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On noncommutative crystalline cohomology

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ABSTRACT. We outline several constructions of a noncommutative version of crystalline cohomology for an associative algebra over a field of positive characteristic.

1. Introduction

In this paper we review several approaches to generalizing cohomology of schemes over a field of characteristic p to the case of noncommutative algebras or, more generally, differential graded categories.

Recall that the classical (commutative) theory uses several ideas. One starts from a lifting of an $\mathbb{F}_p$-algebra to a $\mathbb{Z}_p$-algebra. If a good lifting exists, one defines crystalline cohomology as the De Rham cohomology of the lifting and proves that the result does not depend on a lifting. If not, one has to use other methods [**24**], [**3**]. Alternatively one can define De Rham forms and cohomology extending the method by which Witt vectors are defined [**22**]. In a yet another approach, one uses topological/bornological methods in non-Archimedean geometry [**37**].

Noncommutative versions of all of the above have been advanced in recent years. In noncommutative geometry, the analog of De Rham cohomology is (periodic) cyclic homology [**8**], [**9**], [**10**], [**35**], [**38**], [**47**]. It is more suited in characteristic zero, and it had been long recognized that topological Hochschild and cyclic homology [**5**], [**6**], [**26**], [**39**] works better for arithmetics. However, some versions of the standard theory can be used to give some basic definitions, and sometimes they give an alternative way of looking at the invariants from the topological theory. Below we present four different approaches along these lines.

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2. Hochschild-Witt homology

Here we follow Kaledin's works [**29**], [**30**], [**31**].

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2.1. Noncommutative Witt vectors. For a $\mathbb{Z}$-module V, consider the action of the cyclic group C_{p^n} on $V^{\otimes p^n}$ by permutations. We denote the generator of C_{p^n} by σ and write

$$N = 1 + \sigma + \ldots + \sigma^{p^n - 1} \tag{2.1}$$

We denote the image of N by Norms.

Lemma 2.1. *Let x and y be two elements of V. Then*

$$(x+y)^p - x^p - y^p \in \mathrm{Norms}$$

Corollary 2.2. *In an associative algebra A over $\mathbb{F}_p$, for any x and y in A*

$$(x+y)^p = x^p + y^p$$

in $A/[A,A]$.

Lemma 2.3. *(Noncommutative Dwork's lemma). Let x and y be two elements of V. Then*

$$(x+py)^{p^n} - x^{p^n} \in p\mathrm{Norms}$$

in $V^{\otimes p^n}$.

Proof. Let V be a free $\mathbb{Z}$-module with a basis $\{x_j | j \in J\}$. We will denote a monomial in $V^{\otimes p^n}$ (with respect to this basis) by X. We say a monomial is primitive if it is not a pth power of another monomial. Any monomial is of the form

$$X = Y^{p^{n-k}}$$

where y is a primitive monomial in $V^{\otimes p^k}$. This happens if and only if the C_{p^n}-orbit of X is of order p^k.

Let M_k be a set of representatives of all primitive monomials in $V^{\otimes p^k}$ (two monomials are equivalent if one is obtained from the other by a permutation from C_{p^k}).

Let V be a free $\mathbb{Z}$-module with the basis $\{x, y\}$. Then

$$(x+y)^{p^n} = \sum_{k=0}^{n} \sum_{Y \in M_k} N_{p^k}(Y^{p^{n-k}}) \tag{2.2}$$

where

$$N_{p^k} = 1 + \sigma + \ldots + \sigma^{p^k - 1} \tag{2.3}$$

(in particular $N = N_{p^n}$). Now let y be divisible by p. Then, unless $k = 0$ and $Y = x$, $N_{p^k}(Y^{p^{n-k}})$ is in the image of $p^{p^{n-k}} N_{p^k}$. But the image of $p^{n-k+1} N_{p^k}$ is in the image of pN. Indeed, $n - k + 1 \le p^{n-k}$. This proves Lemma 2.3. Lemma 2.1 follows from (2.2) when $n = 1$. $\square$

Definition 2.4. Let V be a free $\mathbb{Z}$-module. Put

$$W_n(V) = (V^{\otimes p^n})^{C_{p^n}}/\mathrm{Norms}$$

In other words,

$$W_n(V) = \check{H}^0(C_{p^n}, V^{\otimes p^n})$$

(the Tate cohomology of degree zero). Put also

$$W_n'(V) = (V^{\otimes p^n})^{C_{p^n}}/p\mathrm{Norms}$$

LEMMA 2.5.

$$W_n(V) = \bigoplus_{k=0}^{n-1} \bigoplus_{Y \in M_k} (\mathbb{Z}/p^{n-k}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}})$$

$$W_n'(V) = \bigoplus_{k=0}^{n} \bigoplus_{Y \in M_k} (\mathbb{Z}/p^{n-k+1}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}})$$

(Recall that M_k is a set of representatives of primitive monomials of length p^k up to cyclic permutation).

The proof is clear: one only has to compute $M^{C_{p^n}}/N(M)$ and $M^{C_{p^n}}/pN(M)$ for a C_{p^n}-module M induced from a trivial representation of C_{p^k}.

LEMMA 2.6. *Let f and g be two linear maps $V_1 \to V_2$ that differ modulo p. Then $f^{\otimes p^n}$ and $g^{\otimes p^n}$ define the same maps $W_n(V_1) \to W_n(V_2)$ and $W_n'(V_1) \to W_n'(V_2)$.*

This follows from noncommutative Dwork's lemma 2.3.

COROLLARY 2.7. *For a vector space E over $\mathbb{F}_p$ choose a free $\mathbb{Z}$-module $\widetilde{E}$ together with an isomorphism $\widetilde{E}/p\widetilde{E} \xrightarrow{\sim} E$. For any $n \geq 0$, $E \mapsto W_n(\widetilde{E})$ is a well-defined functor from vector spaces over $\mathbb{F}_p$ to modules over $\mathbb{Z}/p^n\mathbb{Z}$.*

We will denote this functor by the same symbol W_n, or $W_n(E)$ where E is a vector space over $\mathbb{F}_p$.

LEMMA 2.8. *There is a natural isomorphism*

$$W_n'(V) \xrightarrow{\sim} W_{n+1}(V)$$

PROOF. First observe that the two sides become isomorphic if one identifies the terms corresponding to the same primitive monomial Y in the decomposition from Lemma 2.5. It remains to see that this isomorphism is natural. We call a linear map $V \to (V^{\otimes p})^{C_p}$ standard if the induced map

$$V/pV \to (V^{\otimes p})^{C_p}/\mathrm{Norms}$$

is the isomorphism sending each v to v^p. From Lemma 2.6 we see that any standard map defines the same map $W_n'(V) \to W_{n+1}(V)$. On the other hand, the map $x_j \mapsto x_j^p, j \in J$, induces precisely the isomorphism above. □

2.1.1. *Restriction and Verschiebung.*

DEFINITION 2.9. Define the natural transformation

$$R : W_{n+1}(V) \to W_n(V)$$

by

$$W_{n+1}(V) \xleftarrow{\sim} W_n'(V) \to W_n(V)$$

where the isomorphism on the left is from Lemma 2.8 and the map on the right is the obvious projection.

In terms of the decomposition from Lemma 2.5, R is the projection

$$(\mathbb{Z}/p^{n+1-k}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}}) \to (\mathbb{Z}/p^{n-k}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}})$$

for every primitive monomial Y. If Y is of length p^n then it maps to zero.

DEFINITION 2.10. Define the natural transformation

$$V : W_n(V^{\otimes p}) \to W_{n+1}(V)$$

by

$$N_p : ((V^{\otimes p})^{\otimes p^n})^{C_{p^n}} \xrightarrow{\sim} (V^{\otimes p^{n+1}})^{C_{p^n}} \to (V^{\otimes p^{n+1}})^{C_{p^{n+1}}}$$

(Recall that

$$N_p = 1 + \sigma + \ldots + \sigma^{p-1};$$

note that V takes norms to norms. Indeed, on the left hand side the norm is given by

$$N = 1 + \sigma^p + \ldots + \sigma^{p(p^n-1)};$$

therefore its composition with N_p is the norm on the right).

2.2. Trace functors. Following Kaledin, we define the trace functor from a monoidal category $(\mathcal{A}, \otimes)$ to a category $\mathcal{K}$ as a functor $\mathrm{Tr} : \mathcal{A} \to \mathcal{K}$ together with a natural transformation

$$\tau_{M,N} : \mathrm{Tr}(M \otimes N) \xrightarrow{\sim} \mathrm{Tr}(N \otimes M) \tag{2.4}$$

such that

$$\tau_{M\otimes N,L}\tau_{N\otimes L,M}\tau_{L\otimes M,N} = \mathrm{id}_{\mathrm{Tr}(L\otimes M\otimes N)} \tag{2.5}$$

and

$$\tau_{M,1} = \tau_{1,M} = \mathrm{id}_{\mathrm{Tr}(M)} \tag{2.6}$$

Given a trace functor Tr from k-modules to a category $\mathcal{K}$, Kaledin defines a cyclic object $\mathrm{Tr}^\natural(A)$ of $\mathcal{K}$ for any k-algebra A. Namely, we put

$$\mathrm{Tr}^\natural(A)[n] = \mathrm{Tr}(A^{\otimes(n+1)}) \tag{2.7}$$

The face maps $d_0, \ldots, d_{n-1}$ are induced by the ones on $A^{\otimes(n+1)}$ and so are the degeneracy maps. The action of the cyclic permutation is by $\tau_{A^{\otimes n},A}$.

More generally, for a k-algebra A with an authomorphism α of order p one defines a p-cyclic object $\mathrm{Tr}^\natural(A, \alpha)$ of $\mathcal{K}$.

2.3. The construction.

LEMMA 2.11. *The cyclic permutation*

$$\sigma : (M \otimes N)^{\otimes p^n} \xrightarrow{\sim} (N \otimes M)^{\otimes p^n},$$

$$v_1 \otimes w_1 \otimes \ldots \otimes v_{p^n} \otimes w_{p^n} \mapsto w_1 \otimes v_{p^n} \otimes \ldots \otimes w_{p^n} \otimes v_1,$$

$v_i \in M,\ w_i \in N,$ *turns* W_n *into a trace functor.*

Now for an $\mathbb{F}_p$-algebra A define the cyclic $\mathbb{Z}/p^n\mathbb{Z}$-module

$$W_n^\natural(A)[k] = W_n(A^{\otimes k+1}) \tag{2.8}$$

DEFINITION 2.12.

$$W_n\mathrm{HH}_\bullet(A) = \mathrm{HH}_\bullet(W_n^\natural(A));\ W_n\mathrm{HC}_\bullet(A) = \mathrm{HC}_\bullet(W_n^\natural(A))$$

The following theorems are from [**29**] and [**30**].

THEOREM 2.13. *For a finitely generated smooth commutative algebra over* $\mathbb{F}_p$ *there is a natural isomorphism*

$$W_n\mathrm{HH}_\bullet(A) \xrightarrow{\sim} W_n\Omega_A^\bullet$$

where the right hand side denotes De Rham -Witt forms of Deligne-Illusie [**22**]. *This isomorphism intertwines the cyclic differential* B *with the De Rham differential.*

THEOREM 2.14. *For any algebra over* $\mathbb{F}_p$ *there is a natural isomorphism*

$$W_n\mathrm{HH}_0(A) \xrightarrow{\sim} W_n^H(A)$$

where the right hand side denotes Hesselholt's generalized Witt vectors [**25**] .

3. The construction of Petrov and Vologodsky

In [**41**] (cf. also [**40**]), another approach to noncommutative crystalline cohomology is proposed. For an algebra (or a DG category) over $\mathbb{F}_p$, a homology theory $\mathrm{HC}_\bullet^{\mathrm{crys}}(A)$ is constructed. It has the following properties.

(1) When A admits a lifting $\widetilde{A}$ to an algebra over $\mathbb{Z}_p$ then

$$\mathrm{HC}_\bullet^{\mathrm{crys}}(A) \xrightarrow{\sim} \widehat{\mathrm{HC}}_\bullet^{\mathrm{per}}(\widetilde{A});$$

(2)

$$\mathrm{HC}_\bullet^{\mathrm{crys}}(A) \xrightarrow{\sim} \widehat{\mathrm{TP}}_\bullet(A)$$

(TP denotes periodic topological cyclic homology; in both cases the hat denotes the p-adic completion).

A few words about the construction in general. When we do not have a lifting, what we can do is compute the periodic cyclic homology *over* $\mathbb{F}_p$. We do this in the derived sense, i. e. replace A by a DG algebra flat over $\mathbb{Z}$ and then computing the p-adically completed standard complex over the DG resolution flat over $\mathbb{Z}$. Explicitly, this resolution is

$$R = (\mathbb{Z}[\xi], p\frac{\partial}{\partial \xi}) \tag{3.1}$$

where ξ is a free commutative variable of degree -1.

There are two problems with this. First, $\mathrm{HC}_\bullet^{\mathrm{per}}(R)$ is too big.

$$\widehat{\mathrm{CC}}_\bullet^{\mathrm{per}}(R) \xrightarrow{\sim} \widehat{\mathrm{CC}}_\bullet^{\mathrm{per}}(\mathbb{Z}[\xi]) \tag{3.2}$$

(with zero differential on the algebra in the right hand side). This may be computed directly, or deduced from section 4. The right hand side projects to $\mathbb{Z}_p$ (the projection induced by $\mathbb{Z}[\xi] \to \mathbb{Z}$, $\xi \mapsto 0$). So the solution would be: tensor over $\widehat{\mathrm{CC}}_\bullet^{\mathrm{per}}(\mathbb{Z}[\xi])$ by $\mathbb{Z}_p$.

But then the second problem arises: neither side of (3.2) is in any natural way a commutative algebra. In fact, for a commutative algebra A, while $\mathrm{CC}_\bullet^{\mathrm{per}}(A)$ is an A_∞ algebra, it is not clear why it is a homotopy commutative ($\mathbb{E}_\infty$-)algebra, and if yes why is there a *preferred* E_∞ structure. Actually the known A_∞ structure [**27**] is manifestly *not* commutative.

The problem is solved as follows. As it is well known, periodic cyclic homology of a cyclic module can be computed in terms of the projection of the module onto the quotient by a certain subcategory. The projection does have a symmetric monoidal structure, and (3.1) can be established at the level of this quotient.

REMARK 3.1. The subtle issue of the symmetric monoidal structure on (periodic) cyclic homology is also studied in [**36**].

4. Operations on (co)chains and noncommutative crystalline cohomology

The contents of this chapter are from [**38**].

DEFINITION 4.1. A lifting of an $\mathbb{F}_p$-algebra A is a torsion-free p-adically complete $\mathbb{Z}_p$-module $\widetilde{A}$ with a p-adically continuous bilinear (not necessarily associative) product together with a product-preserving isomorphism $\widetilde{A}/p\widetilde{A} \xrightarrow{\sim} A$. For two $\mathbb{F}_p$-algebras A and B, a morphism of liftings $\widetilde{A} \to \widetilde{B}$ is a p-adically continuous linear map whose reduction modulo p is a morphism of algebras.

THEOREM 4.2. *For any lifting $\widetilde{A}$ of an $\mathbb{F}_p$-algebra A there exists a complex $\widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A})$ such that:*

(1) *If the product on $\widetilde{A}$ is associative then there is a quasi-isomorphism*
$$\widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A}) \xrightarrow{\sim} \widehat{\mathrm{CC}}_\bullet^{\mathrm{per}}(\widetilde{A})$$
where the right hand side is the p-adic completion of the periodic cyclic complex of the algebra $\widetilde{A}$;

(2) *For any $n \geq 1$ and for any chain of morphisms of liftings*
$$\widetilde{A}_0 \xrightarrow{f_1} \widetilde{A}_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} \widetilde{A}_n$$
there is a map
$$T(f_1, \dots, f_n) : \widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A}_0) \to \widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A}_n)$$
such that
$$[b + uB, T(f_1, \dots, f_{n+1})] = \sum_{j=2}^{n} (-1)^j T(f_1, \dots, \widehat{f_j} \dots, f_{n+1}) -$$
$$\sum_{j=2}^{n} (-1)^j T(f_j, \dots, f_{n+1}) T(f_1, \dots, f_j)$$

Here $b + uB$ denotes the differential in $\widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A})$ for any $\widetilde{A}$.

4.1. Hochschild and cyclic cohomology of DG coalgebras. We start with the Hochschild and cyclic homology of second kind introduced and studied in [**42**], [**43**], [**44**]. Note that the established generality for them is a *conilpotent* (curved) DG coalgebra which ours will automatically be. Let (C, d) be a DG coalgebra with counit. We define Hochschild and cyclic complexes of C in a way dual to what one does for algebras, with one nuance. Namely, we put

$$C_{II}^\bullet(C) = \bigoplus_{n \geq 0} C \otimes \overline{C}[-1]^{\otimes n} \tag{4.1}$$

with the differential $b + d$ and

$$\mathrm{CC}_{II}^\bullet(C) = (C_{II}^\bullet(C)[[u]], b + d + uB) \tag{4.2}$$

with the differential $b + d + vB$ Here $\overline{C}$ is the kernel of the counit and

$$b : C \otimes \overline{C}[-1]^{\otimes n} \to C \otimes \overline{C}[-1]^{\otimes(n+1)};\ B : C \otimes \overline{C}^{\otimes n}[-1] \to C \otimes \overline{C}[-1]^{\otimes(n-1)}$$

are defined dually to the standard Hochschild and cyclic differentials for algebras.

An important feature of both is that they are *not* invariant with respect to quasi-isomorphisms of DG (co)algebras. This suits us well because we are going to consider the example $C = \mathrm{Bar}(A)$ where A is a DG algebra, and C is contractible when A has a unit.

THEOREM 4.3. *For an associative algebra A there are natural quasi-isomorphisms*

$$C^{\bullet}_{II}(\mathrm{Bar}(A)) \xrightarrow{\sim} C_{\bullet}(A);\ \mathrm{CC}^{\bullet}_{II}(\mathrm{Bar}(A)) \xrightarrow{\sim} \mathrm{CC}^{-}_{\bullet}(A)$$

where the right hand side is the standard Hochschild, resp. negative cyclic, complex of A.

This is proven in [**45**] and [**17**].

4.2. The multiplicative structure on cyclic cochains. We start by extending the cyclic Alexander-Whitney constructions from [**2**] and [**34**]. Given two associative algebras A_1 and A_2, there are two homotopy inverse maps ($C_\bullet$ denotes the Hochschild complex):

$$C_\bullet(A_1 \otimes A_2) \rightleftarrows C_\bullet(A_1) \otimes C_\bullet(A_2) \tag{4.3}$$

(i.e. the Alexander-Whitney the Eilenberg-Zilber morphisms). (The left hand side is the total complex of a bisimplicial Abelian group and the right hand side is the complex of the diagonal). The maps satisfy an associativity condition when we consider three algebras. In particular, the Hochschild complex of a bialgebra is a coalgebra. Dualizing this, we see that the Hochschild complex (4.1) of a bialgebra viewed as a coalgebra acquires an algebra structure.

The above can be extended to cyclic (co)chains. For any n coalgebras $C_1 \dots, C_n$ there is a map

$$\mathrm{CC}_{II}(C_1) \otimes \dots \otimes \mathrm{CC}_{II}(C_n) \to \mathrm{CC}_{II}(C_1 \otimes \dots \otimes C_n) \tag{4.4}$$

satisfying the A_∞ relation. This implies that for any bialgebra H its cyclic complex $\mathrm{CC}_{II}(H)$ is an A_∞ algebra.

Therefore for any bialgebra we have an associated A_∞ algebra. It can be constructed explicitly when H is a cocommutative DG Hopf algebra.

4.2.1. *The DG algebra $H \star \mathrm{Cobar}(\overline{H})$.* To start with, there are two subalgebras of $\mathrm{CC}_{II}(H)$ for a bialgebra H. One is H itself (the degree zero cyclic cochains). The other is the subalgebra $k \otimes \overline{H}[-1]^{\otimes \bullet}$ which is identified with the cobar construction of the coalgebra $\overline{H}$. It turns out that the Hochschild complex of a Hopf algebra H is isomorphic to the cross product of the two. For the cyclic complex, the same is true up to an A_∞ quasi-isomorphism and with the addition of a cross term in the differential. Below are the details.

For a bialgebra H and an algebra A, an action of H on A is a linear map $H \otimes A \to A$, $x \otimes a \mapsto \rho(x)a$, such that

$$\rho(xy) = \rho(x)\rho(y);\ \rho(x)(ab) = \sum \rho(x^{(1)})(a)\rho(x^{(2)})(b)$$

If H is a Hopf algebra acting on A then one can define a cross product

$$H \star A = A \otimes H; (a \otimes x)(b \otimes y) = a\rho(x^{(1)})b \otimes Sx^{(2)}y \tag{4.5}$$

Let $A = \mathrm{Cobar}(\overline{H})$. Put

$$\rho(x)(x_1|\dots|x_n) = \sum (x^{(1)}x_1S(x^{(n+1)})|\dots|x^{(n)}x_nS(x^{(2n)})) \tag{4.6}$$

where S is the antipode. The action commutes with the differential on $\mathrm{Cobar}(\overline{\mathrm{H}})$ (which we denote by b), and we get a DG algebra $H\star\mathrm{Cobar}(\overline{H})$.

REMARK 4.4. Note that the comultiplication on $\overline{H}$ is given by

$$\Delta x=\sum x^{(1)}\otimes x^{(2)}-1\otimes x-x\otimes 1$$

In other words, $H\star\mathrm{Cobar}(\overline{H})$ is the DG algebra generated by a subalgebra H and by elements (x), linear in $x\in\overline{H}[1]$, subject to

$$x\cdot(y)=\sum(x^{(1)}yS(x^{(2)}))\cdot x^{(3)};\ bx=0;\ b(x)=\sum(x^{(1)})(x^{(2)}) \tag{4.7}$$

This DG algebra admits a derivation B determined by

$$Bx=0,\ x\in H;\ B(x)=x,\ x\in\overline{H}[-1].$$

It is easy to see that B is well defined and commutes with b. Of course, if H is a DG Hopf algebra, then its differential d induces an extra differential on $H\star\mathrm{Cobar}(\overline{H})$.

PROPOSITION 4.5. *For a cocommutative DG Hopf algebra H,*
1) there is an isomorphism of DG algebras

$$C^{\bullet}_{\mathrm{II}}(H)\xrightarrow{\sim}(H\star\mathrm{Cobar}(\overline{H}),b+d);$$

2) there is a natural $k[[u]]$-linear (u)-adically continuous A_∞ isomorphism

$$\mathrm{CC}^{\bullet}_{\mathrm{II}}(H)\xrightarrow{\sim}((H\star\mathrm{Cobar}(\overline{H}))[[u]],b+d+uB)$$

(An A_∞ isomorphism is an A_∞ morphism whose first term is invertible).

5. The action on the periodic cyclic complex

5.1. The A_∞ action of $\mathrm{CC}^{\bullet}_{\mathrm{II}}(U(\mathfrak{g}_A))$. For an associative algebra A, let $\mathfrak{g}_A$ denote the DG Lie algebra $C^{\bullet+1}(A,A)$ with the Gerstenhaber bracket which we identified with $\mathrm{Coder}(\mathrm{Bar}(A))$. We do not assume that the coderivations preserve the coaugmentation. In other words, the Abelian subalgebra $A[1]$ is contained in $\mathfrak{g}_A$. The latter defines the action of $U(\mathfrak{g}_A)$ on Bar(A) as well as linear maps

$$\mu_N\colon U(\mathfrak{g}_A)^{\otimes N}\otimes\mathrm{Bar(A)}\to\mathrm{Bar(A)} \tag{5.1}$$

(composition of the above action with the n-fold product on $U(\mathfrak{g}_A)$).

LEMMA 5.1. *The above are morphisms of DG coalgebras.*

PROOF. Clear. ☐

COROLLARY 5.2. *The compositions of*

$$\mathrm{CC}^{\bullet}_{\mathrm{II}}(U(\mathfrak{g}_A))^{\otimes N}\otimes\mathrm{CC}^{\bullet}_{\mathrm{II}}(\mathrm{Bar(A)})\longrightarrow\mathrm{CC}^{\bullet}_{\mathrm{II}}(\mathrm{U}(\mathfrak{g}_A)^{\otimes\mathrm{N}}\otimes\mathrm{Bar(A)})$$

(cf. (4.4)*) with the morphism induced by μ_N define on $\mathrm{CC}^{\bullet}_{\mathrm{II}}(\mathrm{Bar(A)})$ a structure of an A_∞ module over the A_∞ algebra $\mathrm{CC}^{\bullet}_{\mathrm{II}}(U(\mathfrak{g}_A))$.*

REMARK 5.3. In classical calculus on a manifold M, the counterpart of the periodic cyclic complex is the De Rham complex $(\Omega^{\bullet}_M((u)),ud)$. If $\mathfrak{g}$ is the algebra of multivector fields then the DG Lie algebra $(\mathfrak{g}[[u]][\epsilon],u\frac{\partial}{\partial\epsilon})$ acts on this complex: an element $X+\epsilon Y$ acts by $L_X+\iota_Y$. In characteristic zero, the DGLA $(\mathfrak{g}_A[[u]][\epsilon],u\frac{\partial}{\partial\epsilon})$ does act on $\mathrm{CC}^{-}_{\bullet}(A)$ as on an L_∞ module. Cf. for example [**48**] and, for a new and different perspective, [**7**]. The A_∞ algebra

$$\mathrm{CC}^{\bullet}_{II}(U(\mathfrak{g}_A))\xrightarrow{\sim}(U(\mathfrak{g}_A)\star\mathrm{Cobar}(\overline{U}(\mathfrak{g}_A))[[u]],b+\delta+uB),$$

while L_∞ quasi-isomorphic to the former in characteristic zero, seems to be better suited to working over $\mathbb{Z}$, and perhaps also in a more general categorical context.

5.2. The construction. Let $\mathfrak{a}$ be the graded Lie algebra with the basis consisting of one element R of degree two. We start by constructing

$$x \in \prod_{n=1}^{\infty} u^{-n}(U(\mathfrak{a}) \star_1 \mathrm{Cobar}(\overline{U}(\mathfrak{a})))^{2n} \tag{5.2}$$

satisfying

$$(\partial_{\mathrm{Cobar}} + uB)x + x^2 = -R \tag{5.3}$$

Here the graded component of degree n is spanned by elements of degree $2n$ with respect to R and of degree 0 with respect to u. We are looking for a solution of the form

$$x_F(R) = \sum_{n=1}^{\infty} x_n(R)(R^n) \text{ where } F(R, y) = \sum_{n=1}^{\infty} x_n(R) y^n \tag{5.4}$$

Equation (5.3) translates into the following two: first,

$$F(y_1 + y_2) - F(y_1) - F(y_2) + F(y_1)F(y_2) = 0$$

which implies

$$x_n(R) = -\frac{1}{n!} f(R)^n$$

for some f, and second,

$$-u \sum_{n=1}^{\infty} \frac{1}{n!} f(R)^n R^n = -R.$$

This implies

$$f(R)R = \log(1 + \frac{R}{u});\ 1 - F(R, y) = \exp(\frac{y}{R}\log(1 + \frac{R}{u})) = (1 + \frac{R}{u})^{\frac{y}{R}};$$

we conclude that

$$F(R, y) = -\sum_{n=1}^{\infty} \frac{u^{-n}}{n!} y(y - R)\ldots(y - (n-1)R) \tag{5.5}$$

A crucial observation for us is that the homogeneous part of $x_F(R)$ ((5.4)) of total degree n in R has the denominator $n!$.

Now assume that we have a chain of morphisms

$$\widetilde{A}_0 \xrightarrow{f_1} \widetilde{A}_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} \widetilde{A}_n \tag{5.6}$$

Here $\widetilde{A}_j$ is a lifting of an algebra A_j and f_j are morphisms modulo p. Let

$$\mathcal{U}(m_0, \ldots, m_n) = U(\mathrm{Lie}(m_0, \ldots, m_n)) \star_1 \mathrm{Cobar}(\overline{U}(\mathrm{Lie}(m_0, \ldots, m_n)) \tag{5.7}$$

where $\mathrm{Lie}(m_0, \ldots, m_n)$ stands for the free Lie algebra with generators $m_0, \ldots, m_n$ each of degree one. We claim that there is a pairing

$$\mathcal{U}(m_0, \ldots, m_n) \otimes \widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A}_0) \to \widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A}_n) \tag{5.8}$$

Moreover: define the DG category as follows. An object is a lifting $\widetilde{A}$ of an $\mathbb{F}_p$-algebra A. A morphism from $\widetilde{A}_0$ to $\widetilde{A}_n$ is a pair of a chain $\{f_j\}$ (5.6) and an element of $\mathcal{U}(m_0, \dots, m_n)$. Composition is concatenation of chains together with the product

$$\mathcal{U}(m_0,\dots,m_n)\otimes\mathcal{U}(m_n,\dots,m_{n+k})\to\mathcal{U}(m_0,\dots,m_{n+k}) \tag{5.9}$$

(embed the two algebras on the left into the one on the right, and multiply there). Then (5.8) extends to an A_∞ functor from this DG category to the category of complexes.

To see that, we need to refer to section 8, namely to 8.1. Define for any chain (5.6) a morphism of coalgebras

$$U(\mathrm{Lie}(m_0))\otimes\dots\otimes U(\mathrm{Lie}(m_n))\to \mathrm{Bar}(\mathbf{C}^\bullet(A_0,A_n)) \tag{5.10}$$

as follows:

$$m_0^{k_0}\otimes\dots\otimes m_n^{k_n}\mapsto f_{1*}\widetilde{m}_0^{\bullet k_0}\bullet\dots\bullet f_{n*}\widetilde{m}_{n-1}^{\bullet k_{n-1}}\bullet\widetilde{m}_n^{\bullet k_n} \tag{5.11}$$

Here $\widetilde{m}_j$ are cochains in $C^2(\widetilde{A}_j,\widetilde{A}_j)$ corresponding to the product. The above intertwines the $\bullet$ product(8.7) with the product in $U(\mathrm{Lie}(m_0))\otimes\dots\otimes U(\mathrm{Lie}(m_{n+k}))$. Now follow (5.10) by

$$U(\mathrm{Lie}(m_0,\dots,m_n))\to U(\mathrm{Lie}(m_0))\otimes\dots\otimes U(\mathrm{Lie}(m_{n+k}))$$

and apply $\mathrm{CC}^\bullet_{II}$. We then get (5.8).

Next, let $\mathfrak{g}$ be the free graded Lie algebra over $k[[u]]$ generated by three elements λ of degree 1 and $\delta\lambda$, R of degree 2. Define a derivation δ of degree one by

$$\delta:\lambda\mapsto\delta\lambda\mapsto[R,\lambda];\ R\mapsto 0.$$

(For us, this is a subalgebra of $\mathrm{Lie}(m_0,m_1)[\frac{1}{2}][[u]]$ with $\lambda=m_1-m_0$, $R=m_0^2$, and $\delta=[m_0,\ ]$).

We assign weight one to λ and $\delta\lambda$, and weight two to R. Extend the weight to the algebra

$$\mathcal{U}=U(\mathfrak{g})\star_1\mathrm{Cobar}(\overline{U}(\mathfrak{g})) \tag{5.12}$$

multiplicatively. Denote by $\mathfrak{g}(n)$, $\mathcal{U}(n)$, etc. the span of all homogeneous elements of weight at least n. Let

$$\widehat{\mathcal{U}}=\prod_{k=0}^{\infty}\frac{u^{-k}k!}{n!}\mathcal{U}(k)[[u]] \tag{5.13}$$

For any $r\in\mathfrak{g}^2(1)$, define an element of

$$x_F(r)\in\widehat{\mathcal{U}} \tag{5.14}$$

by (5.4) with R replaced by r.

Consider the differential

$$\mu(t)=\delta+x(R)t-(-1)^l tx(R-\delta\lambda+\lambda^2) \tag{5.15}$$

for $t\in\mathcal{U}^l$.

We construct an invertible element t_{01} of degree zero in the completion $\widehat{U}$, such that

$$(\mu+\partial_{\mathrm{Cobar}}+uB)t_{01}+t_{01}\lambda=0 \tag{5.16}$$

We write $t_{01}=1+x_1+x_2+\dots$ where x_k is in $\mathcal{U}(k)$. We find x_n recursively just as we did above, using the acyclicity of the differential induced by μ on $\mathcal{U}(n)/\mathcal{U}(n+1)$. For example, $x_1=-\frac{(\lambda)}{u}$.

It remains to prove that $\mathrm{gr}(\mathcal{U})$ is indeed acyclic. Let $\mathfrak{g}_0$ be the graded Lie algebra $\mathfrak{g}$ with the differential δ_0 defined by

$$\delta_0 : \lambda \mapsto \delta\lambda \mapsto 0;\ R \mapsto 0.$$

Define the DGA

$$\mathcal{U}_0 = U(\mathfrak{g}_0) \star_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}_0)) \tag{5.17}$$

with the differential $\delta_0 + \partial_{\mathrm{Cobar}} + uB$. Then

$$\mathrm{gr}(\mathcal{U}) \xrightarrow{\sim} \oplus \frac{u^{-k}}{k!} \mathcal{U}_0(k)$$

Looking at the differential δ_0 as the leading term of a spectral sequence, we see that the above is quasi-isomorphic to itself with $\mathfrak{g}_0$ is replaced by $\mathfrak{a}_0$, the latter being the free graded Lie algebra generated by R of degree two (and weight one). Looking at uB as the leading term in a spectral sequence, we see that our complex is indeed acyclic.

More generally, let $\mathfrak{g}_n$ be the free graded algebra with generators λ_{0j} of degree one and $\delta\lambda_{0j}$ of degree two, $1 \leq j \leq n$, as well as R of degree two. Let the weight of $\delta\lambda_{0j}$ and λ_{0j} be one, and the weight of R be two. Define a derivation δ of degree one by

$$\delta : \lambda_{0j} \mapsto \delta\lambda_{0j} \mapsto [R, \lambda_{0j}]. \tag{5.18}$$

Remark 5.4. Consider the graded Lie algebra generated by elements $m_0, \ldots, m_n$ of degree one and R of degree two, subject to the relation $[m_0, m_0] = 2R$. Let $\delta = [m_0, \ ]$. The span of all monomials of degree > 1 and of $m_0 - m_j$, $1 \leq j \leq n$, is a graded subalgebra stable under δ. It maps to $\mathfrak{g}_n$ via $m_0 - m_j \mapsto \lambda_{0j}$; $R \mapsto R$.

To finish the proofs, recall the A_∞ module structure given by Proposition 4.5 and Corollary 5.2. Together with the above, it gives an A_∞ functor to the category of complexes from the following DG category: objects are liftings of $\mathbb{F}_p$-algebras; morphisms are:

(1) morphisms of liftings (recall: they preserve the product only modulo p, and the products are associative only modulo p);
(2) in addition, every object has an endomorphism of degree one and square zero; and
(3) morphisms in (1) commute with the morphisms in (2).

The value of the A_∞ functor on an object $\widetilde{A}$ is the complex $\widehat{\mathrm{CC}}_\bullet^{\mathrm{PER}}(\widetilde{A})$.

6. Comparisons between different constructiions

Let us start by comparing sections 3 and 4. They obviously give the same result when A admits a lifting to an algebra over $\mathbb{Z}_p$. When this is not the case, the constructions diverge; we strongly expect them to give the same answer. Note that the general construction of section 3 is carried out in terms quite close to the methods of section 4. Clarifying these connections could be quite instructive.

Next we compare sections 2 and 4. We start by giving more details on the approach of section 2.

6.1. More on Frobenius under the trace. Let E be a finite dimensional space over $\mathbb{F}_p$. Corollary 2.2 provides a linear map

$$F : E \to (E^{\otimes p})_{C_p} \tag{6.1}$$

sending e to e^p for any e in E. Passing to dual spaces and taking the adjoint operator, we get

$$C : (E^{\otimes p})^{C_p} \to E \tag{6.2}$$

For a more general perfect field of scalars k, those maps are linear if the action on E is twisted by the Frobenius automorphism of k.

Explicitly: when a basis of E is chosen, F and C act on the corresponding basis of $E^{\otimes p^n}$ as follows: F sends a monomial to its pth power; C sends a monomial X to Y if $X = Y^p$ for some Y; if there is no such Y then X is sent to zero. Applying this to $E^{\otimes p^n}$ instead of E, we get an inverse system

$$C : (E^{\otimes p^n})^{C_{p^n}} \to (E^{\otimes p^{n-1}})^{C_{p^{n-1}}} \tag{6.3}$$

LEMMA 6.1. *For a free $\mathbb{Z}$-module M there is a natural isomorphism*

$$W_{n+1}(M)/pW_{n+1}(M) \xrightarrow{\sim} ((M/pM)^{\otimes p^n})^{C_{p^n}}$$

which intertwines the restriction R with C as in (6.3).

PROOF. This was proven in [**31**], [**29**] and essentially follows from Lemma 2.5. $\square$

6.1.1. *Recollection on cyclic objects.* Fix a unital commutative ring K. For a small category $\mathcal{C}$, a $\mathcal{C}$-module is by definition a functor $\mathcal{C} \to K-\mathrm{mod}$. For a functor $f : \mathcal{C}_1 \to \mathcal{C}_2$, we have the restriction functor

$$f^* : \mathcal{C}_2 - \mathrm{mod} \to \mathcal{C}_1 - \mathrm{mod} \tag{6.4}$$

There are two functors

$$f_!,\ f_* : \mathcal{C}_1 - \mathrm{mod} \to \mathcal{C}_2 - \mathrm{mod} \tag{6.5}$$

The functor $f_!$ is left adjoint and the functor f_* is right adjoint to f^*. A bit informally,

$$f_!M = \mathcal{C}_2 \times_{\mathcal{C}_1} M;\ f_*M = \mathrm{Hom}_{\mathcal{C}_1}(\mathcal{C}_2, M).$$

Recall the definition of the cyclic category Λ ([**11**]) and its generalizations Λ_p for any natural number p ([**16**], [**39**]). For $p = 1$, $\Lambda = \Lambda_p$. By definition, a cyclic K-module is a Λ^{op}-module; a p-cyclic K-module is a Λ_p^{op}-module. One has

$$\mathrm{Aut}_{\Lambda p}([n]) \xrightarrow{\sim} C_{p(n+1)}$$

(the cyclic group). the subgroups C_p for all n form the center of Λ_p.

REMARK 6.2. One also defines the category Λ_∞ ([**14**], [**4**], [**16**]) with the same objects, with

$$\mathrm{Aut}_{\Lambda_\infty}([n]) \xrightarrow{\sim} \mathbb{Z}.$$

For any $p \geq 1$, Λ_p is the quotient (in an easy precise sense) of Λ_∞ over the subgroup $p(n+1)\mathbb{Z}$.

Let Δ be the category whose objects are $[n] = \{0, \ldots, n\}$ for all $n \geq 0$ and whose morphisms are order-preserving maps. By definition, simplicial K-modules are Δ^{op}-modules.There are inclusions $j_p : \Delta \to \Lambda_p$ for all p.

For any associative algebra A over K one defines the Λ^{op}-module $A^\natural$. More generally, for an algebra A together with an automorphism α of degree p one defines the Λ_p^{op}-module $A_\alpha^\natural$.One has

$$A_\alpha^\natural([n]) = A^{\otimes n+1} \tag{6.6}$$

Each group $\mathrm{Aut}_{\Lambda p}([n])$ has a generator t_n that acts by

$$t_n(a_0 \otimes \ldots \otimes a_n) = \alpha(a_n) \otimes a_0 \otimes \ldots \otimes a_{n-1}$$

We put

$$\tau_n = t_n^{n+1} \tag{6.7}$$

For any small category $\mathcal{C}$ define

$$\mathrm{pr}_{\mathcal{C}} : \mathcal{C} \to *$$

to be the unique morphism to the category with one morphism. For a simplicial module M define

$$\mathrm{HH}_\bullet(M) = \mathbb{L}_\bullet \mathrm{pr}_{\Delta!}(M) \tag{6.8}$$

For a p-cyclic module M define

$$\mathrm{HC}_\bullet(M) = \mathbb{L}_\bullet \mathrm{pr}_{\Lambda_p!}(M);\ \mathrm{HH}_\bullet(M) = \mathrm{HH}_\bullet(j_p^* M) \tag{6.9}$$

For an algebra A with an automorphism α of degree p one has

$$\mathrm{HH}_\bullet(A_\alpha^\natural) \xrightarrow{\sim} H_\bullet(A, {}_\alpha A) \tag{6.10}$$

(the Hochschild homology of A with coefficients in A on which A acts by multiplication on the right, and by multiplication twisted by α on the left). Also,

$$\mathrm{HC}_\bullet(A_\alpha^\natural) \xrightarrow{\sim} \mathrm{HC}_\bullet(A, \alpha) \tag{6.11}$$

(the generalized cyclic homology from [**17**]). This is the generalized theorem of Connes [**11**].

6.2. Subdivisions. Here we follow [**28**]. There are two functors

$$\pi_p,\ i_p : \Lambda_p \to \Lambda. \tag{6.12}$$

The functor π_p sends τ_n to 1. One has

$$\pi_{p!}(M) = M_{C_p};\ \pi_{p*}(M) = M^{C_p} \tag{6.13}$$

where C_p is the center. To define i_p, start with any algebra A and pass to the algebra $A^{\otimes p}$ with the automorphism α acting by cyclic permutation. Now construct the p-cyclic module $(A^{\otimes p})_\alpha^\sharp$. We have

$$(A^{\otimes p})_\alpha^\sharp[n] = A^\sharp[p(n+1) - 1]$$

and it is clear from the construction that the rest of the p-cyclic structure on $(A^{\otimes p})_\alpha^\sharp$. is constructed in terms of the cyclic structure on $A^\sharp$. In other words, there exists unique functor $i_p : \Lambda_p \to \Lambda$ such that

$$(A^{\otimes p})_\alpha^\sharp = i_p^*(A^\sharp) \tag{6.14}$$

In particular

$$i_p([n]) = p(n+1) - 1$$

Proposition 6.3. *There are natural isomorphisms*

$$\mathrm{HH}_\bullet(i_p^*M) \xrightarrow{\sim} \mathrm{HH}_\bullet(M);\ \mathrm{HC}_\bullet(i_p^*M) \xrightarrow{\sim} \mathrm{HC}_\bullet(M)$$

where M is any cyclic module.

6.2.1. *Frobenius under the trace and subdivisions.* Let A be an algebra over $\mathbb{F}_p$. Apply Lemma 6.1 and (6.3) to $E = A^{\otimes(n+1)}$. It is straightforward that the morphisms C and F are compatible with the cyclic structures and therefore define morphisms

$$F : A^\natural \to \pi_{p!} i_p^* A^\natural \tag{6.15}$$

$$C : \pi_{p*} i_p^* A^\natural \to A^\natural \tag{6.16}$$

There is an inverse system of morphisms of cyclic modules

$$C : \pi_{p^n *} i_{p^n}^* A^\natural \to \pi_{p^{n-1} *} i_{p^{n-1}}^* A^\natural \tag{6.17}$$

and the diagram where the vertical maps are isomorphisms:

$$\begin{tikzcd} W_{n+1}^\natural(A)/pW_{n+1}^\natural(A) \arrow[r, "R"] \arrow[d] & W_n^\natural(A)/pW_n^\natural(A) \arrow[d] \\ \pi_{p^n *} i_{p^n}^* A^\natural \arrow[r, "C"] & \pi_{p^{n-1} *} i_{p^{n-1}}^* A^\natural \end{tikzcd} \tag{6.18}$$

To summarize: We observe that the periodic version of the Hochschild-Witt complex has some resemblance to the construction in section 4. Both use a lifting of A. Both, when reduced modulo p, become nonstandard (larger) complexes that compute periodic cyclic homology of A or something close to it. We hope that a common framework for operations on higher Hochschild chains and cochains, as discussed in the end of this article, will allow us to compare the two constructions directly.

7. Noncommutative dagger completions

In [**12**] and [**13**] Cortiñas, Cuntz, Meyer, Mukherjee, and Tamme introduce and study another approach to noncommutative crystalline cohomology, the one based on Monsky and Washnitzer's work [**37**]. It would be very interesting to compare this with another approaches outlined in this article.

8. What do DG categories form?

8.1. A category in cocategories. For every two algebras A and B and any two morphisms $f, g : A \to B$ we consider the Hochschild cochain complex $C^\bullet(A, {}_fB_g)$ where B is viewed as an A-bimodule on which A acts on the left via f and on the right via g. We use the picture

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd} \tag{8.1}$$

to refer to such cochains. There is the cup product

$$C^\bullet(A, {}_fB_g) \otimes C^\bullet(A, {}_gB_h) \to C^\bullet(A, {}_fB_h).$$

In other words, given two cochains described by the picture

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, "g"] \arrow[r, bend right, "h"'] & B \end{tikzcd} \tag{8.2}$$

one produces a cochain corresponding to

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] & B \end{tikzcd}$$

Also, given a cochain φ in $C^\bullet(B, {}_{g_1}C_{g_2})$,

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & C \arrow[r, "h"] & D \end{tikzcd} \tag{8.3}$$

one defines a cochain $h_* f^* \varphi$ in $C^\bullet(A, {}_{hg_1f}D_{hg_2f})$.

$$\begin{tikzcd} A \arrow[r, bend left, "hg_1f"] \arrow[r, bend right, "hg_2f"'] & D \end{tikzcd}$$

Given two cochains as shown below, i.e. φ in $C^\bullet(A, {}_{f_1}B_{g_1})$ and ψ in $C^\bullet(B, {}_{f_2}C_{g_2})$,

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] & B \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] & C \end{tikzcd} \tag{8.4}$$

there are two ways to produce a cochain n $C^\bullet(A, {}_{f_2f_1}C_{g_2g_1})$.

$$\begin{tikzcd} A \arrow[r, bend left, "f_2f_1"] \arrow[r, bend right, "g_2g_1"'] & C \end{tikzcd}$$

One is $f_{2*}\varphi \cup g_1^*\psi$ and the other is $f_1^*\psi \cup g_{2*}\varphi$. There is a homotopy between the two, given by the brace operation $\psi\{\varphi\}$.

One may ask whether any two ways to compose the cup product, f_*, and f^* are essentially the same (up to homotopy). Below we outline an affirmative answer.

Let us first look at vertical compositions, i.e. at the cup product. It is associative, so we can define a DG category $\mathbf{C}^\bullet(A, B)$ whose objects are morphisms $A \xrightarrow{f} B$ and whose morphisms are $C^{(}(A, {}_fB_g)$, with the cup product being the composition. Naively we could expect these to form a 2-category, in which case we would have a functor $\mathbf{C}^\bullet(A, B) \otimes \mathbf{C}^\bullet(B, C) \to \mathbf{C}^\bullet(A, C)$ satisfying the associativity condition. As we have seen, there are not one but two candidates for such a functor, and the associativity could be true up to homotopy at best.

In reality, algebras form a category not in categories but in cocategories. Namely, let

$$\mathbf{B}(A, B) = \mathrm{Bar}(\mathbf{C}^\bullet(A, B)) \tag{8.5}$$

be the bar construction. Recall that for a (small, conilpotent) DG category $\mathbf{C}$

$$\mathrm{Bar}(\mathbf{C})(f, g) = \bigoplus_{n=0}^{\infty} \bigoplus_{h_1, \ldots, h_n \in \mathrm{Ob}(\mathbf{C})} \mathbf{C}(f, h_1)[1] \otimes \ldots \otimes \mathbf{C}(h_n, g)[1] \tag{8.6}$$

The coproduct is the deconcatenation. The differential ∂_{Bar} is the usual bar differential plus the one induced by the one in $\mathbf{C}$. The term with $n = 0$ stands for $\mathbf{C}(f, g)[1]$.

Generalizing [**18**] and [**19**], we define a morphism of DG cocategories

$$\bullet : \mathrm{Bar}(\mathbf{C}^\bullet(A, B)) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(B, C)) \to \mathrm{Bar}(\mathbf{C}^\bullet(A, C)) \tag{8.7}$$

which satisfies the associativity property (when four algebras A, B, C, D are chosen).

We can define $\mathrm{CC}^\bullet_{II}(\mathbf{B})$ for any DG cocategory $\mathbf{B}$. We get an A_∞ category whose objects are algebras and whose morphisms are $\mathrm{CC}^\bullet_{II}(\mathbf{B}(A, B))$. We also have an A_∞ module $A \mapsto \mathrm{CC}^\bullet_{II}(\mathrm{Bar}(A))$.

8.2. Category in categories. One can obtain a version of a two-category structure on the category of algebras when one applies to (8.7) the functor Cobar. Now to any two algebras one puts in correspondence the DG category

$$\mathcal{C}(A, B) = \mathrm{Cobar}(\mathrm{Bar}(\mathbf{C}^\bullet(A, B))) \tag{8.8}$$

which is quasi-isomorphic to $\mathbf{C}^\bullet(A, B)$. Those still do not form a strict two-category because Cobar is not compatible to the tensor product in the strictest possible way. Rather, Cobar is a lax monoidal functor which is enough to define a reasonable 2-category structure on algebras ([**38**], [**48**]).

Remark 8.1. Note that, at least in the classical (not derived) sense, algebras form a 3-category, namely, a symmetric monoidal 2-category (the monoidal structure being the tensor product of algebras). The contents of 8.3 below address some aspects of this structure in the derived situation. The other very important aspect of this is the issue of the multiplicative structure on Hochschild and periodic cyclic chains (as we saw in section 3).

8.3. Higher Hochschild complexes. Now, in addition to cochains described by the "bigon" (8.1), let us consider more general $2k$-gons such as the one below (for k=2)

$$\begin{tikzcd} A_1 \arrow[r, "f_{11}"] \arrow[d, "g_{12}"'] & B_1 \\ B_2 & A_2 \arrow[l, "f_{22}"] \arrow[u, "g_{21}"'] \end{tikzcd} \tag{8.9}$$

Namely, for any algebras $A_j, B_j, 1 \le j \le k$, and for any morphisms $f_{jj} : A_j \to B_j$ and $g_{j,j+1} : A_j \to B_{j+1}$, $1 \le j \le k$ (the indices are added modulo k, we define the complex

$$C^\bullet(\otimes_{j=1}^k A_j, \ \ {}_{\otimes f_{jj}}(\otimes_{j=1}^k B_j)_{\otimes g_{j,j+1}}) \tag{8.10}$$

One can generalize the cup product (8.2); namely, for two cochains

$$\begin{tikzcd} A_1 \arrow[r, "f_{11}"] \arrow[d, "g_{12}"'] & B_1 \arrow[d, leftrightarrow, "g_{21}"] \\ B_2 & A_2 \arrow[l, "f_{22}"', "g_{22}=f_{22}"] \arrow[d, "f_{23}"] \\ A_3 \arrow[u, "f_{32}"] \arrow[r, "g_{33}"'] & B_3 \end{tikzcd} \tag{8.11}$$

one produces a cochain

$$\begin{tikzcd} A_1 \arrow[r, "f_{11}"] \arrow[d, "g_{12}"'] & B_1 \\ B_2 & A_2 \arrow[u, "g_{21}"'] \arrow[d, "f_{23}"] \\ A_3 \arrow[u, "f_{32}"] \arrow[r, "g_{33}"'] & B_3 \end{tikzcd}$$

Also, for a morphism $f : A'_j \to A_j$, resp. $B_j \to B'_j$, one defines f^*, resp. g_*.

Furthermore, one can generalize (8.4) and define the $\bullet$ product that takes two cochains as shown below

$$\begin{tikzcd} & C_1 & B_3 \arrow[l, "g_{31}"'] \arrow[d, "f_{32}"] \\ A_2 \arrow[r, "f_{22}"] \arrow[d, "g_{21}"'] & B_2 \arrow[u, "f_{21}"] \arrow[r, "g_{22}"] & C_2 \\ B_1 & A_1 \arrow[l, "f_{11}"] \arrow[u, "g_{12}"'] & \end{tikzcd} \tag{8.12}$$

and produces a cochain described by

$$\begin{tikzcd} & C_1 & B_3 \arrow[l, "g_{31}"'] \arrow[d, "f_{32}"] \\ A_2 \arrow[ur, "f_{21}f_{22}"] \arrow[d, "g_{21}"'] & & C_2 \\ B_1 & A_1 \arrow[l, "f_{11}"] \arrow[ur, "g_{22}g_{21}"'] & \end{tikzcd}$$

As in (8.4), this product is a homotopy between two different ways to compose the two cochains using the cup product and the operations of direct and inverse image. When one takes $A_j = B_j = A$ and $f_{jj} = g_{j,j+1} = \mathrm{id}_A$ for all j, one defines the Kontsevich-Vlassopoulos bracket on

$$\prod_{k=1}^{\infty} C^{\bullet+1}(A^{\otimes k},\ {}_{\alpha}(A^{\otimes k}))^{C_k} \tag{8.13}$$

of degree -1 in k. As above, α is the cyclic permutation.

One would expect a generalization of the construction mentioned in 8.2. Namely, a strict structure would be as follows. We have defined a complex $C^\bullet(K)$ corresponding to a $2k$-gon K. For any picture which is a union of $2k$-gons such as (8.11) or (8.12),

$$K = \cup_{j=1}^m K_j \tag{8.14}$$

there should be an operation

$$\mathrm{Op}(K_1, \ldots, K_m) : \otimes_{i=1}^m C^\bullet(K_j) \to C^\bullet(K) \tag{8.15}$$

and an associativity condition for $\mathrm{Op}(K_1, \ldots, K_n)$ and $\mathrm{Op}(K_{j1}, \ldots, K_{j,nj})$ for every "double subdivision"

$$K = \cup_{i=1}^m K_j;\ K_j = \cup_{i=1}^{n_j} K_{j,i},\ 1 \le j \le m \tag{8.16}$$

More realistically, there should be a version of the notion of an operad (related to and generalizing Batanin's two-operads):

(1) a collection of complexes $\mathcal{O}(K; K_1, \ldots, K_m)$ for any subdivision (8.14);
(2) compositions

$$\mathcal{O}(K; \{K_j\}) \otimes \otimes_{j=1}^m \mathcal{O}(K_j; \{K_{ji}\}) \to \mathcal{O}(K; \{K_{ji}\}) \tag{8.17}$$

for any "double subdivision" (8.16);
(3) an associativity condition for any "triple subdivision"

$$K = \cup_{i=1}^m K_j;\ K_j = \cup_{i=1}^{n_j} K_{j,i},\ 1 \le j \le m;\ K_{j,i} = \cup_\ell K_{\ell,j,i} \tag{8.18}$$

An algebra over such a generalized operad will be

(1) A complex $C^\bullet(K)$ for each K;
(2) A morphism

$$\mathcal{O}(K; \{K_j\}) \otimes \otimes_{j=1}^m C^\bullet(K_j) \to C^\bullet(K)$$

for every subdivision (8.14) which is compatible with composition for any (8.16).

We expect the higher Hochschild complexes (8.10) to form an algebra over a generalized operad $\mathcal{O}$ which is homotopically constant, i.e. such that $\mathcal{O}(K; \{K_j\})$ are all weakly homotopy equivalent to the scalar ring k. This would generalize Tamarkin's theorem [**46**].

Furthermore, for a $2k$-gon $\{A_j, B_j, f_{jj}, g_{j,j+1}\}$ there is also the chain complex

$$C_\bullet(\otimes_{j=1}^k A_j,\ {}_{\otimes f_{jj}}(\otimes_{j=1}^k B_j)_{\otimes g_{j,j+1}}) \tag{8.19}$$

(In fact, when $k > 1$, there are also mixed chain-cochain complexes). The above should generalize to this situation, within the context of (generalized) multi-colored operads.

Remark 8.2. This is not quite straightforward because chains have different functoriality properties. For example, given morphisms as in (8.3), at the level of cochains we get morphisms of complexes $C^\bullet(B, {}_{g_1}C_{g_2}) \to C^\bullet(A, {}_{hg_1f}D_{hg_2f})$ (as we saw earlier); but at the level of chains we get

$$C^\bullet(A, {}_{g_1f}C_{g_2f}) \to C^\bullet(B, {}_{hg_1}D_{hg_2})$$

In [**38**] we proved two results about the structure of chains and cochains in a two-categorical language. Firstly, we showed that the category in cocategories $\mathrm{Bar}(\mathbf{C}^\bullet(\_,\_))$ admits a trace functor up to homotopy (in a precise sense); this structure involves only the Hochschild chain complexes $\mathrm{TR}_A(f) = C_\bullet(A,_f A)$ for endomorphisms f of algebras. The structure on all $C_\bullet(A, \ {}_fB_g)$ is what we call a twisted tetramodule structure over $\mathrm{Bar}(\mathbf{C}^\bullet(\_,\_))$.

To summarize: we expect a rich but homotopically constant structure on higher Hochschild chains and cochains. When restricted to the case when all algebras are the same and all morphisms are identities, the structure stops being homotopically constant (not unlike passing from EG to BG for a group G). We then should recover a package consisting of the Kontsevich-Vlassopoulos Lie algebra structure on Hochschild cochains, the action(s) of this Lie algebra on Hochschild chains, and the subdivision isomorphism of Proposition 6.3 (which produces the Frobenius in characteristic p).

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