mathbb{C}_z \times \mathbb{C}_\zeta$.

For a meromorphic flat bundle (M, ∇) on $\mathbb{C}_z$, we have

$$\mathfrak{F}(M, \nabla) := p_{2+}\Big(p_1^*(M, \nabla) \otimes \big(\mathscr{O}_{\mathbb{C}_z \times \mathbb{C}_\zeta}, d + d(z\zeta)\big)\Big) \in D^b(\mathscr{D}_{\mathbb{C}_\zeta})$$

For $\mathfrak{F}$, a local Fourier transform and an explicit stationary phase formula were studied by Arinkin, Beilinson, Bloch, Deligne, Esnault, Fang, Fu, Graham-Squire, Laumon, Malgrange, Sabbah,....

$$\mathfrak{F}(M, \nabla)_{|\widehat{\infty}} = \bigoplus_{\substack{\alpha \in \mathbb{C} \\ \text{pole}}} \mathfrak{F}^{(\alpha,\infty)}\big((M, \nabla)_{|\widehat{\alpha}}\big) \oplus \mathfrak{F}^{(\infty,\infty)}\big((M, \nabla)_{|\widehat{\infty}}\big)$$

$$\mathfrak{F}^{(\alpha,\infty)}(M, \nabla)_{|\widehat{\alpha}} \in \mathrm{Conn}^1(\widehat{X}, \{\alpha\}).$$

An explicit stationary phase formula for FM^{LR}.

Let $(M, \mathbb{D}^\lambda)$ be a meromorphic λ-flat bundle on T. For simplicity, we assume that $(M, \mathbb{D}^\lambda)$ is simple with $\mathrm{rank}\, M \neq 1$. We obtain a locally free $\mathscr{O}_{\overline{\mathscr{M}}^\lambda}(*T^\lambda_\infty)$-module $\mathrm{FM}^{LR}(M, \mathbb{D}^\lambda)$ on $\overline{\mathscr{M}}^\lambda$.
Let $\mathfrak{s} \subset T$ be the set of poles of $(M, \mathbb{D}^\lambda)$.

Theorem

- There exists a lattice $E \subset \mathrm{FM}^{LR}(M, \mathbb{D}^\lambda)$ such that $E \in \mathrm{VB}_0^{ss}(\overline{\mathscr{M}}^\lambda, \mathfrak{s})$.
- The formal completion $\mathrm{FM}^{LR}(M, \mathbb{D}^\lambda)_{|\widehat{\mathscr{X}}^\lambda}$ depends only on the formal completion of $(M, \mathbb{D}^\lambda)$ along the poles.
- The corresponding object in $\mathrm{Conn}^\lambda(\widehat{X}, \widetilde{\mathfrak{s}})$ is described by the stationary phase formula of local Fourier transform.

Asymptotic analysis

We come back to the study of $E \in \mathrm{VB}_0^{ss}(\overline{\mathscr{X}}^\lambda, \mathfrak{s})$, where $X = \{y \in \mathbb{C} \,|\, |y| \geq R\}$, $\overline{X} = X \cup \{\infty\}$ and $\overline{\mathscr{X}}^\lambda = \Psi_1^{-1}(\overline{X})$.
There exists $(V, \mathbb{D}^\lambda) \in \mathrm{Conn}^\lambda(\overline{X}, \widetilde{\mathfrak{s}})$ such that

$$\Psi_1^*(V, \mathbb{D}^\lambda)_{|\widehat{\mathscr{X}}^\lambda} \simeq E_{|\widehat{\mathscr{X}}^\lambda}. \tag{1}$$

As in the case of meromorphic flat bundles, the isomorphism is not convergent, in general.

Theorem For any small sector $S \subset X$, there exists a holomorphic isomorphism $E_{|\Psi_1^{-1}(S)} \simeq \Psi_1^*(V, \mathbb{D}^\lambda)_{|\Psi_1^{-1}(S)}$, asymptotic to (1).
(It is called an admissible trivialization in this talk.)

A sector is $S = \{w \in \mathbb{C} \,\big|\, |w| \geq R,\, \theta_0 \leq \arg(w) \leq \theta_1\}$.
This is an analogue of the classical asymptotic analysis for meromorphic flat bundles.

Classical asymptotic analysis for a meromorphic flat bundle

Let (V,∇) be a meromorphic flat bundle on $\{z \mid |z|<1\}$ with the pole at $z=0$. We have a formal isomorphism

$$(V,\nabla)_{|\infty} \otimes \mathbb{C}((z^{1/e})) \simeq \bigoplus_{\mathfrak{a}\in z^{-1/e}\mathbb{C}[z^{-1/e}]} L_{\mathfrak{a}} \otimes R_{\mathfrak{a}}. \qquad (2)$$

Here $R_{\mathfrak{a}}$ is regular singular, and $L_{\mathfrak{a}} = (\mathscr{O}, \lambda\, d + d\mathfrak{a})$. It is not convergent in general.
But, for any small sector $S = \{0<|z|<r_0,\ \theta_0 \le \arg(z) \le \theta_1\}$, we have a flat isomorphism, asymptotic to (2)

$$(V,\nabla)_{|S} \simeq \Big(\bigoplus_{\mathfrak{a}} L_{\mathfrak{a}} \otimes R_{\mathfrak{a}}\Big)_{|S}$$

Ambiguity of admissible trivializations

For $\alpha,\beta \in \mathbb{C}$, we consider

$$V_\alpha = \mathscr{O}_{\overline{X}} e_\alpha \quad \mathbb{D}^\lambda_y e_\alpha = \alpha\, e_\alpha, \qquad V_\beta = \mathscr{O}_{\overline{X}} e_\beta \quad \mathbb{D}^\lambda_y e_\beta = \beta\, e_\beta$$

We put $\Psi_1^{-1}(e_\alpha) := \widetilde{e}_\alpha$ and $\Psi_1^{-1}(e_\beta) := \widetilde{e}_\beta$.

For $\chi \in \Lambda$, we have the C^∞-function $\rho_\chi(\tau) = \exp\big(2\sqrt{-1}\,\mathrm{Im}(\chi\overline{\tau})\big)$ on $T^\vee$.

A C^∞-morphism $f : \Psi_1^*(V_\alpha,\mathbb{D}^\lambda)_{|\Psi_1^{-1}(S)} \longrightarrow \Psi_1^*(V_\beta,\mathbb{D}^\lambda)_{|\Psi_1^{-1}(S)}$ is expressed as

$$f = \sum_{\chi\in\Lambda} f_\chi(y)\rho_\chi(\tau)\widetilde{e}_\alpha^\vee \otimes \widetilde{e}_\beta$$

f is holomorphic $\iff \overline{\partial}_y f_\chi = 0$ and $\lambda\partial_y f_\chi(y) + (\chi-\alpha+\beta) f_\chi(y) = 0$
$\iff f_\chi(y) = a_\chi \exp\big(-(\chi-\alpha+\beta)y/\lambda\big)$ for some $a_\chi \in \mathbb{C}$.

$|f| = O(|y|^{-N})$ for $\forall N>0$ on $S \iff f_\chi = 0$ unless $\mathrm{Re}\big((-\chi+\alpha-\beta)y/\lambda\big) < 0$ on S.

Such holomorphic morphisms cause ambiguity of admissible trivializations.

$$\exists F : \Psi_1^*\big(V_\alpha + V_\beta, \mathbb{D}^\lambda\big) \longrightarrow \Psi_1^*\big(V_\alpha + V_\beta, \mathbb{D}^\lambda\big) \quad \text{s.t. } F \sim \mathrm{id},\ F \neq \mathrm{id}$$

Stokes filtration

For $E \in \mathrm{VB}_0^{ss}(\overline{\mathscr{X}}^\lambda, \mathfrak{s})$, we have $(V,\mathbb{D}^\lambda) \in \mathrm{Conn}^\lambda(\overline{X}, \widetilde{\mathfrak{s}})$ such that

$$E_{|\widehat{\mathscr{X}}^\lambda} \simeq \Psi_1^*(V,\mathbb{D}^\lambda)_{|\widehat{\mathscr{X}}^\lambda}$$

An admissible trivialization $E_{|\Psi_1^{-1}(S)} \simeq \Psi_1^*(V,\mathbb{D}^\lambda)_{|\Psi_1^{-1}(S)}$ is not unique. We would like to obtain something canonically determined for E.

We have a $\mathscr{C}^\infty_{\overline{X}}$-module (infinite dimensional bundle) $\Psi_{1*}(E)$ with the meromorphic λ-connection induced by $\overline{\partial}_\xi$ and $\overline{\partial}_\eta$.
For a small sector $S \subset X$, we use the partial order $\le_S$ on $\mathbb{C}$ given by

$$\alpha \le_S \beta \overset{\text{def}}{\iff} -\mathrm{Re}(\alpha y/\lambda) \le -\mathrm{Re}(\beta y/\lambda) \quad (\forall y \in S)$$

We shall introduce a filtration $\mathscr{F}^{(1)}$ of $\Psi_{1*}(E)_{|S}$ indexed by $(\widetilde{\mathfrak{s}}+\Lambda, \le_S)$.

For simplicity, we assume $(V,\mathbb{D}^\lambda) = \bigoplus_{\alpha\in\widetilde{\mathfrak{s}}}(V_\alpha,\mathbb{D}^\lambda)$ for $(V_\alpha,\mathbb{D}^\lambda) \in \mathrm{Conn}^\lambda(\overline{X},\{\alpha\})$.
Let $v_1,\dots,v_r$ be a frame of V, obtained from frames of V_α. ($v_i \in V_{\alpha_i}$.)
Let $U \subset S$ be any open subset. A C^∞-section f of $\Psi_1^*(V,\mathbb{D}^\lambda)$ on $\Psi_1^{-1}(U)$ is expressed as

$$f = \sum_{i,\chi} f_{\chi i}(y)\rho_\chi(\tau) v_i.$$

We set $\mathscr{F}^{(1)}_\beta \Psi_{1*}(\Psi_1^*(V))_{|\overline{S}}(U) := \{f \mid f_{\chi,i} = 0 \text{ unless } \alpha_i + \chi \le_S \beta\}$.

We define a filtration $\mathscr{F}^{(1)}\Psi_{1*}(E)_{|S}$ by using an admissible trivialization.

Proposition

- The filtration is independent of the choice of an admissible trivialization. It is characterized in terms of the growth order.
- The filtration is preserved by the λ-connection.
- For $S' \subset S$, we have $\Big(\mathscr{F}^{(1)}_\alpha \Psi_{1*}(E)_{|S}\Big)_{|S'} \subset \mathscr{F}^{(1)}_\alpha \Psi_{1*}(E)_{|S'}$, and $\Big(\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)_{|S}\Big)_{|S'} \simeq \mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)_{|S'}$.
  (We put $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)_{|S} := \mathscr{F}^{(1)}_\alpha \Psi_{1*}(E)_{|S} / \mathscr{F}^{(1)}_{<\alpha}\Psi_{1*}(E)_{|S}$)

The construction $\mathrm{Gr}^{(1)}\Psi_{1*}$

- By varying sectors S and gluing $\mathrm{Gr}^{(1)}_\alpha\big(\Psi_{1*}(E)_{|S}\big)$, we obtain a λ-flat bundle $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)_X$ on X.
- By the construction on the real blow up $\widetilde{X}(D)$, we obtain a natural extension of $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)_X$ to a vector bundle $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)$ on $\overline{X}$ with a meromorphic flat λ-connection, for which

$$\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)_{|\widehat{X}} \simeq (V_\alpha,\mathbb{D}^\lambda)_{|\widehat{X}} \quad (\alpha \in \widetilde{\mathfrak{s}})$$

We obtain a functor $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*} : \mathrm{VB}_0^{ss}(\overline{\mathscr{X}}^\lambda, \mathfrak{s}) \longrightarrow \mathrm{Conn}^\lambda(\overline{X}, \{\alpha\})$ for $\alpha \in \widetilde{\mathfrak{s}}+\Lambda$.

- $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E)$ may have non-trivial Stokes structure. It is not necessarily isomorphic to $(V,\mathbb{D}^\lambda)$.
- We have a similar classical construction $\mathrm{Gr}^{(1)}_\alpha : \mathrm{Conn}^\lambda(\overline{X},\widetilde{\mathfrak{s}}) \longrightarrow \mathrm{Conn}^\alpha(\overline{X},\{\alpha\})$ for $\alpha \in \widetilde{\mathfrak{s}}$. We have $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}\Psi_1^* = \mathrm{Gr}^{(1)}_\alpha$.

"Riemann-Hilbert-Birkhoff correspondence"

Let E^* be a holomorphic vector bundle on $\mathscr{X}^\lambda = \overline{\mathscr{X}}^\lambda \setminus T^\lambda_\infty$.
(We obtain an infinite dimensional λ-flat bundle $\Psi_{1*}(E^*)$ on X.)

- $\widetilde{\mathfrak{s}} \subset \mathbb{C}$: a finite subset such that $\widetilde{\mathfrak{s}} \longrightarrow T$ is injective.
- For each small sector $S \subset X$, let $\mathscr{F}^{(1)}$ be a filtration of $\Psi_{1*}(E^*)_{|S}$ indexed by $(\widetilde{\mathfrak{s}}+\Lambda, \le_S)$ which can be "trivialized", satisfying some compatibility condition. (We obtain a λ-flat bundle $\mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E^*)$ on X.)
- For each $\alpha \in \widetilde{\mathfrak{s}}$, let $(V_\alpha,\mathbb{D}^\lambda) \in \mathrm{Conn}^\lambda(\overline{X},\{\alpha\})$ s.t. $(V_\alpha,\mathbb{D}^\lambda)_{|X} \simeq \mathrm{Gr}^{(1)}_\alpha \Psi_{1*}(E^*)$.

$(\{\mathscr{F}^{(1)}\}, \{(V_\alpha,\mathbb{D}^\lambda)\})$ is called a Stokes structure of E^* of type $\widetilde{\mathfrak{s}}$.

Theorem An object in $\mathrm{VB}_0^{ss}(\overline{\mathscr{X}}^\lambda, \mathfrak{s})$ naturally corresponds to a holomorphic vector bundle on $\mathscr{X}^\lambda$ with Stokes structure of type $\widetilde{\mathfrak{s}}$.

Application to instantons on $T^\vee \times \mathbb{C}$

Instanton

We use the metric $dz d\overline{z} + dw d\overline{w}$ on $T^\vee \times \mathbb{C}$. Let $X := \{w \in \mathbb{C} \,|\, |w| \geq R\}$.
Let E be a C^∞-bundle on $\Psi_0^{-1}(X) = T^\vee \times X$ with a hermitian metric h and a unitary connection ∇. The curvature of ∇ is denoted by $F(\nabla)$.

The connection ∇ is called anti-self dual, if $*F(\nabla) = -F(\nabla)$, where $*$ denotes the Hodge star operator. In this case, (E, ∇, h) is called an instanton.

It is equivalent to the following:

- The $(0,1)$-part of ∇ gives a holomorphic structure.
- For the expression $F(\nabla) = F_{z\overline{z}} dz d\overline{z} + F_{w\overline{w}} dw d\overline{w} + F_{z\overline{w}} dz d\overline{w} + F_{w\overline{z}} dw d\overline{z}$, we have $F_{z\overline{z}} + F_{w\overline{w}} = 0$.

We would like to explain how to use the Stokes structure of vector bundles on $T^\vee \times X$ for the study of instantons on $\mathscr{X}^\lambda$ such that $F(\nabla)$ is L^2.

Harmonic bundle

Let (E, ∇, h) be an instanton on $T^\vee \times X$ which is $T^\vee$-equivariant.

- We obtain a C^∞-bundle E_1 on X with a hermitian metric h_1 such that $\Psi_0^*(E_1, h_1) = (E, h)$.
- We also have a unitary connection ∇_1 of (E_1, h_1) such that $\Psi_0^*(\nabla_1)(v) = \nabla(v)$ for $v = a\partial_w + b\partial_{\overline{w}}$.
- Because ∇ is $T^\vee$-equivariant, $\nabla - \Psi_0^* \nabla_1 = \Psi_0^* f\, d\overline{z} - \Psi_0^* f^\dagger\, dz$ for $f, f^\dagger \in \mathrm{End}(E_1)$.

The anti-self duality condition is reduced to the Hitchin equation

$$F(\nabla_1) + \left[f\,dw, f^\dagger d\overline{w} \right] = 0$$

$(E_1, \overline{\partial}_{E_1}, f\,dw)$ with the metric h is called a harmonic bundle, where $\overline{\partial}_{E_1}$ is the $(0,1)$-part of ∇_1.

Hitchin
$T^\vee$-equivariant instanton on $T^\vee \times X$ is equivalent to a harmonic bundle on X.

Nahm transform

For a closed subgroup $\Gamma \subset \mathbb{R}^4$, let $\Gamma^\vee := \{\chi \in (\mathbb{R}^4)^\vee \,|\, \chi(\Gamma) \subset \mathbb{Z}\}$.
It is believed and established in some degree

$$\begin{pmatrix} \Gamma\text{-equivariant instanton} \\ \text{satisfying some condition} \\ \text{with some singularity} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \Gamma^\vee\text{-equivariant instanton} \\ \text{satisfying some condition} \\ \text{with some singularity} \end{pmatrix}$$

An instanton on $T^\vee \times \mathbb{C}$ is $\Lambda^\vee$-equivariant instanton.

- ADHM construction (Atiyah-Drinfeld-Hitchin-Manin) in the case $\Gamma = \{1\}$ and $\Gamma^\vee = \mathbb{R}^4$.
- Nahm studied the case $\Gamma = \mathbb{R}$ and $\Gamma^\vee = \mathbb{R}^3$. It was refined by Hitchin and Nakajima.

Since then, the other cases were also studied by many people.

What I would like to do?

The case $\Gamma = \Lambda^\vee$ and $\Gamma^\vee = \Lambda \times \mathbb{C}^2$ was previously studied by Jardim collaborated with Biquard. They established the Nahm transform between

- Harmonic bundles on T with tame singularity.
- Instantons on $T^\vee \times \mathbb{C}$ satisfying the quadratic decay condition. i.e., $|F(\nabla)| = O(|w|^{-2})$ with respect to h and $dz d\overline{z} + dw d\overline{w}$.

My goals

1. Refine the condition from "quadratic decay" to "L^2", and establish the correspondence between
   - Harmonic bundles on T with wild singularity
   - Instantons on $T^\vee \times \mathbb{C}$ such that $F(\nabla)$ is L^2.

   (We do not explain this anymore in this talk.)
2. Refine the study by using the twistor viewpoint.
   - Stokes structure naturally appears.
   - We obtain wild harmonic bundle as a graduation of instanton with respect to the Stokes structure.

Let (E, ∇, h) be an instanton on $T^\vee \times X$ such that $F(\nabla)$ is L^2. Let $\overline{\partial}_E$ be the $(0,1)$-part of ∇, with which $(E, \overline{\partial}_E)$ is a holomorphic vector bundle on $T^\vee \times X$.

Lemma $\exists R > 0$ such that $(E, \overline{\partial}_E)_{|T^\vee \times \{w\}}$ are semistable of degree 0 for any $w \in X$ with $|w| > R$.

We may assume that $(E, \overline{\partial}_E)_{|T^\vee \times \{w\}}$ are semistable of degree 0 from the beginning.

By the relative Fourier-Mukai transform, we obtain a coherent sheaf $FM(E)$ on $T \times X$. The support $\mathscr{Z} \subset T \times X$ is relatively 0-dimensional over X.

Proposition $\mathscr{Z}$ is naturally extended to a subvariety $\overline{\mathscr{Z}}$ in $T \times \overline{X}$.

Let $\mathbb{C} \times \overline{X} \longrightarrow T \times \overline{X}$ be the morphism induced by a universal covering $\mathbb{C} \longrightarrow T$.
We fix a lift $\widetilde{\mathscr{Z}} \subset \mathbb{C} \times \overline{X}$ of $\mathscr{Z}$, and put $\tilde{\mathfrak{s}} := \iota^*\big(\widetilde{\mathscr{Z}} \cap (\mathbb{C} \times \{\infty\})\big)$.

Lemma $\exists\, (V^0, \mathbb{D}^0) \in \mathrm{Conn}^0(\overline{X}, \tilde{\mathfrak{s}})$ such that $\Psi_0^*(V^0, \mathbb{D}^0) = (E, \overline{\partial}_E)$.

We obtain the following theorem.

Theorem We have an induced harmonic metric h_0 of $(V^0, \mathbb{D}^0)$, for which

$$\Psi_0^*(h_0) - h = O\big(\exp(-C|w|^\delta)\big)$$

for some $C, \delta > 0$.

We would like to explain how to obtain a harmonic bundle $(V^0, \mathbb{D}^0, h_0)$, or equivalently $T^\vee$-equivariant instanton $\Psi_0^*(V^0, \mathbb{D}^0, h_0)$, by using the previous consideration on the Stokes structure of objects in $\mathrm{VB}_0^{ss}(\overline{\mathscr{M}}^\lambda)$.

Deligne-Hitchin space

We recall the construction of Deligne-Hitchin space

- We have the natural family $\mathscr{M} \longrightarrow \mathbb{C}$ such that the fiber $\mathscr{M} \times_{\mathbb{C}} \{\lambda\}$ is $\mathscr{M}^{\lambda}$.
- We also have the natural family $\mathscr{M}^{\dagger} \longrightarrow \mathbb{C}$ such that the fiber $\mathscr{M}^{\dagger\mu} = \mathscr{M}^{\dagger} \times_{\mathbb{C}} \{\mu\}$ is the moduli of line bundles with flat μ-connection on $T^{\dagger}$, where $T^{\dagger}$ denotes the conjugate of T.
- We have the natural holomorphic isomorphism $\mathscr{M} \times_{\mathbb{C}} \mathbb{C}^* \simeq \mathscr{M}^{\dagger} \times_{\mathbb{C}} \mathbb{C}^*$. ($\lambda^{-1} = \mu$.)
- By gluing, we obtain a complex manifold $\mathscr{M}_{DH}$ with a morphism $\mathscr{M}_{DH} \longrightarrow \mathbb{P}^1_{\lambda}$. (The twistor space of the hyperkähler manifold $T^{\vee} \times \mathbb{C}$.)

We recall some basic facts.

- We have a C^{∞}-isomorphism $\mathscr{M}_{DH} \simeq \mathbb{P}^1_{\lambda} \times T^{\vee} \times \mathbb{C}$.
- The twistor lines $C_Q := \mathbb{P}^1_{\lambda} \times \{Q\}$ are complex submanifolds for any $Q \in T^{\vee} \times \mathbb{C}$.
- We have an anti-holomorphic involution $\sigma : \mathscr{M}_{DH} \longrightarrow \mathscr{M}_{DH}$, compatible with $\sigma : \mathbb{P}^1_{\lambda} \longrightarrow \mathbb{P}^1_{\lambda}$ given by $\sigma(\lambda) = -\overline{\lambda}^{-1}$.

Twistor description of an instanton

- We have the C^{∞}-map $\Psi_{DH} : \mathscr{M}_{DH} = \mathbb{P}^1_{\lambda} \times T^{\vee} \times \mathbb{C} \longrightarrow \mathbb{C}$.
- For $X = \{w \in \mathbb{C} \,|\, |w| \geq R\}$, we set $\mathscr{X}_{DH} := \Psi_{DH}^{-1}(X)$.

Recall the following well known fact.

An instanton on $T^{\vee} \times X$ is equivalent to a holomorphic vector bundle $\mathscr{E}_{DH}$ on $\mathscr{X}_{DH}$ with a holomorphic pairing $P : \mathscr{E}_{DH} \times \sigma^* \mathscr{E}_{DH} \longrightarrow \mathscr{O}_{\mathscr{X}_{DH}}$ satisfying the following for any $Q \in T^{\vee} \times X$.

- $(\mathscr{E}_{DH}, P)_{|C_Q}$ are polarized pure twistor structure of weight 0, i.e., $\mathscr{E}_{DH,Q} := \mathscr{E}_{DH|C_Q}$ are isomorphic to $\mathscr{O}_{\mathbb{P}^1}^{\oplus r}$, and P_Q induces a positive definite hermitian metric of $H^0(C_Q, \mathscr{E}_{DH,Q})$.

Prolongation

Let (E, h, ∇) be an L^2-instanton on $T^{\vee} \times X$. Let $\mathscr{E}$ be the corresponding holomorphic vector bundle on $\mathscr{X}_{DH}$. For $\lambda \in \mathbb{P}^1_{\lambda} \setminus \{\infty\}$, we set $\mathscr{E}^{\lambda} := \mathscr{E}_{|\mathscr{X}^{\lambda}}$.

Proposition $(\mathscr{E}^{\lambda}, h)$ is acceptable, i.e., the curvature of $(\mathscr{E}^{\lambda}, h)$ is bounded with respect to h and the Poincaré like metric of $\mathscr{M}^{\lambda}$.

For each $a \in \mathbb{R}$, we obtain an $\mathscr{O}_{\overline{\mathscr{M}}^{\lambda}}$-module $\mathscr{P}_a \mathscr{E}^{\lambda}$ such that $\mathscr{P}_a \mathscr{E}^{\lambda}_{|\mathscr{X}^{\lambda}} = \mathscr{E}^{\lambda}$.

For each open $U \subset \overline{\mathscr{X}}^{\lambda}$, we set

$$\mathscr{P}_a \mathscr{E}^{\lambda}(U) = \left\{ f \in \mathscr{E}^{\lambda}(U \setminus T^{\lambda}_{\infty}) \,\middle|\, |f|_h = O(|w|^{-a-\varepsilon}) \text{ locally on } U \ \forall \varepsilon > 0 \right\}$$

By the above proposition and a general theory for acceptable bundles, $\mathscr{P}_a \mathscr{E}^{\lambda}$ is locally free $\mathscr{O}_{\overline{\mathscr{M}}^{\lambda}}$-module.

Proposition $\mathscr{P}_a \mathscr{E}^{\lambda}$ is an object in $\mathrm{VB}_0^{ss}(\overline{\mathscr{X}}^{\lambda}, \mathfrak{s})$.

Taking Gr

We obtain a vector bundle with a meromorphic flat λ-connection on $\Psi_1(\overline{\mathscr{X}}^{\lambda})$

$$(\mathrm{V}^{\lambda}, \mathbb{D}^{\lambda}) := \bigoplus_{\alpha \in \bar{\mathfrak{s}}} (\mathrm{Gr}^{\mathscr{F}}_{\alpha} \Psi_{1*} \mathscr{P}_a \mathscr{E}^{\lambda}, \mathbb{D}^{\lambda})$$

We obtain a vector bundle $\mathscr{E}_0^{\lambda} := \Psi_1^*(\mathrm{V}^{\lambda}, \mathbb{D}^{\lambda})_{|\mathscr{X}^{\lambda}}$ on $\mathscr{X}^{\lambda}$.

Proposition $\bigcup_{\lambda \in \mathbb{P}^1_{\lambda} \setminus \{\infty\}} \mathscr{E}_0^{\lambda}$ naturally gives a holomorphic vector bundle $\mathscr{E}_0$ on $\mathscr{X}_{DH} \cap \mathscr{M}$. (Recall $\mathscr{M}_{DH} = \mathscr{M} \cup \mathscr{M}^{\dagger}$.)

- By considering the conjugate, we obtain $\mathscr{E}_0^{\dagger}$ on $\mathscr{X}_{DH} \cap \mathscr{M}^{\dagger}$ over $\mathbb{P}^1_{\lambda} \setminus \{0\}$.
- We have a natural isomorphism $\mathscr{E}_{0|\mathscr{M} \cap \mathscr{M}^{\dagger} \cap \mathscr{X}_{DH}} \simeq \mathscr{E}^{\dagger}_{0|\mathscr{M} \cap \mathscr{M}^{\dagger} \cap \mathscr{X}_{DH}}$.
- By gluing $\mathscr{E}_0$ and $\mathscr{E}_0^{\dagger}$, we obtain a holomorphic vector bundle $\mathscr{E}_{0,DH}$ on $\mathscr{X}_{DH}$.
- We have a naturally induced pairing $P_0 : \mathscr{E}_{0,DH} \times \sigma^* \mathscr{E}_{0,DH} \longrightarrow \mathscr{O}_{\mathscr{X}_{DH}}$.

Theorem After X is shrank appropriately, $(\mathscr{E}_{0,DH}, P_0)$ gives an instanton. It is $T^{\vee}$-equivariant.