This is a draft of a book.

Cyclic homology

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Contents

CHAPTER 1

Introduction

Many geometric objects associated to a manifold M can be expressed in terms of an appropriate algebra A of functions on M (measurable, continuous, smooth, holomorphic, algebraic, ...). Very often those objects can be defined in a way that is applicable to any algebra A, commutative or not. Study of associative algebras by means of such objects of geometric origin is the subject of noncommutative geometry [**97**], [**?**]. The Hochschild and cyclic (co)homology theory is the part of noncommutative geometry which generalizes the classical differential and integral calculus. The geometric objects being generalized to the noncommutative setting are differential forms, densities, multivector fields, etc.

In later developments, starting roughly from the nineties, it became clear that the notion of a "noncommutative space" should be significantly generalized to include differential graded or A_∞ categories, ringed spectra, or stable infinity categories ****refs?*** The scope of this book is within linear algebra and includes the first two generalizations mentioned above. It tries, however, to align itself with recent developments in the context of the other two. Also, in the spirit of Loday's book and of Kaledin's words "we want a calculus, not a theory", we do not use triangulated categories, model categories, and infinity categories, as well as operads. (Nore are we mentioning Fréchet algebras or C^* algebras, for that matter). There are many excellent sources on those, and our exposition is often closely motivated by these concepts and provides building blocks for statements involving them (as for example in 7 and 6.

In our exposition, the primary object is the negative cyclic complex $CC^-_\bullet(A)$. Other complexes, namely the Hochschild chain complex $C_\bullet(A)$, the periodic cyclic complex $CC^{\mathrm{per}}_\bullet(A)$, and the cyclic complex $CC_\bullet(A)$, are defined as results of some natural procedure applied to $CC^-_\bullet(A)$. The cyclic homology is the homology of the cyclic complex $CC_\bullet(A)$. It was originally defined using another standard complex which we denote by $C^\lambda_\bullet(A)$. The study of this latter complex has a distinctly different flavor, mainly coming from the fact that it is related to the Lie algebra homology.

The above complexes are noncommutative versions of the space of differential forms (the Hochschild chain complex) and of the De Rham complex. One also defines the Hochschild cochain complex $C^\bullet(A, A)$ which is a noncommutative analogue of the space of multivector fields.

Hochschild and cyclic homology may be defined from several starting points. Firstly, we can start from the explicit complexes mentioned above and relate them to differential forms when our algebra is commutative. One can also interpret the Hochschild and cyclic homology in terms of Connes' cyclic objects and their homology (chapter 6). This approach is well suited to developing analogies between this homology and De Rham cohomology in positive characteristic, and to eventually replacing rings by ring spectra and stable infinity categories. Finally, one can start with defining our homology in terms of noncommutative forms (chapter 1).

In much of the book we study rather systematically various algebraic structures on the above complexes. These structures are supposed to generalize the classical algebraic structures arising in calculus. Sections 2 and **??** are devoted to the structures on the complex $CC_\bullet^-(A)$ and related complexes; section 4 deals with the cyclic complex $C_\bullet^\lambda(A)$.

These algebraic structures can be studied on three levels of complexity. One can start by defining basic algebraic operations generalizing the product and the bracket of multivector fields and various pairings between multivectors and forms; these are operations at the level of complexes. They acquire the standard prorerties from calculus only if one passes from complexes to their cohomology. When the algebra A is the algebra of functions on a manifold, then the homology of the Hochschild chain complex is the space of forms, the cohomology of the Hochschild cochain complex is the space of multivectors, etc. So, in a more naive sense, even at the first level of noncommutative calculus we already achieve our goal: to generalize the basic definitions and structures of calculus to the noncommutative setting (subsections 1, 5). But this generalization is not interesting when the algebra becomes noncommutative. For example, for the algebra of differential operators on $\mathbb{R}^n$ both the spaces of "noncommutative forms" and "noncommutative multivectors" are one-dimensional.

To make noncommutative calculus useful for applications, one has to pass to the second level of complexity. Namely, one tries to reproduce more of the standard algebraic structure from the classical calculus not at the level of cohomology but rather at the level of complexes. Without using transcendental methods this plan succeeds partially, in the following sense: one can define certain algebraic structures on Hochschild chains and cochains. These structures are only part of what one could expect from the classical calculus. The advantage of these structures is that they are defined by explicit, canonical constructions (**??**, **??**). They are also adequate for some applications, namely for the index theorems for symplectic deformations.

*** SOMETHING NOT IN THE BOOK

To construct a richer algebraic structure on Hochschild chains and cochains, one that generalizes a fuller version of the classical calculus, one has to go to a different level of complexity. The work in this direction was started by Tamarkin in [**?**] and then continued in [**?**], [**?**], [**?**]. These methods are outside the scope of this book. They provide a considerable refinement of the results of **??**, **??**, but there is no canonical and explicit construction anymore. The "non-comutative differential calculus" can be constructed using some inexplicit formulas; a choice of coefficients in these formulas depends on a choice of a Drinfeld associator [**?**]. In particular, the Grothendieck-Teichmüller group acts on the space of all such calculi. This version of noncommutative calculus allows one to generalize the index theorem from symplectic deformations to arbitrary deformations.

Note that in [**97**], [**116**],[**115**],[**114**] a different, though clearly related, version of noncommutative calculus is used. In particular, there the renormalization group appears as a hidden group of symmetries of the calculus, whereas in our construction the Grothendieck-Teichmüller group acts on the space of universal formulas defining a calculus. This group is closely related to the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note that finding a unified symmetry group incorporating both the renormalization group and the Galois group of $\mathbb{Q}$ is one of the important aims of Connes' noncommutative geometry program. In light of this, it seems to be an interesting problem to find a unified framework to the two approaches to noncommutative calculus (the second author would like to thank Alain Connes for his remarks on this subject).

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The key to constructing noncommutative differential calculus is an answer to Drinfeld's question "What do DG categories form?" [] It is well-known that rings form not only a category but a two-category, bimodules playing the role of one-morphisms and morphisms of bimodules the role of two-morphisms. (In addition, tensor product of rings makes this a monoidal two-category, or a three-category with one object). The crucial point for us is being able to say this correctly for DG categories, in the derived context. Our construction here is intended to serve as a bridge between the theory of Lurie and the constructions that are more traditional in the operadic approach to formality theorems. It is, essentially, based on a specialized and simplified version of Lurie's definitions (chapter 17).

In chapter 2 we give the main definitions, both of the standard chain and cochain complexes, in the generality of A_∞ algebras. We also introduce, following Wodzicki, the notion of an H-unital algebra and prove excision for H-unitalideals.

In chapter 3 we start the study of operations on Hochschild and cyclic complexes. We define and study the Eilenberg Zilber product, the Alexander-Whitney coproduct, and the pairings between chains and cochains. All these are classical operations of homological algebra that are extended from Hochschild to cyclic chains when appropriate. We present first applications of operations, namely, to the simplest cases of Morita equivalence and of homotopy invariance of periodic cyclic homology.

In chapter 4 we study the other definition of the cyclic homology, namely, via the complex $C^\lambda_\bullet$. It gives the same result as above when the ground ring contains $\mathbb{Q}$. We prove the theorem relating cyclic homology of A to Lie algebra homology of matrices over A.

In chapter 5 we advance with our study of operations. Roughly speaking, we introduce the algebra (let us call it $\mathcal{U}_A$ here) of operations on the negative cyclic complex and define by explicit formulas a pairing of *complexes* $\mathcal{U} \otimes \mathrm{CC}^-_\bullet(A) \to \mathrm{CC}^-_\bullet(A)$. Using this, we prove Goodwillie's rigidity theorem for periodic cyclic homology. We prove a more elaborate version later on. We also prove Cuntz and Quillen's excision theroem for periodic cyclic homology. ***IS THERE A MORE ELABORATE VERSION FOR THAT?***

In chapter 6 we give an exposition of Connes' theory of cyclic objects. We explain there relation to spaces with an S^1 action. Following Kaledin, we develop various tools needed for noncommutative geometry in positive characteristic, in particular the Frobenius morphism and ***SOME INTRO VERSION OF?*** the cyclotomic structure.

In chapter 7 we explain how Hochschild and cyclic homology can be defined using Quillen's language of non-Abelian derived functors.

In chapter 8, we study various examples of computations of the Hochschild and cyclic homology. The examples that we choose in this presentation revolve mostly around several related classes of algebras: functions on manifolds or on algebraic varieties; operators on functions; deformed algebras of functions; group algebras. *** Not included yet)*** The second class of examples is relevant to representation theory of quivers and other topics (preprojective algebras, CY algebras). Many interesting examples, in particular the ones related to algebraic topology, are not considered here.

In chapter 9 we study characteristic classes in noncommutative geometry. We start with the Chern character of Connes and Karoubi and then define the Karoubi regulator for topological algebras. As a version of that, we get a Goodwillie morphism from relative K theory to relative cyclic homology of a nilpotent ideal over the rationals, as well as a version of a more refined Beilinson morphism over the p-adics. Then we extend the Chern character K_0 from projective modules over an algebra to perfect complexes over a sheaf of algebras ***RELY ON MATERIAL LATER IN THE TEXT***. ***SHOULD BE THERE: Chern character on K theory of DG categories, Karoubi regulator on DG categories over $\mathbb{C}$ (using Blanc's topological K theory).***

In chapter 10 we introduce an important generalization of a sheaf of algebras, namely, an algebroid stack. Algebroid stacks are concrete and explicit realizations of sheaves of categories. They are used, in particular, in deformation quantization. The constructions of cyclic and cyclic complexes, perfect complexes, and the Chern character generalize to this context.

In chapter 11 we express the Hochschild and cyclic homology of an algebra in terms of its bar construction. We essentially follow Cuntz and Quillen.

In chapter 12 we use the results of 11 to advance our study of operations. We construct two A_∞ algebras and prove that they act on the negative cyclic complex. One has a motivation in classical calculus on manifolds. Namely, recall that multivector fields act on forms in two ways: by contraction ι_X and by Lie derivative $L_X = [d, \iota_X]$. if $\mathfrak{g}_M$ is the graded Lie algebra of multivector fields on a manifold M with the Schouten bracket, construct a new graded Lie algebra over $\mathbb{C}[[u]]$ generated by operators ι_X and L_X for $X \in \mathfrak{g}_M$. This algebra acts on $\Omega^\bullet_M[[u]]$. Take the universal enveloping algebra, and equip it with the differential induced by the commutator with ud_{DR}. The result is an associative algebra over $\mathbb{C}[[u]]$ that can be defined starting with any differential graded Lie algebra $\mathfrak{g}$. Apply this construction to $\mathfrak{g}_A$, the algebra of Hochschild cochains on A. This is our first algebra of operations on $\mathrm{CC}^-_\bullet(A)$. The other, larger A_∞ algebra of operations on the same complex is the negative cyclic complex of the *associative* differential graded algebra of Hochschild cochains. The fact that it is an algebra, and that it acts on $\mathrm{CC}^-_\bullet(A)$, is explained later in chapter

In chapter 13 we use the above results to prove the rigidity property of periodic cyclic homology and to construct the Gauss Manin connection on the periodic cyclic complex of a family of algebras. We generalize the theorems, respectively, of Goodwillie and Getzler. Our results over p-adic integers, not the rationals, and at the level of complexes, not homologies.

In chapter 14 we systematically develop the theory of Hochschild and cyclic complexes for DG categories. The new elements, as compared to the case of DG algebras, are as follows. First of all, the notion of (weak) equivalence becomes more delicate. Second, there is a notion of a quotient by a full subcategory, due to Drinfeld. We prove Keller's excision theorem stating that a categorical quotient gives rise to a homotopy fibre sequence of Hochschild and cyclic complexes, as well as other invariance properties, such as invariance under week equivalence and a form of Morita invariance, also essentially due to Keller. We extend our constructions to A_∞ categories.

In chapter 15 we study Frobenius algebras and their generalizations. A Frobenius algebra is an algebra with a trace τ such that the pairing $\langle a, b\rangle = \tau(ab)$ is nondegenerate. Frobenius algebras have several interconnected generalizations in the context of DG categories and A_∞ categories. ***MORE*** Homotopy BV algebra; also oh Hochschild-Tate complexes, Rivera-Wang... relation to representation schemes...

In chapter 16 we compute the Hochschild and cyclichomology of the Drinfeld quotient of the DG category of perfect complexesby the full DG subcategory of acyclic complexes.

In chapter 17 we show that DG categories form a homotopy category in DG categories, in the sense of Leinster. The DG category $\mathbf{C}^\bullet(\mathcal{A}, \mathcal{B})$ for two DG categories $\mathcal{A}$ and $\mathcal{B}$ is defined already in chapter 14; the main ingredient in the definition is the brace structure on Hochschild cochains. Then we extend the homotopy category structure to Hochschild chains. We show that, taken together, Hochschild cochains and chains form a homotopy category with a trace functor. Trace functors are central to Kaledin's work on noncommutative generalization of Witt vector and De Rham Witt complexes.

Remark 0.0.1. It looks like the correct answer to the question: What do DG categories form? should unify the construction above with the constructions in 6, as well as in 15. This should be a structure on all Hochschild chains and cochains of $A_1 \otimes \ldots \otimes A_n$ with coefficients in bimodules $B_1 \otimes \ldots \otimes B_n$; the bimodule structure is given by morphisms $f_j : A_j \to B_j$ and $g_j : A_j \to B_{j+1}$, $1 \geq j \geq n$ (and $B_{n+1} = B_1$). The constructions of chapter 17 should correspond to the case $n = 1$; the full structure should incorporate the Frobenius and the cyclotomic structure of 6, as well as the weak CY structure of Kontsevich-Vlassopoulos ***MENTIONED?*** in 15.

In chapter 18 we study the approach to cyclic homology via noncommutative differential forms. We follow Cuntz-Quillen, Karoubi, and Ginzburg-Schedler. In particular, we show that the standard HKR map, previously defined for commutative algebras, generalizes to any algebra if we use noncommutative forms. This HKR morphism maps Hochschild chains $C_\bullet(A)$ to noncommutative forms $\Omega^\bullet_A$. It intertwines the cyclic differential B with the De Rham differential d. and the Hochschild differential b with the Ginzburg-Schedler differential ι_Δ. We interpret ι_Δ as a homotopy between id and f^* for any homomorphism f, in case when $f = \mathrm{id}_A$. (In other words: noncommutative De Rham cohomology is trivial for *any* algebra A; in particular, any morphism of algebras acts trivially on this cohomology; construct a homotopy for it, and then evaluate it on id_A; we get a new differential that automatically commutes with d). In conclusion, we show how this can be used to generalize quantum moment map and quantum Hamiltonian reduction from Lie algebra actions on associative algebras to Hopf algebra actions.

In chapters 19, 20, and 21 we study the link between Hochschild and cyclic homology of an algebra A and various versions of the representation scheme of A. Note that, similarly to defining the maximal spectrum of a commutative algebra over $\mathbb{C}$ as the space of its one-dimensional representations, one can develop parts of noncommutative geometry by studying spaces of representations of a noncommutative algebra A. Cyclic homology theory initially took a different road, namely it defined various invariants as complexes of forms on an imaginary non-existent space that could be thought of as a noncommutative spectrum of our algebra A. It was later that connections were established between these invariants and actual functions and forms on the algebraic variety of finite dimensional representations of A. The approach with noncommutative forms (cf. 18) is related to these developments. We study both the usual and derived versions of representation varieties. ***A BIT MORE? CONSOLIDATE THE 3 CHAPTERS? RELATE TO MODULI SCHEMES OF OBJECTS OF A DG CATEGORY?***

In chapter 22 we discuss noncommutative Hodge theory...

In chapter 24 we discuss Hochschild and cyclic (co)homology of the second kind. The notion is due to Possitselsky and Polishchuk. We have already encountered it once, in the dual context ofcoalgebras (chapter 11). Cyclic cohomology of the second kind of the *coalgebra* $U(\mathfrak{g})$ where $\mathfrak{g}$ is a DG Lie algebra is central in our studies of operations on the negative cyclic complex. ***MORE***

In chapter 25 we study Hochschild and cyclic homology of the DG category which is the Drinfeld quotient of the category of (bounded from above) complexes of A-modules by the full subcategory of perfect complexes. We establish that it is (weakly) equivalent to the category of matrix factorizations, and then prove Efimov's theorem about their Hochschild and cyclic homology. Complexes of the second kind are used.

CHAPTER 2

Hochschild and cyclic homology of algebras

1. Basic homological complexes

**********Assumptions: when characteristic zero?***

Let k denote a commutative unital ring and let A be a flat k-algebra with unit, not necessarily commutative. Let $\overline{A} = A/k \cdot 1$, and let

DEFINITION 1.0.1.

$$\widetilde{C}_p(A) \stackrel{def}{=} A \otimes_k A^{\otimes_k p}.$$
$$C_p(A) \stackrel{def}{=} A \otimes_k \overline{A}^{\otimes_k p}.$$

We call elements of $\widetilde{C}_\bullet$ *non-normalized and the elements of* $C_\bullet$ *normalized Hochschild chains of* A.

DEFINITION 1.0.2. *Define*

$$\begin{aligned} b : A \otimes A^{\otimes p} &\to A \otimes A^{\otimes p-1} \\ a_0 \otimes \cdots \otimes a_p &\mapsto (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \\ &\quad \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \end{aligned} \tag{1.1}$$

$$\begin{aligned} B : A \otimes \overline{A}^{\otimes p} &\to A \otimes \overline{A}^{\otimes p+1} \\ a_0 \otimes \cdots \otimes a_p &\mapsto \sum_{i=0}^{p} (-1)^{pi} 1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{i-1} \end{aligned} \tag{1.2}$$

LEMMA 1.0.3. *The map* b *descends to the map*

$$b : C_\bullet(A) \to C_{\bullet-1}(A)$$

PROPOSITION 1.0.4. *The maps*

$$b : C_\bullet(A) \to C_{\bullet-1}(A) \text{ and } B : C_\bullet(A) \to C_{*+1}(A).$$

satisfy the identities $b^2 = B^2 = bB + Bb = 0$

PROOF. We will leave the proof of this claim to the reader. ****OR...**** □

DEFINITION 1.0.5. The complex $(C_\bullet(A), b)$ is called *the (normalized) standard Hochschild complex of* A and its homology is denoted by $H_\bullet(A, A)$ or by $HH_\bullet(A)$. We sometimes write $C_\bullet(A, A)$ instead of $C_\bullet(A)$.

REMARK 1.0.6. For any algebra A we will use A^{op} to denote the opposite algebra, i. e.

$$\begin{aligned} &A^{op} = \{a^\circ \mid a \in A\} \text{ as a } k\text{-module} \\ &a^\circ b^\circ = (ba)^\circ. \end{aligned} \tag{1.3}$$

We will set

$$A^e = A \otimes A^{op}. \tag{1.4}$$

In particular, an A^e-module is the same as an A-bimodule. Suppose that A is unital. The Hochschild complex $(C_\bullet(A), b)$ is just the tensor product $A \otimes_{A^e} \mathcal{B}_\bullet(A)$, where

$$\mathcal{B}_\bullet(A) = A^e \otimes_k \overline{A}^{\otimes \bullet} \tag{1.5}$$

is the standard free resolution of A as an A^e-module. In particular, $H_\bullet(A, A)$ is the same as the the left derived tensor product $A \otimes^{\mathbb{L}}_{A \otimes A^{op}} A$ in the category of A-bimodules. More precisely, if we identify P_n with $A \otimes \overline{A}^{\otimes n} \otimes A$ via

$$(a_0 \otimes a^\circ_{n+1}) \otimes a_1 \ldots \otimes a_n \mapsto a_0 \otimes \ldots \otimes a_{n+1},$$

then the differential is as follows:

$$b'(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{j=0}^{n} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots a_{n+1} \tag{1.6}$$

We have

$$H_\bullet(A, A) = \mathrm{Tor}_\bullet^{A \otimes A^{op}}(A, A).$$

The identity $Bb + bB$ means that the map B induces a morphism of complexes

$$B : (C_\bullet(A), b) \to (C_\bullet(A)[-1], -b)\ .$$

LEMMA 1.0.7. *The morphism of complexes*

$$(A \otimes A^{\otimes \bullet}, b) \to (A \otimes \overline{A}^{\otimes \bullet}, b)$$

induces an isomorphism on homology.

PROOF. Let $(\widetilde{\mathcal{B}}(A), b)$ be the free resolution of A given by

$$\widetilde{\mathcal{B}}_\bullet(A) = A^e \otimes A^{\otimes \bullet},$$

where b is given by the formula 1.1. Then the quotient map $A \to A/k1$ induces a morphism of resolutions of A:

$$(\widetilde{\mathcal{B}}_\bullet(A), b) \to (\mathcal{B}_\bullet(A), b).$$

In particular, the induced map

$$A \otimes_{A^e} \widetilde{\mathcal{B}}_\bullet(A) \to A \otimes_{A^e} \mathcal{B}_\bullet(A)$$

induces an isomorphism in homology. □

DEFINITION 1.0.8. *For $i, j, p \in \mathbb{Z}$ let*

$$CC^-_p(A) = \prod_{\substack{i \ge p \\ i = p \mod 2}} C_i(A)$$

$$CC^{per}_p(A) = \prod_{i = p \mod 2} C_i(A)$$

$$CC_p(A) = \bigoplus_{\substack{i \le p \\ i = p \mod 2}} C_i(A)$$

The associated complexes are:

(1) *the negative cyclic complex $(CC^-_\bullet(A), B + b)$;*

(2) *the periodic cyclic complex* $(CC^{per}_\bullet(A), B+b)$ *and*
(3) *the cyclic complex* $(CC_\bullet(A), b+B)$

The homology of these complexes is denoted by $HC^-_\bullet(A)$, *respectively by* $HC^{per}_\bullet(A)$, *respectively by* $HC_\bullet(A)$.

In what follows we will use the notation of Getzler and Jones ([**?**]). Let u denote a variable of degree -2. Then the negative and periodic cyclic complexes are described by the following formulas:

$$CC^-_\bullet(A) = (C_\bullet(A)[[u]], b+uB) \tag{1.7}$$

$$CC^{\mathrm{per}}_\bullet(A) = (C_\bullet(A)[[u,u^{-1}], b+uB) \tag{1.8}$$

$$CC_\bullet(A) = (C_\bullet(A)[[u,u^{-1}]/uC_\bullet(A)[[u]], b+uB) \tag{1.9}$$

Remark 1.0.9. Here and in the future we will always consider the algebra of formal power series $k[[u]]$ in its u-adic topology.

The following is a good picture to keep in mind:

$$\begin{tikzcd}
{} & {} \\
{} \arrow[u, "b"] \arrow[r, "uB"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
& & & & & {} & \\
& & & & & u^2C_0(A) \arrow[u] \arrow[r] & {} \\
& & & & uC_0(A) \arrow[r] & u^2C_1(A) \arrow[u] \arrow[r] & {} \\
& & & C_0(A) \arrow[r] \arrow[dr, squiggly, "diag"] & uC_1(A) \arrow[u] \arrow[r] & u^2C_2(A) \arrow[u] \arrow[r] & {} \\
& & u^{-1}C_0(A) \arrow[r] & C_1(A) \arrow[u] \arrow[r] \arrow[dr, squiggly, "diag"] & uC_2(A) \arrow[u] \arrow[r] \arrow[dr, squiggly, "diag"] & u^2C_3(A) \arrow[u] \arrow[r] & {} \\
& u^{-2}C_0(A) \arrow[r] & u^{-1}C_1(A) \arrow[u] \arrow[r] & C_2(A) \arrow[u] \arrow[r] & uC_3(A) \arrow[u] \arrow[r] \arrow[dr, squiggly, "diag"] & u^2C_4(A) \arrow[u] \arrow[r] \arrow[dr, squiggly, "diag"] & {} \\
{} \arrow[r] & u^{-2}C_1(A) \arrow[u] \arrow[r] & u^{-1}C_2(A) \arrow[u] \arrow[r] & C_3(A) \arrow[u] \arrow[r] & uC_4(A) \arrow[u] \arrow[r] & u^2C_5(A) \arrow[u] \arrow[r] \arrow[dr, squiggly, "diag"] & {} \\
& {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {}
\end{tikzcd}$$

As immediately seen from the picture, there are inclusions of complexes

$$CC^-_\bullet(A)[-2] \hookrightarrow CC^-_\bullet(A) \hookrightarrow CC^{per}_\bullet(A) \tag{1.10}$$

and short exact sequences:

$$0 \to CC^-_\bullet(A)[-2] \xrightarrow{S} CC^-_\bullet(A) \to C_\bullet(A) \to 0 \tag{1.11}$$

$$0 \to C_\bullet(A) \to CC_\bullet(A) \xrightarrow{S} CC_\bullet(A)[2] \to 0 \tag{1.12}$$

and

$$0 \to CC_\bullet^-(A) \to CC_\bullet^{per}(A) \to CC_\bullet(A)[2] \to 0. \tag{1.13}$$

The periodicity map map S is just the multiplication by u.

REMARK 1.0.10. The long exact sequence of homology induced by the short exact sequence of complexes 1.12 has the form

$$\dots \longrightarrow H_k(A,A) \xrightarrow{I} HC_k(A) \xrightarrow{S} HC_{k-2}(A) \xrightarrow{B} H_{k-1}(A,A) \xrightarrow{I} \dots$$

and is sometimes called the *Connes-Gysin exact sequence*. More generally, let $\mathcal{F}_p$ be the to horisontal filtration of the double complex $CC_\bullet(A)$:

$$\mathcal{F}_p(CC_\bullet(A)) = \bigoplus_{l-k=p} u^{-k} C_l(A).$$

The associated spectral sequence has the E^2-term

$$E^2_{pq} = H_{p-q}(A,A) \tag{1.14}$$

and converges to $HC_{p+q}(A)$.

PROPOSITION 1.0.11. *The quotient map $CC_n^{per}(A) \to CC_n(A)$ induces a short exact sequence*

$$0 \to \underset{\leftarrow}{\lim}{}^1 HC_\bullet(A) \to HC_\bullet^{per}(A) \to \underset{\leftarrow}{\lim} HC_\bullet(A)$$

PROOF. The claim follows immediately from the fact that

$$CC_\bullet^{per}(A) = \underset{\leftarrow}{\lim} CC_\bullet(A).$$

∎

REMARK 1.0.12.
All the three complexes can be just as well thought of as covariant functors from the category of unital algebras over k to the category of complexes of vector spaces over k.

EXAMPLE 1.0.13. Suppose that $A = k$. Then

$$C_n(k) = \begin{cases} k & n = 0 \\ 0 & n > 0 \end{cases}$$

and hence

$$HC_\bullet(k) = k[u^{-1}];\ HC_\bullet^{per}(k) = k[u^{-1}, u]] \text{ and } HC_\bullet^-(k) = k[[u]].$$

with the grading given by $|u| = -2$.

PROPOSITION 1.0.14. *Suppose that A_1 and A_2 are unital algebras over k. Then the inclusion*

$$(C_\bullet(A_1), b) \oplus (C_\bullet(A_2), b) \hookrightarrow (C_\bullet(A_1 \oplus A_2), b)$$

is a quasiisomorphism of complexes.

Proof. Since A_1 and A_2 are unital, $A_1^e \oplus A_2^e$ is a projective $(A_1 \oplus A_2)^e$-module (see 1.4 for the notation) and hence

$$\mathcal{B}_\bullet = (A_1^{\otimes\bullet} \oplus A_2^{\otimes\bullet}) \otimes_k (A_1^e \oplus A_2^e)$$

is a subcomplex of $(\mathcal{B}_\bullet(A_1 \oplus A_2)^e, b)$ contractible in positive degrees and such that each term is again a projective $(A_1 \oplus A_2)^e$-module. Hence the inclusion

$$(\mathcal{B}_\bullet, b) \hookrightarrow (\mathcal{B}_\bullet(A_1 \oplus A_2), b)$$

is a quasiisomorphism. As a corollary, the inclusion ι of complexes

$$\begin{gathered} C_\bullet(A_1) \oplus C_\bullet(A_2) = \mathcal{B}_\bullet \otimes_{(A_1 \oplus A_2)^e} (A_1 \oplus A_2) \hookrightarrow \\ (\mathcal{B}_\bullet(A_1 \oplus A_2) \otimes_{(A_1 \oplus A_2)^e} (A_1 \oplus A_2) = C_\bullet(A_1 \oplus A_2) \end{gathered} \tag{1.15}$$

is a quasiisomorphism. □

Corollary 1.0.15. *Hochschild, cyclic, negative cyclic and periodic homologies are additive, i.e.* $\mathrm{HC}_\bullet^\#(A_1 \oplus A_2) = \mathrm{HC}_\bullet^\#(A_1) \oplus \mathrm{HC}_\bullet^\#(A_2)$ *whenever A and B are unital algebras (where # stands for cyclic, negative and resp. periodic homology).*

Proof. The part of the claim about the Hochschild homology follows from the above proposition.

Let e_1 denote the unit of A_1 and e_2 denote the unit if A_2. Set

$$C'_n(A_1) = \begin{cases} A_1, \text{ for } n = 0 \\ (A_1 \oplus ke_2) \otimes A_1^{\otimes n} \text{ for } n > 0. \end{cases}$$

and similarly for $C'_\bullet(A_2)$. Then

$$C'_\bullet(A_1) \oplus C'_\bullet(A_2)$$

is a subcomplex of $C_\bullet(A)$ invariant under B. The summands $ke_2 \otimes A_1^{\otimes\bullet}$ and $ke_1 \otimes A_2^{\otimes\bullet}$ are both contractible, the contracting homotopies given by

$$e_2 \otimes a_1 \otimes \ldots \otimes a_n \to e_2 \otimes e_1 \otimes a_1 \otimes \ldots \otimes a_n$$

and

$$e_1 \otimes a_1 \otimes \ldots \otimes a_n \to e_2 \otimes e_2 \otimes a_1 \otimes \ldots \otimes a_n$$

respectively. Together with the lemma 1.0.7 this implies that, say,

$$C'_\bullet(A_1)[u^{-1}, u]] \oplus C'_\bullet(A_2)[u^{-1}, u]], b + uB) \to (CC^{per}(A), b + uB)$$

is a morphism of double complexes which induces quasiisomorphism on the columns and hence is a quasiiomorphism of double complexes. This proves the claimed result for the periodic cyclic homology. The other two versions of the claim follow from the same argument (replacing Lorenz series in u^{-1} by polynomials in u^{-1} and formal power series in u respectively). □

Definition 1.0.16. *For any associative algebra A set $A^+ = A + k \cdot 1$, where*

$$\forall a \in A \ a \cdot 1 = a,$$

The cyclic complex of A is the subcomplex of $(CC_\bullet(A^+), b + uB)$ given by

$$\overline{CC}_\bullet(A^+) = \mathrm{Ker}(CC_\bullet(A^+) \to CC_\bullet(k)$$

and similarly for the negative and periodic cyclic complexes.

If A is a unital algebra then the inclusion $A \hookrightarrow A^+$ induces a morphism of complexes

$$CC_\bullet(A) \to \overline{CC}_\bullet(A^+)$$

which, by the corollary above, is a quasiisomorphism and hence this definition agrees with the one given in 1.0.8.

2. H-unitality and excision

Suppose that A is a non-unital algebra. We set

$$b' : C_p(A) \to C_{p-1}(A) \tag{2.1}$$

$$a_0 \otimes \cdots \otimes a_p \mapsto \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \tag{2.2}$$

Definition 2.0.1. *A is H-unital if the complex $(C_\bullet(A), b')$ is acyclic.*

Theorem 2.0.2 (Excision in Hochschild homology). *Given a short exact sequence*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0,$$

where I is H-unital, there exists a long exact sequence
(2.3)

$$\cdots \xrightarrow{\partial} H_k(I) \xrightarrow{H(\iota)} H_k(A) \xrightarrow{\pi} H_k(A/I) \xrightarrow{\partial} H_{k-1}(I) \xrightarrow{H(\iota)} H_{k-1}(A) \xrightarrow{\pi} \cdots$$

$$\xrightarrow{\partial} H_0(I) \xrightarrow{H(\iota)} H_0(A) \xrightarrow{\pi} H_0(A/I) \longrightarrow 0$$

Sketch of the proof. The standard way of proving this kind of result consists of proving that the map

$$C_\bullet(I) \to \text{ Ker } (H_\bullet(\pi))$$

induced by ι is a quasiisomorphism. Instead we will sketch the construction of the boundary map ∂. A complete proof of the exactness of the sequence (2.3) follows the same pattern.

Construction of the boundary map

Let B_0 be the contracting homotopy for the complex $(C_\bullet(I), b')$. The boundary map is given by the following recipe.

Let $x = \sum a_0 \otimes \dots a_n$ be a b-cycle in $C_n(A/I)$. Let $\tilde{x} = \sum \tilde{a}_0 \otimes \dots \tilde{a}_n$ be its lift to a chain in $C_n(A)$. Then, provided that we can choose $\tilde{x}$ so that $b\tilde{x} \in C_{n-1}(I)$,

$$\partial(x) = b\tilde{x}.$$

So suppose that we have an $x \in C_n(A/I)$ satisfying $bx = 0$ and let $\tilde{x}$ be a lift of x to $C_c(A)$. Since $bx = 0$,

$$b\tilde{x} \in \bigoplus_{k+l=n-1} A^{\otimes k} \otimes I \otimes A^{\otimes l}.$$

Using B_0 on the I factor, we get an element

$$X_1 \in \left(\bigoplus_{k+l=n-1} A^{\otimes k} \otimes I \otimes I \otimes A^{\otimes l} \right) \oplus \left(I \otimes A^{\otimes n-1} \otimes I \right)$$

such that, if we set $\tilde{x}_1 = \tilde{x} - X_1$,

$$\pi(\tilde{x}_1) = x \text{ and } b(\tilde{x}_1) \in \left(\bigoplus_{k+l=n-2} A^{\otimes k} \otimes I^{\otimes 2} \otimes A^{\otimes l} \right) \oplus \left(I \otimes A^{\otimes n-2} \otimes I \right).$$

One checks readily that b'_I, i. e. b' used on the $I^{\otimes 2}$ factor, kills $b(\tilde{x}_1)$ where, in the last summand, we will order the I-factors as $i_1 \otimes a_1 \otimes \dots a_{n-2} \otimes i_0$. It implies that, if we again use B_0 on the $I^{\otimes 2}$ factor in $b\tilde{x}_1$, we get an X_2 in

$$\bigoplus_{k+l=n-1} A^{\otimes k} \otimes I^{\otimes 3} \otimes A^{\otimes l}$$

and such that $\tilde{x}_2 = \tilde{x}_1 - X_2$ satisfies

$$\pi(\tilde{x}_2) = x \text{ and } b(\tilde{x}_2) \in \bigoplus_{k+l=n-3} A^{\otimes k} \otimes I^{\otimes 3} \otimes A^{\otimes l}.$$

An obvious induction on the number of succesive I factors on $b\tilde{x}_\bullet$ completes the construction.

For the details of the proof we will refer the reader to the original paper [**?**]. □

Corollary 2.0.3. *Given a short exact sequence*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0,$$

where I is H-unital, there exists a long exact sequence
(2.4)

$$\dots \xrightarrow{\partial} HC_k^{\#}(I) \xrightarrow{HC^{\#}(\iota)} HC_k^{\#}(A) \xrightarrow{HC^{\#}(\pi)} HC_k^{\#}(A/I) \xrightarrow{\partial} HC_{k-1}^{\#}(I) \xrightarrow{HC^{\#}(\iota)} HC_{k-1}^{\#}(A) \xrightarrow{HC^{\#}(\pi)} \dots$$

$$\xrightarrow{\partial} HC_0^{\#}(I) \xrightarrow{HC^{\#}(\iota)} HC_0^{\#}(A) \xrightarrow{HC^{\#}(\pi)} HC_0^{\#}(A/I) \longrightarrow 0$$

where $HC^{\#}$ stands for cyclic and negative cyclic homology. The corresponding two periodic version in cyclic periodic homology has the form of an exact triangle

(2.5)
$$\begin{tikzcd} HC_\bullet^{per}(I) \arrow[rr] & & HC_\bullet^{per}(A) \arrow[dl] \\ & HC_\bullet^{per}(A/I) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Sketch of the proof. Follows essentially from the act that, according to the above theorem, the inclusion of complexes

$$(C_\bullet(I), b) \to (\operatorname{Ker} H_\bullet(\pi), b)$$

is a quasiisomorphism, hence the same holds for the inclusion of the double complexes computing cyclic homologies. □

Remark 2.0.4. We will see later that the excision in periodic cyclic homology holds without the H-unitality assumption on the ideal.

3. Homology of differential graded algebras

One can easily generalize all the above constructions to the case when A is a differential graded algebra (DGA). For future reference we will recall the definition.

DEFINITION 3.0.1. *A differential graded algebra (DGA) is a pair (A, d), where A is a $\mathbb{Z}$-graded algebra and d is a derivation of degree 1 such that $d^2 = 0$.*

So suppose that (A, d) is a DGA. The action of d extends to an action on Hochschild chains of A by the Leibniz rule:

$$d(a_0 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p} (-1)^{\sum_{k<i} (|a_k|+1)+1} (a_0 \otimes \cdots \otimes \delta a_i \otimes \cdots \otimes a_p)$$

The maps b and B are modified to include signs:

$$b(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p-1} (-1)^{\sum_{i=0}^{k} (|a_i|+1)+1} a_0 \ldots \otimes a_k a_{k+1} \otimes \ldots a_p \tag{3.1}$$

$$+(-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)} a_p a_0 \otimes \ldots \otimes a_{p-1}$$

$$B(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p} (-1)^{\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)} 1 \otimes a_{k+1} \otimes \ldots \otimes a_p \otimes \tag{3.2}$$

$$\otimes a_0 \otimes \ldots \otimes a_k$$

The complex $C_\bullet(A)$ now becomes the total complex of the double complex with the differential $b + d$. In other words:

$$\widetilde{C}_\bullet(A) = (\bigoplus_{n\ge 0} A \otimes A^{\otimes n}, d + b);\ C_\bullet(A) = (\bigoplus_{n\ge 0} A \otimes \overline{A}^{\otimes n}, d + b) \tag{3.3}$$

The cyclic, negative cyclic, and the periodic cyclic complexes are defined as before using the new definition of $C_\bullet(A)$.

4. Cyclic cohomology

. *****Maybe a bit more***** The definitions of cyclic, negative cyclic and cyclic periodic cohomology follow the usual pattern of replacing the associated complexes with their linear duals and the boundary maps b and B with their transpose.

Note however that, since

$$\mathrm{Hom}_k(k[[u]], k) \simeq k[u^{-1}],$$

the cocycles are given by *finite sums* of cochains. So, for example, the complex computing periodic cyclic cohomology of a unital algebra A becomes the complex of continuos cochains, i. e.

$$(\mathrm{Hom}_k(CC_\bullet(A), k)[u^{-1}, u], b^t + u^{-1}B^t).$$

5. The Hochschild cochain complex

As usual, for any graded k-module E, $E[1]^p = E^{p+1}$ for all p; for any two graded k-modules E and F,

$$\underline{\mathrm{Hom}}^p(E,F) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_k(E^n, F^{n+p}) \tag{5.1}$$

$$(E\otimes F)^p = \bigoplus_{n\in\mathbb{Z}} E^n \otimes_k F^{p-n} \tag{5.2}$$

Definition 5.0.1. *Let $A = \oplus_{n\in\mathbb{Z}} A^n$ be a graded module over a commutative unital ring k. The k-module of (non-normalized) Hochschild cochains of A is by definition*

$$\widetilde{C}^\bullet(A,\, A) = \prod_{n\geq 0} \underline{\mathrm{Hom}}(A[1]^{\otimes n}, A)$$

If 1 is a chosen element of A^0 then the k-module of (normalized) Hochschild cochains of A is

$$C^\bullet(A,A) = \prod_{n\geq 0} \underline{\mathrm{Hom}}(\overline{A}[1]^{\otimes n}, A)$$

where $\overline{A} = A/k\cdot 1$.

We will often shorten the notation and write $\widetilde{C}^\bullet(A)$ or $C^\bullet(A)$.

Definition 5.0.2. *Suppose that D and E are homogeneous cochains on A. Set*

$$D\circ E)(a_1,\ldots,a_{d+e-1}) = \sum_{j\geq 0} (-1)^{(|E|+1)\sum_{i=1}^{j}(|a_i|+1)} D(a_1,\ldots,a_j, E(a_{j+1},\ldots,a_{j+e}),\ldots);$$

and

$$[D,\ E] = D\circ E - (-1)^{(|D|+1)(|E|+1)} E\circ D.$$

The above bracket is called the Gerstenhaber bracket.

Proposition 5.0.3. *Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be a graded module over a commutative unital ring k. Then*

$$(C^\bullet(A)[1], [\ ,\ ])$$

is a graded Lie algebra.

Suppose moreover that A is a differential graded algebra. Define a non-normalized Hochschild 2-cochain m by

$$m = m_1 + m_2;\ m_1(a_1) = da_1;\ m_2(a_1,a_2) = (-1)^{|a_1|} a_1 a_2;$$

m_p vanishes on $A[1]^{\otimes q}$ with $n = 1$, 2 and $q \neq p$.

Lemma 5.0.4. *One has*

$$m\circ m = 0$$

Lemma 5.0.5. *The maps*

$$d : \widetilde{C}^\bullet(A) \to \widetilde{C}^{\bullet+1}(A);\ dD = [m_1, D]$$

and

$$\delta : \widetilde{C}^\bullet(A) \to \widetilde{C}^{\bullet+1}(A);\ \delta D = [m_2, D]$$

descend to the k-module of normalized cochains $C^\bullet(A)$.

LEMMA 5.0.6.
$(C^\bullet(A)[1], [\ ,\ ], d+\delta)$ *is a differential graded Lie algebra.*

DEFINITION 5.0.7. The cohomology of the complex $(C^\bullet(A), d+\delta)$ is called the Hochschild cohomology of the differential graded algebra A with coefficients in the A-bimodule A and will be denoted by $H^\bullet(A, A)$.

REMARK 5.0.8. Explicitly, one has

$$\begin{aligned}(\delta D)(a_1,\ldots,a_{d+1}) = (-1)^{|a_1||D|+|D|+1} a_1 D(a_2,\ldots,a_{d+1}) + \\ + \textstyle\sum_{j=1}^{d}(-1)^{|D|+1+\sum_{i=1}^{j}(|a_i|+1)} D(a_1,\ldots,a_j a_{j+1},\ldots,a_{d+1}) \\ +(-1)^{|D|\sum_{i=1}^{d}(|a_i|+1)} D(a_1,\ldots,a_d)a_{d+1}\end{aligned}$$

and

$$(dD)(a_1,\ldots,a_d) = dD(a_1,\ldots,a_d) - \sum_{j=1}^{p} \epsilon_j D(a_1,\ldots,da_j,\ldots,a_d)$$

where $\epsilon_j = (-1)^{\sum_{p<j}(|a_p|+1)}$.

In the case when A is an ordinary unital algebra, $H^\bullet(A, A)$ coincides with $Ext^\bullet_{A\otimes A^{op}}(A, A)$.

DEFINITION 5.0.9. *Suppose that A is a graded associative algebra. For homogeneous cochains D and E from $C^\bullet(A, A)$ we set*

$$(D \smile E)(a_1,\ldots,a_{d+e}) = (-1)^{|E|\sum_{i\le d}(|a_i|+1)} D(a_1,\ldots,a_d)E(a_{d+1},\ldots,a_{d+e}).$$

Extending this by linearity to all of $C^\bullet(A)$ we get the cup product

$$\smile : C^i(A) \times C^j(A) \to C^{i+j}(A)$$

PROPOSITION 5.0.10. *Let A be a graded associative algebra. Then $(C^\bullet(A, A)\ , \smile\ , d+\delta)$ is a differential graded associative algebra.*

PROOF. Since the proof is a pure bookkeeping just like in the case of the previous proposition, we will refer the reader to the standard references [**84**] and [**223**]). □

REMARK 5.0.11. Under the isomorphism $H^\bullet(A, A) \simeq Ext^\bullet_{A\otimes A^0}(A, A)$, the cup product induces the Yoneda product on Hochshild cohomology.

6. Braces

The following definition is essentially due to Gerstenhaber (see [**223**],[**237**]).

DEFINITION 6.0.1 (**Braces**). *Suppose that A is graded k-module and*

$$D_i,\ i = 0,\ldots,m$$

are Hochschild cochains on A. The following formula defines a new Hochschild cochain on A:

$$\begin{aligned}&D_0\{D_1,\ldots,D_m\}(a_1,\ldots,a_n) = \\ &\textstyle\sum_{i_1,\ldots,i_m} \epsilon_{i_1,\ldots,i_m} D_0(a_1,\ldots,a_{i_1}, D_1(a_{i_1+1},\ldots),\ldots,D_m(a_{i_m+1},\ldots),\ldots,a_n)\end{aligned}$$

where the sign is given by

$$\epsilon_{i_1,\ldots,i_m} = (-1)^{\sum_p \sum_{k\le i_p}(|a_k|+1)(|D_p|+1)}$$

PROPOSITION 6.0.2. *One has*

$$(D\{E_1,\dots,E_k\})\{F_1,\dots,F_l\} = \sum(-1)^{\sum_{q\le i_p}(|E_p|+1)(|F_q|+1)}\times$$
$$\times D\{F_1,\dots,E_1\{F_{i_1+1},\dots,\},\dots,E_k\{F_{i_k+1},\dots,\},\dots,\}$$

PROOF. The proof of the statement reduces immediately to the question of bookkeeping and is left as an exercise to the reader. ***OR: in terms of NC diff ops...*** □

The above proposition can be restated as follows.

PROPOSITION 6.0.3. *Suppose that A is an associative algebra and endow both $C^\bullet(A)$ and $C^\bullet(C^\bullet(A))$ with the differential graded algebra structure induced by the cup product. For a cochain D on A let $D^{(k)}$ be the following k-cochain on $C^\bullet(A)$:*

$$D^{(k)}(D_1,\dots,D_k) = D\{D_1,\dots,D_k\}$$

Then the map

$$C^\bullet(A) \to C^\bullet(C^\bullet(A))$$

given by

$$D \mapsto \sum_{k\ge 0} D^{(k)}$$

is a morphism of differential graded algebras.

6.1. Hochschild cochains as coderivations. Let V be a ($\mathbb{Z}$-)graded k-module. The tensor algebra

$$TV = \oplus_{n\ge 1}V^{\otimes n}$$

has the structure of the universal counital coalgebra (co-)generated by V. We give T^cV the standard grading, i. e.

$$|v_1\otimes\ldots\otimes v_n| = \sum_k |v_k|.$$

The reduced tensor algebra

$$T^cV = \oplus_{n\ge 1}V^{\otimes n}$$

is the quotient of TV by the image of the counit.

For any graded coalgebra B we denote by $\mathrm{Coder}(B)$ the graded Lie algebra of its coderivations.

LEMMA 6.1.1. *For a graded k-module A there is an isomorphism of differential graded Lie algebras*

$$\widetilde{C}^\bullet(A)[1] \xrightarrow{\sim} \mathrm{Coder}(T(A[1]))$$

PROOF. Recall that the universal coalgebra generated by a vector space V is a coalgebra $C(V)$ together with a linear map $\pi : C(V) \to V$ such that, given any coalgebra C, every linear map $\phi : C \to V$ has a unique extension to a coalgebra morphism $\tilde{\phi} : C \to C(V)$ such that $\phi = \pi\circ\tilde{\phi}$. By universality, a coderivation D of $T(A[1])$ is uniquely determined by the composition

$$m : T(A[1]) \xrightarrow{D} T(A[1]) \to A[1],$$

where the second map is the projection of the tensor coalgebra on its first direct summand. It is straightforward that the Gerstenhaber Lie bracket from Definition 5.0.2 corresponds to commutator of coderivations. □

7. A_∞ algebras and their Hochschild complexes

DEFINITION 7.0.1. *An A_∞-algebra structure on A is a degree 1 coderivation D of $T^c(A[1])$ satisfying $D^2 = 0$.*

DEFINITION 7.0.2. *For an A_∞ algebra A, the DG coalgebra $(T(A), D)$ is called the bar construction of A and denoted by* $\mathrm{Bar}(A)$.

Let us record the following alternative definition.

LEMMA 7.0.3. *An A_∞ structure on a graded k-module V is given by a Hochschild cochain m on V of degree 2 satisfying the identity*

$$m \circ m = 0.$$

PROOF. Follows from Lemma 6.1.1. A little bit more precisely, the $D^2 = 0$ condition is easily seen to be equivalent to the associativity condition $m \circ m = 0$. □

REMARK 7.0.4. A Hochschild cochain as in the lemma above has the form of infinite sum

$$m = m_1 + m_2 + m_3 + \dots,$$

where

$$m_p \in \mathrm{Hom}_k^1(V[1]^{\otimes p}, V[1]) = \prod_{n_1+\dots+n_p-n=p-2} \mathrm{Hom}(V^{n_1} \otimes \dots V^{n_p}, V^n)$$

We set

$$d = m_1$$

and, for homogeneous elements a_1 and a_2 of V,

$$m(a_1, a_2) = (-1)^{|a_i|} m_2(a_1, a_2).$$

Then

- d is a differential of degree one on V;
- m is a graded bilinear product on V which is associative up to homotopy determined by m_3 and such that $[d, m] = 0$;
- m_3 satisfies, up to homotopy m_4, the pentagonal identity

$$m_2(m_3(a_1, a_2, a_3), a_4) \pm m_2(a_1, m_3(a_2, a_3, a_4)) =$$

$$m_3(m_2(a_1, a_2), a_3, a_4) \pm m_3(a_1, m_2(a_2, a_3), a_4) \pm m_3(a_1, a_2, m_2(a_3, a_4)) = 0,$$

  etc.

In particular, the following holds:

PROPOSITION 7.0.5 (Quillen). *****Are we sure it is Quillen?**** An A_∞-structure on a graded vector space V of the form $m = m_1 + m_2$ is the same as the structure of a differential graded algebra (V, m, d) (in the notation above).*

Once we have description of an A_∞ structure in the terms of the lemma 7.0.3, the following definition is quite natural.

DEFINITION 7.0.6. *An A_∞-module over an A_∞ algebra (A, m_A) is a graded k-module M and a degree one element*

$$m_M \in \mathrm{Hom}(M[1] \otimes T^c(A[1]), M[1])$$

satisfying

$$m_M \circ m = 0,$$

where the right hand m stands for m_A or m_M depending on whether its arguments include an element of M.

Before continuing we need a bit of notation.

DEFINITION 7.0.7. *Let A be a graded vector space and D a Hochschild cochain on A. We set*

$$\begin{aligned} L_D(a_0 \otimes \ldots \otimes a_n) = D(a_0 \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n + \\ \textstyle\sum_{k=0}^{n-d} \epsilon_k a_0 \otimes \ldots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \ldots \otimes a_n + \\ \textstyle\sum_{k=n+1-d}^{n} \eta_k D(a_{k+1}, \ldots, a_n, a_0, \ldots) \otimes \ldots \otimes a_k, \end{aligned}$$

where the second sum in the above formula is taken over all cyclic permutations such that a_0 is inside D. The signs are given by

$$\epsilon_k = (-1)^{(|D|+1)\sum_{i=0}^{k}(|a_i|+1)}$$

and

$$\eta_k = (-1)^{|D|+1+\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)}$$

PROPOSITION 7.0.8.

$$[L_D, L_E] = L_{[D,E]} \textit{ and } [L_D, B] = 0.$$

PROOF. We will leave the proof as an exercise for the reader. □

DEFINITION 7.0.9. *Suppose that (A, m) is an A_∞-algebra. Then*

(1) *The non-normalized Hochschild chain complex of A is*

$$(\widetilde{C}_\bullet(A), L_m),$$

(2) *The non-normalized Hochschild cochain complex of A is*

$$(\widetilde{C}^\bullet(A), [m,\ ]).$$

DEFINITION 7.0.10. Let (A, m) is a unital A_∞ algebra, i.e. assume there is an element $1 \in A$ satisfying $m_2(1, a) = (-1)^{|a|} m_2(a, 1) = a$ for all homogeneous $a \in A$ and $m_k(\ldots, 1, \ldots) = 0$. Then the differential $[m,\ ]$ descends to $C_\bullet(A)$. We define the (normalized) Hochschild cochain, resp. chain, complex of A to be

$$(C^\bullet(A), [m,\ ]), \text{resp.} (C_\bullet(A), L_m).$$

Let u be an element of degree -2. Then $[L_m, B] = 0$ and the negative cyclic complex of A is defined by

$$CC_*^-(A) = (C_*(A)[[u]], L_m + uB)$$

and similarly for the periodic cyclic and cyclic complexes.

A simple modification (using full and not reduced Hochschild complexes) can be given for non-unital A_∞ algebras.

7.1. A_∞ morphisms. Given two A_∞ algebras A and B, an A_∞ morphism $T : A \to B$ is a morphism of differential graded coalgebras

$$(T^c(A[1]), D_A) \to (T^c(B[1]), D_B)$$

As in the proof of Lemma 6.1.1, any morphism of graded coalgebras is determined by its composition with the projection $T^cB[1] \to B[1]$ which amounts to a collection of

$$F_n : A^{\otimes n} \to A \tag{7.1}$$

of degree $1 - n$, $n \geq 1$. Intertwining m_A with m_B is equivalent to the relation

$$\begin{aligned}\sum_{j,k} \pm\epsilon_{jk} F_{n-k}(a_1, \ldots, m(a_{j+1}, \ldots, a_{j+k}), \ldots, a_n) + \\ \sum_{p\geq 1; n_1,\ldots,n_{p-1}} \eta_{n_1,\ldots,n_{p-1}} m_p(F_{n_1}(a_1, \ldots, a_{n_1}), \ldots, F_{n_p}(a_{n_{p-1}+1}, \ldots, a_n)) = 0\end{aligned} \tag{7.2}$$

where the signs are ****** Clearly, for two A_∞ morphisms $A \to B \to C$, their composition $A \to C$ is defined.

7.1.1. *A_∞ morphisms acting on Hochschild and cyclic complexes.* For an A_∞ morphism $F : A \to B$, define

$$F_* : a_0 \otimes \ldots \otimes a_n \mapsto \sum \pm F_{n_0}(A_0) \otimes \ldots \otimes F_{n_p}(A_p) \tag{7.3}$$

where $(A_0, \ldots, A_p)$ run through all subdivisions of some cyclic permutation $(a_{j+1}, \ldots, a_j)$ into $p + 1$ segments so that A_0 contains a_0. The sign is

$$(-1)^{\sum_{k>j}(|a_k|+1)\sum_{k\leq j}(|a_k|+1)}.$$

PROPOSITION 7.1.1. *Formula* (7.3) *defines a morphism of Hochschild complexes*

$$F_* : \widetilde{C}_\bullet(A) \to \widetilde{C}_\bullet(B)$$

commuting with the cyclic differential B.

PROOF. This can be done by direct computation or using the interpretation of the Hochschild complexes given in 11. □

PROPOSITION 7.1.2. *Assume that F is an A_∞ morphism such that F_1 is a quasi-isomorphism. Then F_* is a quasi-isomorphism.*

PROOF. Indeed, F_* preserves the filtration

$$\mathcal{F}_n = \bigoplus_{m\leq n} A \otimes A^{\otimes m} \tag{7.4}$$

and induces a quasi-isomorphism on differential graded quotients. □

If A and B are A_∞ algebras with unit then an A_∞ morphism F is called unital if $F_1(1) = 1$ and $F_n(\ldots, 1, \ldots) = 0$ for $n \geq 2$. It is easy to see that in this case F_* descends to a morphism

$$F_* : C_\bullet(A) \to C_\bullet(B) \tag{7.5}$$

An analogue of Proposition 7.1.2 is true in this case.

REMARK 7.1.3. The projection $\widetilde{C}_\bullet(A) \to C_\bullet(A)$ is a quasi-isomorphism. Indeed, consider the spectral sequence associated to the filtration (7.4). Its E_1 term is $\widetilde{C}_\bullet(H^*(A))$, resp.

$$\bigoplus_{n\geq 0} H^*(A) \otimes H^*(\overline{A})^{\otimes n}$$

If the above were $C_\bullet(H^*(A))$, the projection would be an isomorphism of the E_2 terms and we would be done, but...********

7.2. The bialgebra structure on $\mathrm{Bar}(C^\bullet(A,A))$**.** Let us first recall the product on the bar construction $\mathrm{Bar}(C^\bullet(A,A))$ where $C^\bullet(A,A)$ is the algebra of Hochschild cochains of an algebra A with coefficients in A (cf. [**?**], [**234**]). For cochains D_i and E_j, define

$$(D_1|\ldots|D_m)\bullet(E_1|\ldots|E_n) = \sum \pm(\ldots|D_1\{\ldots\}|\ldots|D_m\{\ldots\}|\ldots)$$

Here the space denoned by $\ldots$ inside the braces contains $E_{j+1},\ldots,E_k$; outside the braces, it contains $E_{j+1}|\ldots|E_k$. The factor $D_i\{E_{j+1},\ldots,E_k\}$ is the brace operation as in (**??**). The sum is taken over all possible combinations for which the natural order of E_j's is preserved. The signs are computed as follows: a transposition of D_i and E_j introduces a sign $(-1)^{(|D_i|+1)(|E_j|+1)}$. In other words, the right hand side is the sum over all tensor products of $D_i\{E_{j+1},\ldots,E_k\}$, $k\geq j$, and E_p, so that the natural orders of D_i's and of E_j's are preserved. For example,

$$(D)\bullet(E) = (D|E) + (-1)^{(|D|+1)(|E|+1)}(E|D) + D\{E\}$$

PROPOSITION 7.2.1. *The product $\bullet$ together with the comultiplication Δ make* $\mathrm{Bar}(C^\bullet(A,A))$ *an associative bialgebra.*

8. Homotopy and homotopy equivalence

8.1. The case of DG algebras. Let $C^\bullet(\Delta^1)$ be the algebra of (nondegenerate) cochains with coefficients in k of the one-simplex Δ^1 with the standard triangulation. Explicitly, this algebra has the basis e_0, e_1, ξ where

$$|e_0| = |e_1| = 0;\ |\xi| = 1;\ de_0 = \xi = -de_1;\ e_0^2 = e_0;\ e_1^2 = e_1;$$

$$e_0e_1 = e_1e_0 = 0;\ e_0\xi = \xi e_1 = \xi; e_1\xi = \xi e_0 = 0$$

There are two DGA morphisms

$$\mathrm{ev}_0,\ \mathrm{ev}_1 : C^\bullet(\Delta^1) \to k$$

given by $\mathrm{ev}_i(e_j) = \delta_i^j$ and $\mathrm{ev}_i(\xi) = 0$.

DEFINITION 8.1.1. *Two morphisms of DG algebras f_0, $f_1 : A \to B$ are homotopic if there exists a morphism $f : A \to B \otimes C^\bullet(\Delta^1)$ such that $\mathrm{ev}_j \circ f = f_j$ for $j = 0, 1$. A homotopy equivalence between A and B is a pair of morphisms $f : A \to B$ and $g : B \to A$ such that gf is homotopic to id_A and fg is homotopic to id_B.*

Explicitly, a homotopy between A and B is a k-linear map $A \to B$ of degree one satisfying

$$D(ab) = D(a)g(b) + (-1)^{|a|}f(a)D(b); \tag{8.1}$$

$$[d, D] = f_0 - f_1. \tag{8.2}$$

Indeed, D satisfies the two equalities above if and only if

$$f(a) = f_0(a)e_0 + f_1(a)e_1 + D(a)\xi$$

is a DGA morphism.

8.2. The case of A_∞ algebras. We start by rewriting the definition of two homotopic morphisms of DG algebras from 8.1.1 in a way that works for A_∞ morphismsof A_infty algebras. First note that a pair of A_∞ morphisms $f, g : A \to B$ turn B into an A_∞ bimodule ${}_fB_g$. If B is unital, then one has a zero-cochain $\mathbf{1}$ in $C^\bullet(A,_f B_g)$ given by

$$ {}_f\mathbf{1}_g = 1 \in B^0 \tag{8.3}$$

One has

$$\delta_m({}_f\mathbf{1}_g) = f - g \tag{8.4}$$

Definition 8.2.1. *A homotopy between two A_∞ morphisms f_0, $f_1 : A \to B$ is a cochain D of total degree one in $C^\bullet(A,_{f_0} \mathcal{B}_{f_1})$ that is supported on the product of terms with $n \geq 1$ (9.1) and satisfies*

$$\delta_m D = f_0 - f_1 \tag{8.5}$$

In other words, if B is unital, a homotopy between f_0 and f_1 is an extension ${}_{f_0}\mathbb{I}_{f_1}$ of ${}_{f_0}\mathbf{1}_{f_1}$ to a Hochschild cocycle.

Lemma 8.2.2. *Being homotopic is an equivalence relation on A_∞ morphisms $A \to B$.*

Proof. Start with B being a unital DG algebra. In terms of Definition 8.2.1 we can define

$${}_{f_0}\mathbb{I}_{f_0} = {}_{f_0}\mathbf{1}_{f_0};\ {}_{f_0}\mathbb{I}_{f_2} = {}_{f_0}\mathbb{I}_{f_1} \cup {}_{f_1}\mathbb{I}_{f_2};\ {}_{f_1}\mathbb{I}_{f_0} = ({}_{f_0}\mathbb{I}_{f_1})^{-1}$$

(the inverse is with respect to the cup product). If there is no unit then we can formally attach it. If B is A_∞ then nothing changes: the cup product is still defined exactly as before in terms of m_2 (now it is not associative but is a part of an A_∞ structure, in particular a morphism of complexes which is enough for us). □

Definition 8.2.3. *An A_∞ homotopy equivalence between A and B is a pair $F : A \to B$ and $G : B \to A$ such that gf is homotopic to id_A and fg is homotopic to id_B.*

We also say that each f_0 and f_1 is an A_∞ homotopy equivalence.

Lemma 8.2.4. *Being homotopy equivalent is an equivalence relation.*

Proof. The relation is obviously reflexive and symmetric. As for transitivity, observe that, given

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

We claim that if g_0 is homotopic to g_1 then hg_0f is homotopic to hg_1f. Indeed, we can put

$${}_{hg_0f}\mathbb{I}_{hg_0f} = h_*f^*{}_{g_0}\mathbb{I}_{g_1}$$

Now assume that we are given

$$\begin{tikzcd} A \arrow[r, bend left, "f_0"] & B \arrow[l, bend left, "f_1"] \arrow[r, bend left, "g_0"] & C \arrow[l, bend left, "g_1"] \end{tikzcd}$$

such that $f_1 f_0$ is homotopic to Id_A and $g_1 g_0$ is homotopic to Id_B. Then $f_1 g_1 g_0 f_0$ is homotopic to $f_1 f_0$ which is homotopic to Id_A. Similarly in the opposite direction. $\square$

If we use the $\bullet$ product ***REF*** we can prove more:

LEMMA 8.2.5. *For A_∞ algebras and A_∞ morphisms as shown on the diagram below, assume that f_0 is homotopic to f_1 and g_0 is homotopic to g_1. Then $g_0 f_0$ is homotopic to $g_1 f_1$.*

$$\begin{tikzcd} A \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & B \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & C \end{tikzcd}$$

PROOF. In fact, we can put

$$_{g_0 f_0}\mathbb{I}_{g_1 f_1} = {}_{f_0}\mathbb{I}_{f_1} \bullet {}_{g_0}\mathbb{I}_{g_1}$$

$\square$

CHAPTER 3

Operations on Hochschild and cyclic complexes, I

We start our analysis of operations with the classical Eilenberg-Zilber and Alexander-Whitney exterior products and coproducts that we extend from simplicial to cyclic situation. We will later show that both the product and (dual version of) the coproduct are parts of a more general operation.

0.1. The Eilenberg-Zilber exterior product on Hochschild complexes.

DEFINITION 0.1.1. *For two algebras A_1 and A_2 define the shuffle product*

$$\mathrm{sh}: C_p(A_1)\otimes C_q(A_2)\to C_{p+q}(A_1\otimes A_2) \tag{0.1}$$

as follows.

(0.2)

$$(a_0^{(1)}\otimes\ldots\otimes a_p^{(1)})\otimes(a_0^{(2)}\otimes\ldots\otimes_q^{(2)})\mapsto a_0^{(1)}a_0^{(2)}\otimes\mathrm{sh}_{pq}(a_1^{(1)},\ldots,a_p^{(1)},\,a_1^{(2)}\;\ldots,a_q^{(2)})$$

where

$$\mathrm{sh}_{pq}(x_1,\ldots,x_{p+q})=\sum_{\sigma\in\mathrm{Sh}(p,q)}\mathrm{sgn}(\sigma)x_{\sigma^{-1}1}\otimes\ldots\otimes x_{\sigma^{-1}(p+q)} \tag{0.3}$$

and

$$\mathrm{Sh}(p,q)=\{\sigma\in\Sigma_{p+q}\,|\sigma 1<\ldots<\sigma p;\;\sigma(p+1)<\ldots<\sigma(p+q)\}$$

(We identify $a_j^{(1)}$ with $a_j^{(1)}\otimes 1$ and $a_j^{(2)}$ with $1\otimes a_j^{(2)}$).
In the graded case, $\mathrm{sgn}(\sigma)$ gets replaced by the sign computed by the following rule: in all transpositions, the parity of a_i is equal to $|a_i|+1$ if $i>0$, and similarly for c_i. A transposition contributes a product of parities.
Put also

$$m_{\mathrm{EZ}}(c_1,c_2)=(-1)^{|c_1|}\mathrm{sh}(c_1\otimes c_2)$$

THEOREM 0.1.2. *For two unital algebras A_1 and A_2*

$$m_{\mathrm{EZ}}: C_\bullet(A_1)\otimes C_\bullet(A_2)\to C_\bullet(A_1\otimes A_2)$$

is a quasi-isomorphism.

SKETCH OF THE PROOF.
Recall the free bimodule resolution $\mathcal{B}_\bullet(A)\to A$ of an algebra A as an A-bimodule given by (1.5). Let us recall their construction from [**84**]. For any algebra C, let $\mathcal{B}_\bullet(C)$ be the bar resolution for C. We use the notation

$$c_0\otimes\ldots\otimes c_{p+1}=c_0[c_1\ldots c_p]c_{p+1}$$

For any two algebras A and B, define

$$\mathrm{EZ}:\mathcal{B}_\bullet(A_1)\otimes\mathcal{B}_\bullet(A_2)\to\mathcal{B}_\bullet(A_1\otimes A_2) \tag{0.4}$$

to be the $A_1 \otimes A_2$-bimodule morphism such that

$$[a_1^{(1)}|\ldots|a_p^{(1)}] \otimes [a_1^{(2)}|\ldots|a_q^{(2)}] \mapsto \mathrm{sh}_{p,q}(a_1^{(1)} \otimes 1, \ldots, a_p^{(1)} \otimes 1, 1 \otimes a_1^{(2)}, \ldots, 1 \otimes a_q^{(2)})$$

This gives a quasiisomorphism of complexes of free $A_1 \otimes A_2$-bimodules

$$\bigoplus_{k+l=\bullet} \mathcal{B}_k(A_1) \otimes \mathcal{B}_l(A_2) \to \mathcal{B}_\bullet(A_1 \otimes A_2)$$

In particular, after tensoring with $A_1 \otimes A_2$, we get a quasiisomorphism of complexes

$$\bigoplus_{k+l=\bullet} (\mathcal{B}_k(A_1) \otimes \mathcal{B}_l(A_2)) \otimes_{A_1^e \otimes A_2^e} A_1 \otimes A_2 \to \mathcal{B}_\bullet(A_1 \otimes A_2) \otimes_{A_1^e \otimes A_2^e} A_1 \otimes A_2$$

The right hand side computes Hochschild homology of $A_1 \otimes A_2$. The obvious spectral sequence identifies the homology of the left hand side complex with

$$\bigoplus_{k+l=\bullet} H_k(A_1) \otimes H_l(A_2)$$

and, in particular, we get an isomorphism

$$\bigoplus_{k+l=\bullet} H_k(A_1) \otimes H_l(A_2) \to H_\bullet(A_1 \otimes A_2).$$

We leave it to the reader to check that the shuffle product satisfies

$$b(sh(x \times y)) = sh(bx \times y) + (-1)^{|x|} sh(x \times by) \tag{0.5}$$

and implements the isomorphism in question. □

1. The Hood-Jones exterior product on negative cyclic complexes

For any n unital algebras $A_1, \ldots, A_n$, $n \geq 2$, we will construct a $k[[u]]$-linear map of degree $n-2$ such that $m(A_1) = b + uB$ and

$$m(A_1, \ldots, A_N) \colon \mathrm{CC}_\bullet^-(A_1) \otimes_{k[[u]]} \ldots \otimes_{k[[u]]} \mathrm{CC}_\bullet^-(A_N) \to \mathrm{CC}_\bullet^-(A_1 \otimes \ldots \otimes A_N) \tag{1.1}$$

such that $m(A_1) = b + uB$ and the following A_∞ relation holds:

$$\sum_{k \geq 1, k+l \leq n} \pm m(A_1, \ldots, A_{k+1} \otimes \ldots \otimes A_{k+l}, \ldots, A_n) \circ m(A_{k+1}, \ldots, A_{k+l}) = 0 \tag{1.2}$$

(compare to **??**). In particular, for a commutative algebra A, $\mathrm{CC}_\bullet^-(A)$ is an A_∞ algebra over $k[[u]]$. We will later substantially enlarge the class of algebras A for which this is the case.

Definition 1.0.1. *Let A be an algebra. The map*

$$\mathrm{sh}'_n : C_{p_1}(A_1) \otimes \ldots \otimes C_{p_n}(A_n) \to C_{p_1+\ldots+p_n+n}(A_1 \otimes \ldots \otimes A_n) \tag{1.3}$$

as follows. Consider the embeddings

$$i_j : A_j \to A_1 \otimes \ldots \otimes A_n,\ a \mapsto 1 \otimes \ldots \otimes a_j \otimes \ldots \otimes 1.$$

Identify algebras A_j with their images under these embeddings. Denote

$$(x_1, \ldots, x_{n+\sum p_j}) = (a_0^{(1)}, \ldots, a_{p_1}^{(1)}, \ldots, a_0^{(n)}, \ldots, a_{p_n}^{(n)})$$

For $c_j = a_0^{(j)} \otimes \ldots \otimes a_{p_1}^{(1)}$, $1 \leq j \leq n$, set

$$\mathrm{sh}'_n(c_1, \ldots, c_n) = 1 \otimes \sum \mathrm{sgn}(\sigma) x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(\sum p_j + n)}$$

where σ runs through the set $\mathrm{Sh}'(p_1+1, \ldots, p_n+1)$ of all permutations such that:

a) the cyclic order of every group $(a_0^{(j)}, \ldots, a_{p_j}^{(j)})$ *is preserved;*
b) if $j < k$ *then* $a_0^{(j)}$ *appears to the left of* $a_0^{(k)}$.
In the graded case, the sign rule is as follows: any $a_i^{(j)}$ *has parity* $|a_i^{(j)}| + 1$.

DEFINITION 1.0.2.

$$m_1 = b + uB;$$

$$m_2(c_1, c_2) = (-1)^{|c_1|}(\mathrm{sh}(c_1, c_2) + u\mathrm{sh}'(c_1, c_2));$$

$$m_n(c_1, c_2) = (-1)^{***}u\mathrm{sh}'(c_1, \ldots, c_n)),\ n > 2.$$

THEOREM 1.0.3. *The above* m_n *satisfy the* A_∞ *relations* (1.2).

PROOF. □

THEOREM 1.0.4. *The map* $\mathrm{sh} + u\mathrm{sh}'$ *defines a* $k[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphisms of complexes*

$$C_\bullet(A_1) \otimes C_\bullet(C_2))[[u]] \to CC_\bullet^-(A_1 \otimes A_2),$$

$$(C_\bullet(A_1) \otimes C_\bullet(A_2))[u^{-1}, u]] \to CC_\bullet^{\mathrm{per}}(A_1 \otimes A_2)$$

and

$$(C_\bullet(A_1) \otimes C_\bullet(A_2))[u^{-1}, u]]/u(C_\bullet(A_1) \otimes C_\bullet(A_2))[[u]] \to CC_\bullet(A_1 \otimes A_2).$$

The differentials on the left hand sides are equal to

$$b \otimes 1 + 1 \otimes b + u(B \otimes 1 + 1 \otimes B).$$

SKETCH OF THE PROOF. We already know that $sh + ush'$ is (up to a sign) a morphism of total complexes. If we think of the left hand side as a double complex with the vertical boundary map $b \otimes 1 + 1 \otimes b$, Theorem 0.1.2 implies that all three morphisms of double complexes are quasiisomorphisms on the columns and hence are quasiisomorphisms on the total complexes. □

As a corollary we get the following Künneth formula for the cyclic homology.

THEOREM 1.0.5 (Künneth Theorem).
There is a long exact sequence

$$\cdots \xrightarrow{\times} HC_n(A \otimes C) \xrightarrow{\Delta} \bigoplus_{p+q=n} HC_p(A_1) \otimes HC_q(A_2) \xrightarrow{S\otimes 1 - 1 \otimes S}$$

$$\longrightarrow \bigoplus_{p+q=n-2} HC_p(A_1) \otimes HC_q(A_2) \xrightarrow{\times} HC_{n-1}(A_1 \otimes A_2) \xrightarrow{\Delta} \cdots$$

where Δ *is induced by the diagonal embedding*

$$u^{-p} c \otimes c' \mapsto (u^{-1} \otimes 1 + 1 \otimes u^{-1})^p c \otimes c'.$$

SKETCH OF THE PROOF. One checks that Δ is an embedding whose cokernel is the kernel of the multiplication by $u \otimes 1 - 1 \otimes u$ which, in turn, is the same as the kernel of $S \otimes 1 - 1 \otimes S$ (S is as in (1.12)). □

2. The Alexander-Whitney exterior coproduct on the Hochschild complex

For two algebras A_1 and A_2 define

$$\Delta_{\mathrm{AW}} : C_\bullet(A_1 \otimes A_2) \to C_\bullet(A_1) \otimes C_\bullet(A_2) \tag{2.1}$$

$$a_0^{(1)} a_0^{(2)} \otimes \ldots \otimes a_n^{(1)} a_n^{(2)} \mapsto$$

$$\sum_{j=0}^{n} (a_0^{(1)} \ldots a_j^{(1)} \otimes a_{j+1}^{(1)} \otimes \ldots \otimes a_n^{(1)}) \otimes (a_{j+1}^{(2)} \ldots a_n^{(2)} a_0^{(2)} \otimes a_1^{(2)} \otimes \ldots \otimes a_j^{(2)})$$

Similarly to EZ, the morphism AW is induced by a morphism of bar resolutions. Namely, define

$$[a_1^{(1)} a_1^{(2)} | \ldots | a_m^{(1)} \otimes a_m^{(2)}] \mapsto \sum_{j=0}^{m} [a_1^{(1)} | \ldots | a_j^{(1)}] a_{j+1}^{(1)} \ldots a_m^{(1)} \otimes a_1^{(2)} \ldots a_j^{(2)} [a_{j+1}^{(2)} | \ldots | a_m^{(2)}]$$

This gives a morphism

$$\mathcal{B}_\bullet(A_1 \otimes A_2) \to \mathcal{B}_\bullet(A_1) \otimes \mathcal{B}_\bullet(A_2) \tag{2.2}$$

THEOREM 2.0.1. *Δ_{AW} is a quasi-isomorphism of complexes. It is homotopy inverse to m_{EZ} from Theorem 0.1.2.*

PROOF. One checks that $\mathrm{EZ} \circ \mathrm{AW} = \mathrm{id}$.

LEMMA 2.0.2. *Let*

$$t([a_1^{(1)} a_1^{(2)} | \ldots | a_n^{(1)} a_n^{(2)}]) = \sum_j \sum_{k>j} \pm [a_1^{(1)} a_1^{(2)} | \ldots | a_j^{(1)} a_j^{(2)} | a_{j+1}^{(1)} \ldots a_k^{(1)} | C_{jk}] a_{k+1}^{(2)} \ldots a_n^{(2)}$$

where

$$C_{jk} = \mathrm{sh}([a_{k+1}^{(1)} | \ldots | a_n^{(1)}], [a_{j+1}^{(2)} | \ldots | a_k^{(2)}]).$$

Then t is a homotopy between id *and* $\mathrm{AW} \circ \mathrm{EZ}$.

The proof is a direct computation and we leave it to the reader ***OR NOT?*** The homotopy t is the one constructed by Eilenberg and Zilber in [**?**]. □

3. Exterior coproduct on $\mathrm{CC}_\bullet^-$

THEOREM 3.0.1. *For any n algebras $A_1, \ldots, A_n$, $n \geq 2$, there is a natural $k[[u]]$-linear map of degree $n-2$*

$$\Delta(A_1, \ldots, A_N) \colon \mathrm{CC}_\bullet^-(A_1 \otimes \ldots \otimes A_N) \to \mathrm{CC}_\bullet^-(A_1) \otimes_{k[[u]]} \ldots \otimes_{k[[u]]} \mathrm{CC}_\bullet^-(A_N) \tag{3.1}$$

such that

a)

$$\Delta(A_1) = b + uB;\ \Delta(A_1, A_2) = \eta \Delta_{\mathrm{AW}} \bmod u$$

where $\eta(a^{(1)} \otimes a^{(2)}) = (-1)^{|a^{(1)}|} a^{(1)} \otimes a^{(2)}$;

b) the following dual A_∞ relation is satisfied

$$\sum_{k \geq 1, k+l \leq n} \pm \Delta(A_{k+1}, \ldots, A_{k+l}) \circ \Delta(A_1, \ldots, A_{k+1} \otimes \ldots \otimes A_{k+l}, \ldots, A_n) = 0$$

*(again, compare to **??**). In particular, for a bialgebra A, $\mathrm{CC}_\bullet^-(A)$ is an A_∞ coalgebra over $k[[u]]$.*

PROOF. Denote by $\Lambda([m],[n])$ the set of all natural operations $A^{\otimes(m+1)} \to A^{\otimes(n+1)}$ for a unital monoid A that are compositions of:

a) $a_0 \otimes \ldots \otimes a_n \mapsto a_0 a_1 \otimes \ldots \otimes a_n$;

b) $a_0 \otimes \ldots \otimes a_n \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_n$;

c) $a_0 \otimes \ldots \otimes a_n \mapsto a_1 \otimes \ldots \otimes a_n \otimes a_0$.

For any operations

$$\lambda(j) : A^{\otimes(m+1)} \to A^{\otimes(n_j+1)}$$

and using notation

$$\lambda(j)(a_0^{(j)} \otimes \ldots \otimes a_m^{(j)}) = c_0^{(j)} \otimes \ldots \otimes c_{n_j}^{(j)},$$

define an (odometer-like) operation

$$\mathrm{Op}(\lambda(1), \ldots, \lambda(N)) : C_m(A_1 \otimes \ldots \otimes A_N) \to C_{n_1}(A_1) \otimes \ldots \otimes C_{n_N}(A_N); \tag{3.2}$$

$$a_0^{(1)} \ldots a_0^{(N)} \otimes \ldots \otimes a_m^{(1)} \ldots a_m^{(N)} \mapsto (c_0^{(1)} \otimes \ldots \otimes c_{n_1}^{(1)}) \otimes \ldots \otimes (c_0^{(N)} \otimes \ldots \otimes c_{n_N}^{(N)}) \tag{3.3}$$

Denote the k-linear span of these operations by $\mathcal{P}([m]; [n_1], \ldots, [n_N])$. Put

$$\mathcal{P}_\bullet^{(N)}([m]) = (\bigoplus_{n_1+\ldots+n_N=\bullet} \mathcal{P}([m]; [n_1], \ldots, [n_N]), \, b = (\sum_{j=1}^{N} b_{A_j}) \circ \_) \tag{3.4}$$

We also denote $\mathcal{P}_\bullet^{(1)}([m])$ by $\mathcal{P}_\bullet([m])$.

We claim (cf. also **??**) that the homology of $(\mathcal{P}_\bullet, b)$ is concentrated in degrees zero and one only, and is of rank 1 over k. It is easy to write the morphisms of complexes

$$\mathcal{P}_\bullet[m] \underset{i}{\overset{p}{\rightleftarrows}} (k \xrightarrow{0} k) \tag{3.5}$$

together with $s : \mathcal{P}_\bullet[m] \to \mathcal{P}_{\bullet+1}([m])$ such that $pi = \mathrm{id}$, $ip = [s, b]$. Explicilly:

Denote the generators of degree zero and one of $k \xrightarrow{0} k$ by α_0 and α_1 respectively. Then

$$i(\alpha_0) = a_0 \ldots a_m; \; i(\alpha_1) = \sum_{j=0}^{m} a_{j+1} \ldots a_{j-1} \otimes a_j \tag{3.6}$$

$$p(a_{j+1} \ldots a_j) = \alpha_0; \; p(a_{j+1} \ldots a_k \otimes a_{k+1} \ldots a_j) = \alpha_1 \tag{3.7}$$

if a_0 is a factor of $a_{k+1} \ldots a_j$ and zero otherwise; $p = 0$ on $\mathcal{P}_n([m])$ for $n \geq 2$;

$$s(a_{j+1} \ldots a_j) = \sum_{k=j+1}^{n} a_{k+1} \ldots a_{k-1} \otimes a_k; \tag{3.8}$$

for $n > 0$,

$$s(r_0 \otimes r_1 \otimes \ldots \otimes r_n) = \sum_{p=j}^{k} r_0 a_j \ldots a_{p-1} \otimes a_p \otimes a_{p+1} \ldots a_{k+1} \otimes \ldots r_2 \ldots r_n \tag{3.9}$$

where r_i are monomials, i.e. products of consecutive a_l in the cyclic order, and $r_1 = a_j \ldots a_{k+1}$. (Note that r_1 is also understood as a product in the cyclic order, i.e. it may contain a_0 as a factor).

For every N there are homotopy equivalences

$$\mathcal{P}_\bullet^{(N)}[m] \underset{i^{\otimes N}}{\overset{p^{\otimes N}}{\rightleftarrows}} (k \xrightarrow{0} k)^{\otimes N} \tag{3.10}$$

One has

$$p^{\otimes N} i^{\otimes N} = \mathrm{id};\ \mathrm{id} - i^{\otimes N} p^{\otimes N} = [b, s^{(N)}]$$

where

$$s^{(N)} = \sum_{j-1}^{N} (-1)^j p^{\otimes(j-1)} \otimes s \otimes \mathrm{id}^{\otimes(N-j)} \tag{3.11}$$

□

We will write Δ_N instead of $\Delta(A_1, \ldots, A_N)$. We want to construct

$$\Delta_N = \sum_{k=0}^{\infty} u^k \Delta_N^{(k)}$$

Start with $\Delta_2^{(0)}$. Define it as the AW coproduct from 2. Now compute $[B, \Delta_2^{(0)}]$. It is equal to zero on $\mathcal{P}_0$:

$$\Delta_2^{(0)} B(a_0^{(1)} a_0^{(2)}) = \Delta_2^{(0)}(1 \otimes a_0^{(1)} a_0^{(2)}) =$$
$$(a_0^{(1)}) \otimes (1 \otimes a_0^{(2)}) + (1 \otimes a_0^{(1)}) \otimes (a_0^{(2)}) = B\Delta_2^{(0)}(a_0^{(1)} a_0^{(2)})$$

Furthermore, it sends $\mathcal{P}_n$, $n > 0$, to $\ker(p \otimes p)$. Indeed, this is enough to check for the component $\mathcal{P}_1 \to \mathcal{P}_1 \otimes \mathcal{P}_1$, in which case there are four terms of $[B, \Delta_2^{(0)}]$:

$$(a_0^{(1)} \otimes a_1^{(1)}) \otimes (a_1^{(2)} \otimes a_0^{(2)});\ (a_1^{(1)} \otimes a_0^{(1)}) \otimes (a_0^{(2)} \otimes a_1^{(2)});$$
$$(1 \otimes a_0^{(1)} a_1^{(1)}) \otimes (a_0^{(2)} \otimes a_1^{(2)});\ (a_0^{(1)} \otimes a_1^{(1)}) \otimes (1 \otimes a_1^{(2)} a_0^{(2)})$$

But p only detects terms in C_1 that have a factor a_0 on the right, and none of the four terms have that in both tensor factors.

Therefore there is $\Delta_2^{(1)}$ such that $[b, \Delta_2^{(1)}] + [B, \Delta_2^{(0)}] = 0$. By degree considerations, its commutator with B lands in $\ker(p \otimes p)$, etc. We construct Δ_2 by recursion. Now we have to construct $\Delta_3^{(j)}$ starting with $j = 1$ (because $\Delta_2^{(0)}$ is coassociative). We proceed by recursion, on every step finding $\Delta_N^{(j)}$ from the condition that its commutator with b is a given morphism $\mathcal{P}_\bullet \to \ker(p^{\otimes N})$.

Lemma 3.0.2. *There are no nonzero components of $\Delta^{(k)}(A_1, \ldots, A_N)$ of the form*

$$C_m(A_1 \otimes \ldots \otimes A_N) \to C_{n_1}(A_1) \otimes \ldots \otimes C_{n_N}(A_N)$$

with any of the n_j being equal to zero, unless $N = 2$ and $k = 0$.

Proof. Let $\Delta_N^{(k)}(m)$ be the restriction of $\Delta_N^{(k)}(A_1, \ldots, A_N)$ to $C_m(A_1 \otimes \ldots \otimes A_N)$. Recall how the construction of $\Delta_N^{(k)}(m)$ goes. We assume that $\Delta_{N'}^{(k')}(m')$ are constructed for

a) all $N' < N$ and all k', m';
b) $N' = N$, $k' < k$, all m';
c) $N' = N$, $k' = k$, $m' < m$.

Then one constructs a particular linear combination of terms of the form

1) $\Delta_{N'}^{k'}(m') \circ \Delta_{N-N'+1}^{k-k'}(m)$;

2) $\Delta_N^{(k)}(m-1)\circ b$;
3) $\Delta_N^{(k-1)}(m+1)\circ B$;
4) $B\circ\Delta_N^{(k-1)}(m)$.

We obtain $\Delta_N^{(k)}(m)$ by applying to this expression the homotopy $s^{(N)}$. By the induction hypothesis, among all these terms, only 1) may contain terms with some $n_j=0$, and only when $k'=0$, $N'=2$ or $k-k'=0$, $N-N'+1=2$.

When $k'=0$ and $m'=2$, we get the operation $\Delta^{(k)}(A_1,\dots,A_j\otimes A_{j+1},\dots,A_N)$

$$C_m(A_1\otimes\dots\otimes A_N)\to C_{n_1}(A_1)\otimes\dots\otimes C_{n_{j,j+1}}(A_j\otimes A_{j+1})\otimes\dots\otimes C_{n_N}(A_N)$$

followed by

$$\Delta^{(0)}(A_j,A_{j+1}):C_{n_{j,j+1}}(A_j\otimes A_{j+1})\to C_{n_j}(A_j)\otimes C_{n_{j+1}}(A_{j+1}).$$

Consider the terms with either $n_j=0$ or $n_{j+1}=0$ (by the inductive hypothesis it cannot be both). When we apply $s^{(N)}$, these terms can be hit by either s or p. But s increases the degree n_j. And applying p transforms such a term into

$$a_0^{(j')}\dots a_m^{(j')} \tag{3.12}$$

where $j'=j$ or $j+1$. So applying p to the position j' is the same as computing

$$\Delta^{(k)}(A_1,\dots,\widehat{A_{j'}},\dots,A_N) \tag{3.13}$$

and then inserting the tensor factor (3.12) in the j'th position. Applying to this the terms of $s^{(N)}$ that hit $C_{n'_j}$ with p becomes the same as applying $s^{(N-1)}$ to (3.12) and then inserting the tensor factor. But (3.12) is itself in the image of $s^{(N-1)}$, therefore the result is zero.

Now consider the case $k-k'=0$ and $N-N'+1=2$. Then we get the composition of $\Delta_2^{(0)}(A_1\otimes\dots\otimes A_{N-1},A_N)$

$$C_m(A_1\otimes\dots\otimes A_N)\to C_n(A_1\otimes\dots\otimes A_{N-1})\otimes C_{n_N}(A_N)$$

composed with $\Delta^{(k)}(A_1,\dots,A_{N-1})$. By the induction hypothesis, the only possibly nonzero terms with $n_j=0$ occur when $j=N$. Applying $s^{(N)}$ to them is the same as applying $s^{(N-1)}$ to $\Delta^{(k)}(A_1,\dots,A_{N-1})$ and then tensoring by $a_0^N\dots a_m^{(N)}$. This is zero because $\Delta^{(k)}(A_1,\dots,A_{N-1})$ is in the image of $s^{(N-1)}$ Similarly for A_1 and $A_2\otimes\dots\otimes A_N$. ☐

Remark 3.0.3. We chose to avoid mentioning cyclic modules in the above proof, not only to make it self-sufficient but because we needed an extra degree of explicitness. Here we would like to relate our construction to 6. Recall the complex of cocyclic k-modules

$$\mathcal{P}_\bullet([m])=\Lambda([m],[\bullet]);\ \lambda\mapsto b\lambda;\ b=\sum_{j=0}^{n}(-1)^j d_j\ \ \text{on}\mathcal{P}_n \tag{3.14}$$

from (4.11). For any $N\geq 1$, define the complexes

$$\mathcal{P}_\bullet^{\otimes N}[[u]]([m])=(\mathcal{P}_\bullet([m])^{\otimes N}[[u]],b+uB) \tag{3.15}$$

(We use the tensor product of complexes combined with the diagonal tensor product of cocyclic modules). Note that, when tensored by $k[[u,u^{-1}]/k[[u]]$ over $k[[u]]$, all of them become resolutions of the constant cocyclic module $k_\sharp$. In particular there is a chain map between them over $k_\sharp$. What is a little more subtle is the question of its linearity over $k[[u]]$. What we have proven is that there are $k[[u]]$-linear morphisms of complexes of cocyclic modules

$$\Delta_N\colon \mathcal{P}_\bullet[[u]] \to \mathcal{P}_\bullet^{\otimes N}[[u]] \tag{3.16}$$

from which the above follows, and that the following is true:

$$[b+uB, \Delta_N] = \sum_{k,l} \pm(\mathrm{id}^{\otimes k} \otimes \Delta_l \otimes \mathrm{id}^{\otimes(n-k-l)}) \circ \Delta_{N-l+1} \tag{3.17}$$

We identify morpisms in

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathcal{P}_n, \mathcal{P}_{n_1} \otimes \ldots \otimes \mathcal{P}_{n_N}) \tag{3.18}$$

with k-linear natural maps

$$C_n(A_1 \otimes \ldots \otimes A_N) \to C_{n_1}(A_1) \otimes \ldots \otimes C_{n_N}(A_N) \tag{3.19}$$

For example, when $N=2$, the map from (3.19)

$$a_0b_0 \otimes a_1b_1 \otimes a_2b_2 \mapsto (a_2a_0 \otimes a_1) \otimes (b_0 \otimes 1 \otimes b_1b_2)$$

corresponds to the only morphism $\mathcal{P}_2 \to \mathcal{P}_1 \otimes \mathcal{P}_2$ for which

$$(\mathrm{id} \in \Lambda([2],[2])) \mapsto d_0 \otimes s_1d_1 \in \Lambda([2],[1]) \otimes \Lambda([2],[3]).$$

4. Multiplication on $\mathrm{CC}^\bullet$

Dually to Δ_{AW}, if $C_1, \ldots, C_n$ are coalgebras there are products

$$m(C_1, \ldots, C_N)\colon \mathrm{CC}_-^\bullet(C_1) \otimes_{k[[u]]} \ldots \otimes_{k[[u]]} \mathrm{CC}_-^\bullet(C_N) \to \mathrm{CC}_\bullet^-(C_1 \otimes \ldots \otimes C_N) \tag{4.1}$$

such that $m(C_1) = b + uB$ and the A_∞ relation (1.2) holds.

For a bialgebra H the compositions of the above maps (when $C_1 = \ldots = C_N = H$) with the morphism of complexes induced by the product on H define an A_∞ algebra structure on $\mathrm{CC}_-^\bullet(H)$. Modulo u, it is an associative graded algebra with the product

(4.2)

$$(x_0 \otimes \ldots \otimes x_p) \otimes (y_0 \otimes \ldots \otimes y_q) = x_0^{(0)} y_0^{(p+1)} \otimes x_0^{(1)} y_1 \ldots \otimes x_0^{(q)} y_q \otimes x_1 y_0^{(1)} \otimes \ldots \otimes x_p y_0^{(p)}$$

where we use the notation

$$\Delta^q x_0 = \sum x_0^{(0)} \otimes \ldots \otimes x_0^{(q)};\ \Delta^p y_0 = \sum y_0^{(1)} \otimes \ldots \otimes y_0^{(p+1)}$$

In general, the A_∞ structure involves the following operations.

Consider all operations $H^{\otimes(p+1)} \to H^{\otimes(q+1)}$ that are compositions of:

a) $x_0 \otimes \ldots \otimes x_p \mapsto \sum x_0^{(1)} \otimes x_0^{(2)} \otimes \ldots \otimes x_p$;

b) $x_0 \otimes \ldots \otimes x_p \mapsto \epsilon(x_0) x_1 \otimes \ldots \otimes x_p$;

c) $x_0 \otimes \ldots \otimes x_p \mapsto x_1 \otimes \ldots \otimes x_p \otimes x_0$.

Consider an element

$$x(j) = x_0(j) \otimes \ldots \otimes x_{p_j}(j) \in C^{p_j}(H),\ j = 1, \ldots, N.$$

For any operations

$$\lambda(j) : H^{\otimes(p_j+1)} \to H^{\otimes(q+1)}$$

and using notation

$$\lambda(j)(x_0(j) \otimes \ldots \otimes x_{p_j}(j)) = y_0(j) \otimes \ldots \otimes y_q(j),$$

define

$$\mathrm{Op}(\lambda(1), \ldots, \lambda(N)) : C^{p_1}(H) \otimes \ldots \otimes C^{p_N}(H) \to C^q(H); \tag{4.3}$$

$$x(1)\otimes\ldots\otimes x(N)\mapsto y_0(1)\ldots y_0(N)\otimes\ldots\otimes y_q(1)\ldots y_q(N) \tag{4.4}$$

The construction in 3 implies that the A_∞ operations are linear combinations of (4.4). In particuar, we get

LEMMA 4.0.1. *Let H be a cocommutative bialgebra. Then*
1) $H=C^0(H)$ is a DG subalgebra with respect to m_2.
2) The m_2 multiplication by H from left and right is the standard left and right action of H on tensor powers of H via comultiplication.
3) The A_∞ operations m_N are H-bimodule maps

$$m_N\colon C^\bullet(H)\otimes_H\ldots\otimes_H C^\bullet(H)[[u]].$$

4) Substituting $x\in H$ into m_N gives 0 when $N\geq 3$.

PROOF. 1) is true because of the formulas (4.2) for the product and because $b:C^0\to C^1$ is the cocommutator. 2) follows from (4.2). 3) follow from (4.4), and 4) follows from Lemma 3.0.2. $\square$

4.1. $\mathrm{CC}^\bullet(H)$ for cocommutative Hopf algebras. Let H be a cocommutative Hopf algebra. Let

$$\overline{H}=\ker(\epsilon) \tag{4.5}$$

Use the normalized Hochschild complex

$$C^n(H)=H\otimes\overline{H}^{\otimes n}$$

It is an A_∞ algebra (clearly the structure on the full complex descends to it). Also, the embedding into the full complex is a quasi-isomorphism.

4.2. The DG algebra $H\ltimes\mathrm{Cobar}(\overline{H})$. For a bialgebra H and an algebra A, an action of H on A is a linear map $H\otimes A$, $x\otimes a\mapsto\rho(x)a$, such that

$$\rho(xy)=\rho(x)\rho(y);\ \rho(x)(ab)=\sum\rho(x^{(1)})a\rho(x^{(2)})b$$

If H is a Hopf algebra acting on A then one can define a cross product

$$H\ltimes A=A\otimes H;(a\otimes x)(b\otimes y)=a\rho(x^{(1)})b\otimes Sx^{(2)}y \tag{4.6}$$

Let $A=\mathrm{Cobar}(\overline{H})$. Put

$$\rho(x)(x_1|\ldots|x_n)=\sum(x^{(1)}x_1S(x^{(n+1)})|\ldots|x^{(n)}x_nS(x^{(2n)})) \tag{4.7}$$

where S is the antipode. The action commutes with the differential on $\mathrm{Cobar}(\overline{\mathrm{H}})$ (which we denote by b), and we get a DG algebra $H\ltimes\mathrm{Cobar}(H)$.

REMARK 4.2.1. Note that the comultiplication on $\overline{H}$ is given by

$$\Delta x=\sum x^{(1)}\otimes x^{(2)}-1\otimes x-x\otimes 1$$

In other words, $H\ltimes\mathrm{Cobar}(\overline{H})$ is the DG algebra generated by a subalgebra H and by elements (x), linear in $x\in\overline{H}[1]$, subject to

$$x\cdot(y)=\sum(x^{(1)}yS(x^{(2)}))\cdot x^{(3)};\ bx=0;\ b(x)=\sum(x^{(1)})(x^{(2)}) \tag{4.8}$$

This DG algebra admits a derivation B determined by

$$Bx=0;B(x)=x.$$

It is easy to see that B is well defined and commutes with b. Of course, if H is a DG Hopf algebra, then its differential d induces an extra differential on $H\ltimes\mathrm{Cobar}(\overline{H})$.

We state the next result in the generality that we will need later. The cyclic complex of the second kind of a DG coalgebra is defined in ***REF. For an ordinary coalgebra this is just the usual Hochschild complex.

PROPOSITION 4.2.2. *For a cocommutative DG Hopf algebra H,*
1) there is an isomorphism of DG algebras

$$C^{\bullet}_{\mathrm{II}}(H) \stackrel{\sim}{\to} (H \ltimes \mathrm{Cobar}(\overline{H}), b+d);$$

*2) there are natural $k[[u]]$-linear (u)-adically continuous A_∞ ***ISO? morphism*

$$\mathrm{CC}^{\bullet}_{\mathrm{II}}(H) \stackrel{\sim}{\to} ((H \ltimes \mathrm{Cobar}(\overline{H}))[[u]], b+d+uB)$$

PROOF. Let us prove 1). Note that the product on Hochschild cochains is as follows:

$$(4.9)\quad (1\otimes x_1\otimes\ldots\otimes x_n)(1\otimes y_1\otimes\ldots\otimes y_n) = \pm 1\otimes y_1\otimes\ldots\otimes y_n\otimes x_1\otimes\ldots\otimes x_n$$

Therefore, if we denote $1\otimes x$ by (x), the k-submodule generated by $1\otimes x_1\otimes\ldots\otimes x_n$ for $x\in\overline{H}$ is a DG subalgebra isomorphic to $\mathrm{Cobar}(\overline{H})$. Combining this with the formulas for left and right multiplication by H, we get 1).

To prove 2), note first that $B(x)=x$ which is compatible with 2). But the action of B on $\mathrm{Cobar}(\overline{H})$ does not agree for the algebras in the two sides of 2). In fact, on the right hand side we have

$$B(1\otimes x_1\otimes\ldots\otimes x_n) = \pm B((x_n)\ldots(x_1)) = \sum \pm (x_n)\ldots(x_{j+1})x_j(x_{j-1})\ldots(x_1)) =$$
$$\sum \pm(1\otimes x_{j+1}\otimes\ldots\otimes x_n)(x_j^{(0)}\otimes x_j^{(1)}x_1\otimes\ldots\otimes x_j^{(j-1)}x_{j-1}) =$$
$$\sum x_j^{(0)}\otimes x_j^{(1)}x_1\otimes\ldots\otimes x_j^{(j-1)}x_{j-1}\otimes x_{j+1}x_j^{(j)}\otimes\ldots\otimes x_n x_j^{(n-1)}$$

whereas on the left hand side we have the usual

$$B(1\otimes x_1\otimes\ldots\otimes x_n) = \sum \pm x_j\otimes\ldots\otimes x_{j-1}$$

For $n=1$ the two expressions are equal. For $n=2$ they are not but the difference is cohomologous to zero. In fact, it is equal to the value of the map

$$1\otimes x_1\otimes x_2\otimes x_3\mapsto x_2\otimes x_3x_1$$

at $b(1\otimes x_1\otimes x_2\otimes)$.

We want to prove 2) by constructing a universal A_∞ morphism comprised of the following expressions. Let us write

$$(4.10)\qquad x[m] = \sum x^{(1)}\ldots x^{(m)};$$

for a multi-index $M=(m_1,\ldots,m_n)$, and for $\mathbf{x}=(x_1|\ldots|x_n)$,

$$(4.11)\qquad \mathbf{x}[M] = x_n[m_n]\ldots x_1[m_1];$$

for $\mathbf{x}(1),\ldots,\mathbf{x}(k)\in\mathrm{Cobar}(\overline{H})$, consider linear combinations of the following:

$$(4.12)\quad \varphi(\mathbf{x}(1),\ldots,\mathbf{x}(k)) = \mathbf{x}(1)[M_0^1]\ldots\mathbf{x}(k)[M_0^k]\otimes\ldots\otimes\mathbf{x}(1)[M_0^p]\ldots\mathbf{x}(k)[M_0^p]$$

These are natural linear maps

$$\mathrm{Cobar}(\overline{H})^{\otimes k}\to C^{\bullet}_{\mathrm{II}}(H)$$

Extend them to H-bi-invariant maps

$$(H\ltimes\mathrm{Cobar}(\overline{H}))\otimes_H\ldots\otimes_H(H\ltimes\mathrm{Cobar}(\overline{H}))\to C^{\bullet}_{\mathrm{II}}(H)$$

Because of 1), they can be viewed as Hochschild cochains of $H \ltimes \mathrm{Cobar}(\overline{H})$. They form a subcomplex, meaning, the differential b and the Hochschild differential preserve this class of cochains. We will show that the only cohomology class of this subcomplex in relevant degrees is the derivation B. ***FINISH *** $\square$

5. Pairings between chains and cochains

Definition 5.0.1. *Let a A be a graded algebra. For $D \in C^d(A, A)$ we set*

$$i_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D| \sum_{i \le d}(|a_i|+1)} a_0 D(a_1, \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n \tag{5.1}$$

The following identities are straightforward.

Proposition 5.0.2.

$$[b, i_D] = i_{\delta D}$$

$$i_D i_E = (-1)^{|D||E|} i_{E \smile D}$$

Recall also the L-operations as defined in (7.0.7). The following holds

Proposition 5.0.3. *In particular, the following identities hold*

$$[L_D, L_E] = L_{[D,E]};\ [b, L_D] + L_{\delta D} = 0 \text{ and } [L_D, B] = 0. \tag{5.2}$$

Now let us extend the above operations to the cyclic complex. Define

$$S_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j \ge 0;\ k \ge j+d} \epsilon_{jk} 1 \otimes a_{k+1} \otimes \ldots a_0 \otimes \ldots \otimes \tag{5.3}$$

$$D(a_{j+1}, \ldots, a_{j+d}) \otimes \ldots \otimes a_k$$

(The sum is taken over all cyclic permutations; a_0 appears to the left of D). The signs are as follows:

$$\epsilon_{jk} = (-1)^{|D|(|a_0| + \sum_{i=1}^n (|a_i|+1)) + (|D|+1) \sum_{j+1}^k (|a_i|+1) + \sum_{i \le k}(|a_i|+1) \sum_{i \ge k}(|a_i|+1)}$$

As we will see later, all the above operations are partial cases of a unified algebraic structure for chains and cochains; the sign rule for this unified construction will be explained in 2.

Proposition 5.0.4. *([**?**])*

$$[b + uB, i_D + uS_D] - i_{\delta D} - uS_{\delta D} = L_D$$

Proposition 5.0.5. *([**222**]) There exists a linear transformation $T(D, E)$ of the Hochschild chain complex, bilinear in D, $E \in C^\bullet(A, A)$, such that*

$$[b + uB, T(D, E)] - T(\delta D, E) - (-1)^{|D|} T(D, \delta E) =$$

$$= [L_D, i_E + uS_E] - (-1)^{|D|+1}(i_{[D,E]} + uS_{[D,E]})$$

6. Basic invariance properties of Hochschild and cyclic homology

6.1. Morita invariance.

THEOREM 6.1.1. *The trace map*

$$\# : C_\bullet(M_N(k) \otimes A) \to C_\bullet(A)$$

given by

$$(T_1 \otimes a_1) \otimes (T_2 \otimes a_2) \otimes \ldots \otimes (T_n \otimes a_n) \mapsto Tr(T_1 T_2 \ldots T_n) a_1 \otimes a_2 \otimes \ldots \otimes a_n.$$

descends is a quasiisomorphism of cyclic(resp. periodic and negative periodic) complexes.

PROOF. By the Künneth formula it is sufficient to check the claim for $A = k$, and we will leave as an exercise for the reader. □

6.2. Homotopy invariance.

THEOREM 6.2.1. *Supppose that*

$$t \to \phi(t) : A \to B$$

is a one parameter family of homomorphisms depending polynomially on $t \in \mathbb{R}$. Then the induced family of morphisms of complexes $\phi(t)_* : CC_\bullet^{per}(A) \to CC_\bullet^{per}(B)$ is constant up to homotopy.

PROOF. $\phi(t)$ induces a homomorphism of algebras

$$A \to B \otimes k[t]$$

and, by Künneth formula, it is sufficient to show that the evaluation homomorphism

$$k[t] \ni P \mapsto P(a) \in k$$

induces a map on periodic cyclic homology which is independent of the choice of a. We will see later that the map

$$C_n(k[t]) \ni f_0 \otimes \ldots \otimes f_n \to f_0 df_1 \ldots df_n \in \Omega^n(\mathbb{R})$$

induces a quasiisomorphism of the periodic cyclic complex of $k[t]$ with the de Rham complex of $\mathbb{R}$ with coefficients in $k[t]$ and the Poincare lemma finishes the proof.

An alternative proof can e given using the Cartan formula from the proposition 5.0.4 in the next section applied to the operator L_{∂_t} acting on the cyclic periodic complex of $B[01]$. □

7. Bibliographical notes

CHAPTER 4

The cyclic complex $C^{\lambda}_{\bullet}$

1. Introduction

2. Definition

Recall the original definition of the cyclic complex from [?], [?]. Let $\tau = \tau_p$ denote the endomorphism of $A^{\otimes_k (p+1)}$ given by the formula

$$\tau(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p \otimes a_0 \cdots \otimes a_{p-1}$$

Let

$$C^{\lambda}_p(A) = A^{\otimes_k (p+1)} / \mathrm{Im}(\mathrm{id} - \tau) \ . \tag{2.1}$$

Define

$$\begin{aligned} N : C_p(A) &\rightarrow C_{p-1}(A) \\ N = \mathrm{id} + \tau + \ldots \tau^{p-1} \end{aligned}$$

One has

$$b(\mathrm{id} - \tau) = (\mathrm{id} - \tau) b'; \;\; b' N = N b, \tag{2.2}$$

where $b' : A^{\otimes \bullet} \rightarrow A^{\otimes \bullet - 1}$ is given by

$$b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^k a_0 \otimes \ldots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes \ldots \otimes a_n .$$

Therefore the differential b descends to a map

$$b : C^{\lambda}_p(A) \rightarrow C^{\lambda}_{p-1}(A) \ .$$

Suppose now that A is unital and set

$$B_0(a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_1 \otimes \ldots \otimes a_n . \tag{2.3}$$

Then $[B_0, b'] = id$, i. e. the complex $(A^{\otimes \bullet}, b')$ is contractible.

Proposition 2.0.1. *The complex $C^{\lambda}_{\bullet}(A)$ is quasi-isomorphic to the complex $CC_{\bullet}(A)$.*

Sketch of the proof. Let us look at the double complex $\mathcal{C}_A$

$$
\begin{tikzcd}
 & {} \arrow[d, "b"] & {} \arrow[d, "b'"] & {} \arrow[d, "b"] & {} \arrow[d, "b'"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
\dots \arrow[r, "1-\tau"] & A^{\otimes 3} \arrow[r, "N"] \arrow[d, "b"] & A^{\otimes 3} \arrow[r, "1-\tau"] \arrow[d, "b'"] & A^{\otimes 3} \arrow[r, "N"] \arrow[d, "b"] & A^{\otimes 3} \arrow[r, "1-\tau"] \arrow[d, "b'"] & A^{\otimes 3} \arrow[r, Rightarrow, dashed] \arrow[d, "b"] & C^{\lambda}_2(A) \arrow[d, "b"] \\
\dots \arrow[r, "1-\tau"] & A^{\otimes 2} \arrow[r, "N"] \arrow[d, "b"] & A^{\otimes 2} \arrow[r, "1-\tau"] \arrow[d, "b'"] & A^{\otimes 2} \arrow[r, "N"] \arrow[d, "b"] & A^{\otimes 2} \arrow[r, "1-\tau"] \arrow[d, "b'"] & A^{\otimes 2} \arrow[r, Rightarrow, dashed] \arrow[d, "b"] & C^{\lambda}_1(A) \arrow[d, "b"] \\
\dots \arrow[r, "1-\tau"] & A \arrow[r, "N"] & A \arrow[r, "1-\tau"] & A \arrow[r, "N"] & A \arrow[r, "1-\tau"] & A \arrow[r, Rightarrow, dashed] & C^{\lambda}_0(A)
\end{tikzcd}
$$

Since the rows are acyclic in positive dimensions, the stipled arrows give a quasiisomorphism from the total complex to the $(C^{\lambda}_{\bullet}(A), b)$ complex. On the other hand, since the collumns $(A^{\otimes \bullet}, b')$ are acyclic, the total complex is easily seen to be quasiisomorphic to the complex

$$(A^{\otimes \bullet+1}[u^{-1}, u]/k[u], b + u\tilde{B}),$$

where

$$\tilde{B} = (1-\tau)B_0 N.$$

The quotient map

$$(A^{\otimes \bullet+1}[u^{-1}, u]/k[u], b + u\tilde{B}) \to CC_{\bullet}(A)$$

is a quasiisomorphism on the columns $(A^{\otimes \bullet+1}, b) \to (C_{\bullet}(A), b)$ (see **??**) and hence a quasiisomorphism on the double complexes. The claimed result follows (cf. [**?**] for more detail). □

Lemma 2.0.2. *For algebras without unit, the cyclic homology defined in 1.0.16 is isomorphic to the homology of the complex $C^{\lambda}_{\bullet}(A)$.*

Proof. This follows from tracing what happens for non-unital algebras in the proof of the proposition 2.0.1 above. □

3. The reduced cyclic complex

Let $\overline{A} = A/k$ and let

$$\overline{C}^{\lambda}_p(A) = \overline{A}^{\otimes p+1}/\operatorname{Im}(\mathrm{id} - \tau)\ . \tag{3.1}$$

It is easy to see that the diferential b descends to $\overline{C}^{\lambda}_{\bullet}(A)$. We denote the homology of the complex $\overline{C}^{\lambda}_{\bullet}(A)$ by $\overline{HC}_{\bullet}(A)$.

The following is the reduced analogue of the proposition 2.0.1.

Proposition 3.0.1. *The complex $(\overline{C}^{\lambda}_{\bullet}(A), b)$ is quasiisomorphic to the complex $\overline{CC}_{\bullet}(A)$ given by the cokernel of the inclusion of complexes*

$$CC_{\bullet}(k) \to CC_{\bullet}(A).$$

PROOF. $\overline{CC}_\bullet(A)$ has the form

$$\begin{tikzcd}
 & & & & \overline{A} \\
 & & & \overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{\otimes(n-1)} \arrow[u, dotted] \\
 & & \overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{\otimes(n-1)} \arrow[u] \arrow[r, dotted] & A\otimes\overline{A}^{\otimes n} \arrow[u] \\
 & \overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{\otimes(n-1)} \arrow[u, dotted] \arrow[r] & A\otimes\overline{A}^{\otimes n} \arrow[u] \arrow[r] & A\otimes\overline{A}^{\otimes(n+1)} \arrow[u] \\
\overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{\otimes(n-1)} \arrow[u, dotted] \arrow[r] & A\otimes\overline{A}^{\otimes n} \arrow[u] \arrow[r] & A\otimes\overline{A}^{\otimes(n+1)} \arrow[u] \arrow[r] & A\otimes\overline{A}^{\otimes(n+2)} \arrow[u] \\
\cdots \arrow[u, dotted] & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u]
\end{tikzcd}$$

where, for simplicity of the grafical representation, we did not include the powers of u. We will filter it by subcomplexes of the form

$$\mathcal{F}_n = \begin{tikzcd}
 & & & & \overline{A} \\
 & & & \overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{\otimes(n-1)} \arrow[u, dotted] \\
 & & \overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{n-1} \arrow[u, dotted, no head] \arrow[r] & A\otimes\overline{A}^{\otimes n} \arrow[u] \\
 & \overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{\otimes(n-1)} \arrow[u, dotted] \arrow[r] & A\otimes\overline{A}^{\otimes n} \arrow[u] \arrow[r] & 1\otimes\overline{A}^{\otimes n+1} \arrow[u] \\
\overline{A} \arrow[r, dotted] & A\otimes\overline{A}^{\otimes(n-1)} \arrow[u, dotted] \arrow[r] & A\otimes\overline{A}^{\otimes n} \arrow[u] \arrow[r] & 1\otimes\overline{A}^{\otimes n+1} \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[u, dotted] & \cdots \arrow[u] & \cdots \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The corresponding spectral sequence collapses so that $E^2 = E^\infty$ and, as immediately seen, $\bigoplus_{p+q=n} E^2_{p,q} = \overline{C}^\lambda_n(A)$ and $d_2 = b$. □

PROPOSITION 3.0.2. *There is an exact triangle*

$$C^\lambda_\bullet(k) \to C^\lambda_\bullet(A) \to \overline{C}^\lambda_\bullet(A) \to C^\lambda_\bullet(k)[1] \ . \tag{3.2}$$

PROOF. Since by the proposition 3.0.1, C^λ and $\overline{CC}$ complexes are quasiisomorphic, the claim is just the formulation of the fact that, by the definition of $\overline{CC}$, we have the short exact sequence of complexes

$$0 \to CC_{\bullet}(k) \to CC_{\bullet}(A) \to \overline{CC}_{\bullet}(A) \to 0$$

□

REMARK 3.0.3. The above proposition could also be deduced from the Hochschild-Serre spectral sequence associated to the inclusion $\mathfrak{gl}(k) \to \mathfrak{gl}(A)$ with the E^2-term

$$E^2_{pq} = H_p(\mathfrak{gl}(A), \mathfrak{gl}(k)) \otimes H_q(\mathfrak{gl}(k))$$

which converges to $H_{p+q}(\mathfrak{gl}(A))$ and from theorem 4.0.2).

4. Relation to Lie algebra homology

Let us start by recalling some standard notions from Lie algebra (co-)homology.

DEFINITION 4.0.1. *Let $(\mathfrak{g}, d)$ be a DGLA. The standard Chevalley-Eilenberg complex of chains of $\mathfrak{g}$ with coefficients in the trivial module k has the form*

$$C_{\bullet}(\mathfrak{g}) = (\Lambda^{\bullet}\mathfrak{g}[1], \partial^{Lie} + d),$$

where the Lie boundary operator is defines by

$$\partial^{Lie}(X_1 \wedge \ldots \wedge X_n) = \sum_{i<j} c_{i,j}[X_i, X_j] \wedge X_1 \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_n.$$

Here $\hat{\text{ffl}}$ means that the corresponding element of the product should be omitted, and the sign rule is

$$c_{ij} = (-1)^{(|X_i|+1)\sum_{k<i}(|X_k|+1)+(|X_j|+1)\sum_{k<j,k\neq i}(|X_k|+1)}$$

Let $\mathfrak{h}$ be a DG-subalgebra of $\mathfrak{g}$ acting reductively on $\mathfrak{g}$. The complx of coinvariants of $C_{\bullet}(\mathfrak{g})$ with respect to the adjoint action of $\mathfrak{h}$ will be denoted by

$$C_{\bullet}(\mathfrak{g})_{\mathfrak{h}}$$

and

$$C_{\bullet}(\mathfrak{g}, \mathfrak{h}) = \Lambda(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$$

denotes the complex of relative chains.

For any DG algebra (A, δ) over k let $\mathfrak{gl}(A) = \varinjlim_n \mathfrak{gl}_n(A)$, where the imbeddings $\mathfrak{gl}_n(A) \hookrightarrow \mathfrak{gl}_{n+1}(A)$ are of the form

$$X \longrightarrow \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}.$$

$\mathfrak{gl}(A)$ can be thought of as the Lie algebra of $\mathbb{N} \times \mathbb{N}$-matrices $M_{\infty}(A)$ with finitely many non-zero coefficients in A. Note that $\mathfrak{gl}(k)$ is a DG Lie subalgebra of $\mathfrak{gl}(A)$. Theorem 4.0.2 below identifies the cyclic complex of A, resp. the relative cyclic complex of A, with the subcomplex of primitive elements of the DG coalgebra of coinvariants $C_{\bullet}(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}$, resp. $C_{\bullet}(\mathfrak{gl}(A), \mathfrak{gl}(k))$. Note that these complexes have naturally the structure of Hopf algebras, with the product induced by the diagonal block inclusion

$$\mathfrak{gl}(A) \times \mathfrak{gl}(A) \to \mathfrak{gl}(A) \oplus \mathfrak{gl}(A) \hookrightarrow \mathfrak{gl}(A).$$

It is associative since we work with coinvariants (resp. relative complex). The coproduct is induced by the diagonal map

$$\Delta : \mathfrak{gl}(A) \ni x \to x \oplus x \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(A)$$

using the canonical identification

$$\Lambda^\bullet(\mathfrak{gl}(A) \oplus \mathfrak{gl}(A)) \simeq \Lambda^\bullet(\mathfrak{gl}(A)) \otimes \Lambda^\bullet(\mathfrak{gl}(A)).$$

Let E^a_{pq} denote the elementary matrix with $\left(E^a_{pq}\right)_{pq} = a$ and other entries equal to zero.

THEOREM 4.0.2. *The map*

$$\begin{array}{rcl} A^{\otimes p+1} & \to & \bigwedge^{p+1} \mathfrak{gl}(A) \\ a_0 \otimes \cdots \otimes a_p & \mapsto & E^{a_0}_{01} \wedge E^{a_1}_{12} \wedge \cdots \wedge E^{a_p}_{p-1,0} \end{array}$$

induces isomorphisms of complexes

$$\begin{array}{rcl} C^\lambda_\bullet(A) & \to & \operatorname{Prim} C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}[1] \\ \overline{C}^\lambda_\bullet(A) & \to & \operatorname{Prim} C_\bullet(\mathfrak{gl}(A), \mathfrak{gl}(k))[1] \end{array}$$

SKETCH OF THE PROOF. The basic part of the proof is the identification of the primitive part of the, say, complex

$$C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}.$$

$\mathfrak{gl}(A)$ acts reductively on $\mathfrak{gl}(A)$ and, by basic invariant theory,

$$(M_\infty(k)^{\otimes n})_{\mathfrak{gl}(k)} = k[\Sigma_n].$$

Let σ denote the sign representation of the symmetric group Σ_n. Then

$$\Lambda^n(\mathfrak{gl}(k) \otimes A)_{\mathfrak{gl}(k)} = ((\mathfrak{gl}(k) \otimes A)^n \otimes_{\Sigma_n} \sigma)_{\mathfrak{gl}(k)} = ((\mathfrak{gl}(k)^n_{\mathfrak{gl}(k)} \otimes A)^n \otimes_{\Sigma_n} \sigma) = (k[\Sigma_n] \otimes A^{\otimes n}) \otimes_{k[\Sigma_n]} \sigma.$$

In the left-most term, the symmetric group acts on itself by conjugation. It is now an exercise to check that action of the coproduct on the terms on the right hand side translates into an expression of the form

$$\pi \otimes (a_1 \otimes a_2 \otimes \ldots a_n) \to \sum (\pi|_I \otimes a_I) \otimes (\pi|_J \otimes a_J),$$

where the sum is over all partitions $\{1, 2, \ldots, n\} = I \cup J$ which are invariant under the action of the permutation π and, if

$$I = \{i_1, \ldots i_k\} \subset \{1, 2, \ldots, n\}$$

then

$$a_I = a_{\pi(i_1)} \otimes \ldots \otimes a_{\pi(k)}.$$

In particular, the primitive part of

$$(k[\Sigma_n] \otimes A^{\otimes n}) \otimes_{k[\Sigma_n]} \sigma$$

is given by the conjugacy class of the cyclic permutation $\tau \in \Sigma_n$, i.e.

$$\operatorname{Prim} C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}[1] = A^{\otimes n}/(1-\tau) = C^\lambda_{n-1}$$

Let # denote the trace map

$$(M_\infty(k) \otimes A)^{\otimes \bullet} \to A^{\otimes \bullet}$$

given by

$$(T_1 \otimes a_1) \otimes (T_2 \otimes a_2) \otimes \ldots \otimes (T_n \otimes a_n) \mapsto Tr(T_1 T_2 \ldots T_n) a_1 \otimes a_2 \otimes \ldots \otimes a_n.$$

One checks that # implements the above identification and intertwines the boundary maps completing the proof.

The complete proof can be found in [?], [?], [?] □

5. The connecting morphism

Here we give an explicit formula for the connecting morphism ∂ of the exact triangle from Proposition 3.0.2. Suppose that (A, δ) is a DGA. Let j be a k-linear map $A \to k$ satisfying $j(1) = 1$. The splitting j induces the splitting

$$\boldsymbol{\rho} : \mathfrak{gl}(A) \to \mathfrak{gl}(k)$$

of the inclusion $\mathfrak{gl}(k) \hookrightarrow \mathfrak{gl}(A)$. We set *the curvature of* $\boldsymbol{\rho}$ to be equal to

(5.1)
$$R(\boldsymbol{\rho}) = (\partial^{\mathrm{Lie}} + \delta)\boldsymbol{\rho} + \frac{1}{2}[\boldsymbol{\rho}, \boldsymbol{\rho}] \in Hom_k(\mathfrak{gl}(A)[1], \mathfrak{gl}(k)) \oplus Hom_k(\Lambda^2 \mathfrak{gl}(A)[1], \mathfrak{gl}(k))$$

Let P_n denote the invariant polynomial $X \mapsto \dfrac{1}{n!} tr(X^n)$ on $\mathfrak{gl}(k)$. Set

$$c_n = P_n(R(\boldsymbol{\rho})).$$

Then $c_n \in C^{2n}_{Lie}(\mathfrak{gl}(A); \mathfrak{gl}(k)))$ is a relative Lie algebra cocycle and, by the theorem 4.0.2, defines a linear map

$$ch_n(\boldsymbol{\rho}) : \overline{C}^{\lambda}_{2n+1}(A) \to k \tag{5.2}$$

which descends to homology.We will set

$$1^{(n+1)} = n!(n+1)! \cdot 1^{\otimes 2n+1} \in C^{\lambda}_{2n}(k) \tag{5.3}$$

Proposition 5.0.1. *The morphism* $\mathrm{Br}^A : \overline{C}^{\lambda}_{\bullet}(A) \to C^{\lambda}_{\bullet}(k)[1]$ *given by*

$$Br^A = \sum_n ch_n(\boldsymbol{\rho}) 1^{(n+1)}$$

represents the connecting morphism in the triangle (3.2).

5.1. Explicit formula for the product $HC_p \otimes HC_q \to HC_{p+q+1}$**.** Here we give an explicit formula for the product $\times$ from Theorem 1.0.5 in one of the realizations of the cyclic complex. Note first that, because of (2.2), the map N induces an isomorphism

$$C^{\lambda}(A) \simeq (\mathrm{Ker}(\mathrm{id} - \tau), b') \tag{5.4}$$

where in the right hand side $\mathrm{id} - \tau$ is considered as an operator on $A^{\otimes(\bullet+1)}$.

Proposition 5.1.1. *Suppose that A and C are unital algebras. We will identify them as aubalgebras of $A \otimes C$ using the imbeddings*

$$A \ni a \to a \otimes 1 \in A \otimes C \text{ and } C \ni c \to 1 \otimes c \in A \otimes C.$$

After identifiyng $C^{\lambda}_{\bullet}$ with the right hand side of (5.4), then the shuffle product

$$(a_0 \otimes \ldots \otimes a_p) \times (c_0 \otimes \ldots \otimes c_q) = \mathrm{sh}_{p+1,q+1}(a_0, \ldots a_p, c_0, \ldots, c_q)$$

is a morphism of complexes

$$\times : C^{\lambda}_{\bullet}(A) \otimes C^{\lambda}_{\bullet}(C) \to C^{\lambda}_{\bullet+1}(A \otimes C) \tag{5.5}$$

which induces on homology the $\times$ product from Theorem 1.0.5.

By the same construction one defines the product

$$\times : \overline{C}^{\lambda}_{\bullet}(A) \otimes \overline{C}^{\lambda}_{\bullet}(C) \to \overline{C}^{\lambda}_{\bullet}(A \otimes C)$$

on the reduced cyclic homology.

6. Adams operations

6.1. Euler decomposition. Suppose that $(\mathcal{H}, \mu, \Delta)$ is a Hopf algebra over a field k of characteristic zero, with the product μ, coproduct Δ, unit ϵ and the counit μ. The linear space

$$Hom_k(\mathcal{H}, \mathcal{H})$$

has an associative product given by the convolution:

$$f * g = \mu(f \otimes g)\Delta. \tag{6.1}$$

Assume now that $\mathcal{H}$ is graded commutative, with $\mathcal{H}_k = 0$ for $k < 0$ and $\mathcal{H}_0 = k$. For any

$$f \in Hom_k(\mathcal{H}, \mathcal{H}), \ |f| = 0, \ f(1) = 0,$$

f^{*s} vanishes on $\mathcal{H}_n$ for $n < s$. Moreover the composition $\epsilon \circ \mu$ is the unit of the graded ring $(Hom_k(\mathcal{H}, \mathcal{H}), *)$ and hence the series

$$e^{(1)}(f) = log(\epsilon \circ \mu + f) = \sum_{n>0} (-1)^{n+1} \frac{1}{n} f^{*n}$$

makes sense and is a degree zero endomorphism of $\mathcal{H}$.

Definition 6.1.1. *Suppose that $\mathcal{H}$ is a graded commutative Hopf algebra as above. We set*

$$e^{(k)} = \frac{1}{k!}(e^{(1)}(Id - \epsilon \circ \mu))^{*k}$$

and

$$e_n^{(k)} = e^{(k)}|_{\mathcal{H}_n}.$$

The following is the basic fact about the $e's$.

Proposition 6.1.2.
(1) $e_n^{(k)}$, $k = 1, \ldots, n$ *are pairwise othogonal idempotents;*
(2) $Id^{*k}|_{\mathcal{H}_n} = \sum_{i=1}^n k^i e_n^{(i)}$.

About the proof. The statement reduces to relatively straightforward identities relating the exponential and logarithmic power series and we refere for it to the original papers of Gerstenhaber, Schack and Loday.

□

6.2. λ operations. If A is a vector space, the (non-connected) tensor algebra

$$TA = \oplus_{n\geq 0} A^{\otimes n}$$

has a structure of a graded commutative Hopf algebra, with the product

$$\mu(a_1 \otimes \ldots \otimes a_p, a_{p+1} \otimes \ldots \otimes a_{p+q}) = \sum_{\text{pq shuffles } \sigma} sgn(\sigma) a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(p+q)}$$

and the coproduct

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} a_1 \otimes \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_n.$$

DEFINITION 6.2.1. *The λ_n operations on $A^{\otimes n}$ are defined by*

$$\lambda_n^k = (-1)^{k-1} Id^{*k}|_{A^{\otimes n}}.$$

The Adams operations are given by

$$\psi_n^k = (-1)^{k-1} k \lambda_n^k.$$

THEOREM 6.2.2. *Suppose that A is a commutative unital algebra. The λ operations descend to Hochschild homology.*

PROOF. Note first that

$$Id^{*k} = \mu^k \circ \Delta^k$$

where we, as usual, we use the notation

$$\Delta^k = (\Delta \otimes id^{\otimes k-1}) \circ (\Delta \otimes id^{\otimes k-2}) \circ \ldots \circ \Delta$$

and the similar (dual) definition of μ^k. The identity (0.5) now implies easily that

$$\lambda^k b = b \lambda^k.$$

□

The behaviour of cyclic complexes under the λ operations is controlled by the following result of combinatorial nature.

PROPOSITION 6.2.3. *Suppose that A is a commutative unital algebra. Then*

$$\lambda_n^k B = k B \lambda_{n-1}^k$$

and

$$B e_n^{(k)} = e_{n+1}^{(k+1)} B,$$

where $e_n^{(k)}$ are the Euler idempotents (see the definition (6.1.1)*).*

Since the proof is entirely combinatorial, we will omit it here (see f. ex.[**?**]). The following is the main (and easy) corollary.

THEOREM 6.2.4. *Suppose that A is a commutative unital algebra. The cyclic (negative) complex $CC_\bullet$ of A splits into a direct sum of subcomplexes:*

$$CC_n(A)^{(k)} = \bigoplus_i u^{-i} e_n^{(k-2i)} C_{n-2i}(A).$$

In particular, the cyclic homology of A has the decomposition

$$HC_n(A) = \bigoplus_k HC_n(A)^{(k)}$$

which is called the λ-decomposition.

CHAPTER 5

Other operations on cyclic complexes

1. Introduction

The motivation for the theorem below is the following. If $A = C^\infty(M)$ then $HC^-_\bullet(A)$ is isomorphic to the cohomology of the complex $(\Omega^\bullet(M)[[u]], ud)$; the cohomology of $\mathfrak{g}^\bullet_A$ is the graded Lie algebra $\mathfrak{g}^\bullet_M$ of multivector fields on M; one can define an action of $\mathfrak{g}^\bullet_M[\epsilon][u]$ on $HC^-_\bullet(A)$: for two multivector fields X, Y the action of $X + \epsilon Y$ is given by $L_X + \iota_Y$ where ι_Y is the contraction operator and $L_X = [d, \iota_X]$.

We would like to have a noncommutative analog of the above action. In fact, because of Theorem **??**, we know that there is an L_∞ action of $\mathfrak{g}^\bullet_A[\epsilon][u]$ on $CC^-_\bullet(A)$, i.e., a DGLA which is quasi-isomorphic to the former acts on a complex quasi-isomorphic to the latter. However, the proof of Theorem **??** is inexplicit, and we need explicit operations for index-theoretical applications. Below we provide an explicit formula for an action of the *complex* $U(\mathfrak{g}^\bullet_A[\epsilon])[u]$ on $CC^-_\bullet(A)$.

2. Action of the Lie algebra cochain complex of $C^\bullet(A; A)$

Here we recall, in a modified version, some results from [**?**]. Let us start by introducing some notation.

NOTATION 2.0.1. *Let A ba an associative, unital algebra.*

- *$\mathcal{E}_A$ denotes the differential graded algebra $(C^\bullet(A), \cup, \delta)$;*
- *$\mathfrak{g}_A$ denotes the differential graded Lie algebra $(C^{\bullet+1}(A), [\ ,\ ], \delta)$.*

Recall that, for a differential graded algebra $\mathcal{E}_A$, we constructed in the subsection 2 the following structures.

An A_∞ structure on $C_\bullet(C^\bullet(\mathcal{E}_A)))[[u]]$

$$m^{(1)}_n + um^{(2)}_n,\ n \in \mathbb{N}, \tag{2.1}$$

where $m^{(1)}_1 + um^{(2)}_1 = b + \delta + uB$.

An A_∞-module structure over $C_\bullet(C^\bullet(\mathcal{E}_A))[[u]]$ on $C_\bullet(\mathcal{E}_A)[[u]]$

$$\mu^{(1)}_n + u\mu^{(2)}_n,\ n \in \mathbb{N}. \tag{2.2}$$

DEFINITION 2.0.2.
Let A be a unital associative (differential graded) algebra.

(1) $\star : C_\bullet(\mathcal{E}_A)[[u]] \times C_\bullet(\mathcal{E}_A)[[u]] \to C_\bullet(\mathcal{E}_A)[[u]]$

denotes the binary operation (product) on $C_\bullet(\mathcal{E}_A)[[u]]$ given by restricting the corresponding binary operation

$$a \star b = (-1)^{|a|}(m^{(1)}_2(a, b) + um^{(2)}_2(a, b))$$

constructed on $C_\bullet(C^\bullet(\mathcal{E}_A))[[u]]$ *in the theorem 2.1.1 to the subspace*

$$C_\bullet(\mathcal{E}_A)[[u]] = C_\bullet(C^0(\mathcal{E}_A))[[u]] \subset C_\bullet(C^\bullet(\mathcal{E}_A))[[u]].$$

(2) $\diamond : C_\bullet(A)[[u]] \times C_\bullet(\mathcal{E}_A)[[u]] \to CC_\bullet(A)[[u]]$

denotes the binary pairing part of the A_∞*-module structure of* $C_\bullet(\mathcal{E}_A)[[u]]$ *over* $C_\bullet(C^\bullet(\mathcal{E}_A))[[u]]$, $\mu_2^{(1)} + u\mu_2^{(2)}$, *constructed in the theorem 2.1.2, restricted to*

$$C_\bullet(A)[[u]] \times C_\bullet(\mathcal{E}_A)[[u]] = C_\bullet({\mathcal{E}_A}^0)[[u]] \times C_\bullet(C^0(\mathcal{E}_A))[[u]].$$

Definition 2.0.3. *Set*

$$\mathfrak{g}_A[\epsilon] = \mathfrak{g}_A + \mathfrak{g}_A\epsilon,\ \ \epsilon^2 = 0,\ \ |\epsilon| = 1.$$

Let $U(\mathfrak{g}_A[\epsilon])$ *denote the universal enveloping algebra of* $\mathfrak{g}_A[\epsilon]$ *and let* u *be a formal parameter of degree 2. We will give*

$$U(\mathfrak{g}_A[\epsilon])[u]$$

a differential graded algebra structure with the differential

$$u \cdot \frac{\partial}{\partial\epsilon} + \delta,$$

where δ *denotes the total differential in* $\mathfrak{g}_A$.

Theorem 2.0.4. *The formula*

$$(\epsilon D_1 \cdot \cdots \cdot \epsilon D_m)\bullet\alpha = (-1)^{|\alpha|\sum_{i=1}^m(|D_i|+1)}\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\epsilon_\sigma\alpha\diamond(D_{\sigma_1}\star(D_{\sigma_2}\star(\ldots\star D_{\sigma_m}))\ldots)$$

defines morphisms of complexes of $k[u]$*-modules*

$$\begin{array}{rcl} U(\mathfrak{g}_A[\epsilon])[[u]] \otimes_{U(\mathfrak{g}_A)[u]} CC^-_\bullet(A) & \to & CC^-_\bullet(A) \\ U(\mathfrak{g}_A[\epsilon])[u^{-1}, u]] \otimes_{U(\mathfrak{g}_A)[u,u^{-1}]} CC^{per}_\bullet(A) & \to & CC^{per}_\bullet(A) \end{array}$$

Here $D \in \mathfrak{g}_A$ *acts on the* $CC^{per/\text{-}}_\bullet(A)$ *complexes as the* L_D *operation (see the formula* (7.0.7)*) and the signs* ϵ_σ *are computed according to the rule under which the parity of any* D_i *is* $|D_i| + 1$. *We will also use the notation* $I(D_1, \ldots, D_m)\alpha$ *for the left hand side of the above equation.*

Proof. The proof follows immediately from the fact that, in the A_∞ structures constructed in the theorems 2.1.1 and 2.1.2, the total boundary map commutes with the total binary product structure and

$$[L_D, I(D_1, \ldots, D_m)] = \sum_i (-1)^{(|D|+1)(\sum_{k<i}|D_k|+1)} I(D_1, \ldots, [D, D_i], D_{i+1}, \ldots, D_m).$$

$\square$

The following observation will be useful later

Corollary 2.0.5. *Let* τ *be a tracial functional on an algebra* A *and suppose that*

$$\delta_1, \ldots, \delta_n$$

is a family of commuting derivations of A *satisfying*

$$\tau \circ \delta_i = 0,\ \ i = 1, \ldots, n$$

Then

$$a_0 \otimes \ldots \otimes a_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sgn(\sigma)\tau(a_0 \delta_{\sigma(1)}(a_1) \ldots \delta_{\sigma(n)}(a_n))$$

defines a cyclic cocycle on A.

PROOF. Let U_+ denote the ideal generated by the derivations $\{\delta_i\}_{i=1,\ldots,n}$ in $U(\mathfrak{g}_A[\epsilon])[[u]]$. Under our assumptions,

$$[I(\delta_1, \ldots, \delta_n), b + uB] \in U_+,$$

hence, since τ vanishes on U_+, $\tau \circ I(\delta_1, \ldots, \delta_n)$ is a cyclic cocycle. It is easy to check that

$$I(\delta_1, \ldots, \delta_n)(a_0 \otimes \ldots \otimes a_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sgn(\sigma) a_0 \delta_{\sigma(1)}(a_1) \ldots \delta_{\sigma(n)}(a_n),$$

hence the claimed result holds. □

COROLLARY 2.0.6. *Suppose that ∇ is an odd element of $\mathfrak{g}_A^\bullet$ satisfying the Maurer-Cartan equation*

$$\delta\nabla + \frac{1}{2}[\nabla, \nabla] = 0. \tag{2.3}$$

Set

$$\chi(\nabla) = I(\exp(\frac{\nabla\epsilon}{u})) : C_\bullet(A)[u^{-1}, u] \to C_\bullet(A)[u^{-1}, u]]. \tag{2.4}$$

Then

$$[b + uB, \chi(\nabla)] = L_\nabla \chi(\nabla). \tag{2.5}$$

PROOF. By the theorem 2.0.4, given $D_1, \ldots, D_n$ in $\mathfrak{g}_A$,

$$[b + uB, I(D_1\epsilon \wedge \ldots \wedge D_n\epsilon] = \sum_i \pm L_{D_i} I(D_1\epsilon \wedge \ldots \wedge \hat{D}_i \wedge \ldots \wedge D_n) + \sum_{i<j} \pm I([D_i, D_j]\epsilon \wedge D_1\epsilon \ldots \hat{D_i}\epsilon \ldots \hat{D_j}\epsilon \ldots \wedge D_n\epsilon)$$

where the hat $\hat{D}$ means that the corresponding term is omitted from the argument of I. The signs are explained in the statement of the theorem 2.1.1. The claimed identity follows immediately. □

The complex $(C_\bullet(A)[u^{-1}, u], b + uB)$ is contractible but, in many situations, $\chi(\nabla)$ extends to a large subcomplex of the periodic cyclic complex of A and gives interesting maps. Example are given by the following subsection.

3. Rigidity of periodic cyclic homology

3.1. Nilpotent extensions.

THEOREM 3.1.1. *(Goodwillie) Let (A, m) be an associative algebra over a ring k of characteristic zero and let I be a two-sided nilpotent ideal of A. The natural map $CC_\bullet^{\mathrm{per}}(A) \to CC_\bullet^{\mathrm{per}}(A/I)$ is a quasi-isomorphism.*

PROOF. Using exact sequences

$$0 \longrightarrow I^n/I^{n-1} \longrightarrow A/I^{n-1} \longrightarrow A/I^n \longrightarrow 0$$

the claim reduces to the case $I^2 = 0$. Fix a k-linear isomorphism

$$\phi : A \simeq A/I \oplus I$$

which reduces to identity modulo I. The pull back of the product from $A/I \oplus I$ by ϕ defines on A an associative product, say m_1. Set $\lambda = m - m_1$. This is a Maurer Cartan element of the Hochschild cohomological complex of (A, m)). since $I^2 = 0$, the infinite series $\chi(\lambda)$ converges an, by the corollary 2.0.6, provides an isomorphism of the cyclic periodic complexes of (A, m) and (A, m_1). But, by additivity of cyclic periodic homology,

$$CC^{per}_{\bullet}(A) \simeq CC^{per}_{\bullet}(A/I) \oplus CC^{per}_{\bullet}(I) \simeq CC^{per}_{\bullet}(A/I).$$

□

3.2. Completed Hochschild and cyclic complexes. Let A be an algebra and I an ideal in A. Each tensor power of A has a filtration

$$F^N_I(A^{\otimes(p+1)}) = \sum_{n_0+\ldots n_p \geq N+1} I^{n_0} \otimes \ldots \otimes I^{n_p}$$

The differential b preserves the filtration. We denote the induced filtration on Hochschild chains by $F^N_I C_p(A)$. Put

$$\widehat{C}_{\bullet}(A)_I = \varprojlim C_{\bullet}(A)/F^N_I C_{\bullet}(A)$$

$$\widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(A)_I = (\widehat{C}_{\bullet}(A)_I((u)), b + uB)$$

THEOREM 3.2.1. *Suppose that m_1 and m_2 are two associative products on A with the same unit. Suppose moreover that I is an ideal with respect to m_1 and $m_1(a_1, a_2) - m_2(a_1, a_2) \in I$ for all a_1, a_2 in A. Then there is a natural isomorphism of complexes*

$$\widehat{\mathrm{CC}}^{per}_{\bullet}(A, m_1)_I \simeq \widehat{\mathrm{CC}}^{per}_{\bullet}(A, m_2)_I$$

.

PROOF. Set $\lambda = m_1 - m_2$. Then $\chi(\lambda)$ the infinite series converges $\chi(\lambda)$ and produces a quasiisomorphism of the respective cyclic periodic complexes. □

THEOREM 3.2.2. *The projection*

$$\widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(A)_I \to \mathrm{CC}^{\mathrm{per}}_{\bullet}(A/I)$$

is a quasi-isomorphism.

PROOF. Choose a linear section $A/I \to A$ of the projection. This allows to identify A with $A/I \times I$ as k-modules. Consider two products on A : the original one and the one coming from this identification, with the product on A/I being the product in the quotient algebra and the product on I being zero. These two products satisfy the conditions of Theorem 3.2.1. So we have to prove that the projection is a quasi-isomorphism for the second product, which follows from the Künneth formula and the fact that the algebra with zero multiplication has periodic cyclic homology equal to zero. □

Another corollary is the following.

COROLLARY 3.2.3. *Let (A, m) be an associative algebra, TA the tensor algebra over A and $J(A)$ the ideal in TA generated by*

$$\{a \otimes b - m(a, b) \mid a, b \in A\}.$$

Then

$$CC^{per}_{\bullet}(\overline{TA}^{J(A)}) \simeq CC^{per}_{\bullet}(A).$$

PROOF. The claim follows from the fact that, by the theorem **??**, for $n \geq m$ the quotient maps

$$T(A)/J(A)^n \to T(A)/J(A)^m$$

induce isomorphism on periodic cyclic homology. □

4. Construction of cyclic cocycles

Another example of an application of the corollary 2.0.6 is the following.

Suppose that A is $\mathbb{Z}/2\mathbb{Z}$-graded and D an odd element of A and τ is a tracial functional on A.

$$\gamma = -\mathrm{D}^2 + ad\,\mathrm{D}$$

is a Maurer-Cartan element in $\mathfrak{g}^{\bullet}_{A}$ and, for any formal power series f,

$$\tau \circ I(f(\gamma))$$

is a cocycle in the complex dual to $(CC_{\bullet}(A)[u^{-1}, u], b + uB)$. In certain cases it extends to a cocycle on a subcomplex of $(CC^{per}_{\bullet}(A), b + uB)$ big enough to get interesting information.

EXAMPLE 4.0.1. **The JLO cocycle.** The cocycle

$$\tau \circ I(\exp(\frac{\epsilon}{u}(-\mathrm{D}^2 + ad\,\mathrm{D})).$$

Using Duhamel formula for the exponential we get the an (infinite) even cyclic periodic cocycle with components

(4.1)

$$(a_0, \ldots, a_{2n}) \mapsto \int_{\Delta_{2n}} \tau(a_0 e^{-\lambda_0 \mathrm{D}^2}[\mathrm{D}, a_1] e^{-\lambda_1 \mathrm{D}^2} \ldots [\mathrm{D}, a_{2n}]) e^{-\lambda_{2n} \mathrm{D}^2} d\lambda_1 \ldots d\lambda_{2n}.$$

where Δ_k is the standard simplex

$$\Delta_k = \{(\lambda_0, \ldots . \lambda_k \mid \lambda_0 + \ldots + \lambda_k = 1;\, \lambda_i \geq 0, i = 0, \ldots, k.\}$$

In applications, the exponential factors $e^{-\lambda \mathrm{D}^2}$ regularize the expression under the integral sign so that the total cocycle can be evaluated on non-trivial classes in the periodic cyclic homology of A

5. The characteristic map

Let u be a formal parameter of degree two. Consider the differential graded algebra $A[\eta] = A + A\eta$, $\deg\eta = -1$, $\eta^2 = 0$ with the differential $\frac{\partial}{\partial \eta}$. Consider the complex $\overline{C}^{\lambda}_{\bullet}(A[\eta])[u]$ with the differential $\frac{\partial}{\partial \eta} + u \cdot b$.

THEOREM 5.0.1. *There exist natural pairings of $k[[u]]$-modules*

$$\bullet : \overline{C}^{\lambda}_{\bullet-1}(A[\eta])[[u]] \otimes CC^{-}_{-\bullet}(A) \to CC^{-}_{-\bullet}(A)$$

$$\bullet : \overline{C}^{\lambda}_{\bullet-1}(A[\eta])[[u, u^{-1}] \otimes CC^{per}_{-\bullet}(A) \to CC^{per}_{-\bullet}(A)$$

such that:

(1) $\eta^{\otimes m}\bullet ? = \frac{1}{(m-1)!}Id$ *for* $m>0$.
(2) *For* $x_i \in A$ *the operation* $(x_1\otimes\cdots\otimes x_p)\bullet ?$ *sends* $C_N(A)$ *to* $\sum_{i,j\geq 0} C_{N-p+i}(A)u^j$.
(3) *The component of* $(x_1\otimes\cdots\otimes x_p)\bullet(a_0\otimes\cdots\otimes a_N)$ *in* $C_{N-p}[[u]]$ *is equal to*

$$\frac{1}{p!}\sum_{i=1}^{p}(-1)^{i(p-1)}a_0[x_{i+1},a_1][x_{i+2},a_2]\cdots[x_i,a_p]\otimes a_{p+1}\otimes\cdots\otimes a_N$$

(4) $(x_1\otimes\cdots\otimes x_p)\bullet 1 = \sum_{i=1}^{p}(-1)^{i(p-1)}1\otimes x_{i+1}\otimes x_{i+2}\otimes\cdots\otimes x_i$.

Proof. First note that we have a natural the morphism of DGLA:

$$\mathfrak{gl}(A[\eta])\to\mathfrak{gl}(A)\eta\oplus\mathfrak{gl}(A)/k\hookrightarrow C^{\bullet}(M_\infty(A)),$$

where $\mathfrak{gl}(A)\eta$ is identified with $C^0(M_\infty(A))$ and

$$\mathfrak{gl}(A)/k = Im(\delta|_{C^0(M_\infty(A))})\subset C^1(M_\infty(A)).$$

The theorem 2.0.4 provides us with a morphism of complexes:

$$\bullet : U(\mathfrak{gl}(A[\eta,\epsilon]))[[u]]\otimes_{U(\mathfrak{gl}(k))}CC^-_\bullet(M_\infty(A))\to CC^-_\bullet(M_\infty(A)) \tag{5.1}$$

(in the negative cyclic case). Let $M(A)$ denote the algebra of $\mathbb{N}\times\mathbb{N}$-matrices with entries in A and only finitely non-zero diagonals. Set

$$\iota : A\ni a\to a\cdot 1\in M(A)$$

and

$$\# : M_\infty(A)=M_\infty(\mathbb{C})\otimes A\ni T\otimes a\to Tr(T)a\in A. \tag{5.2}$$

It is easy to check that the composition

$$\#\bullet(id\otimes\iota)$$

is well defined and induces a morphism of complexes

$$k\otimes_{\mathfrak{gl}(k[\eta])}U(\mathfrak{gl}(A[\eta,\epsilon]))[[u]]\otimes_{\mathfrak{gl}(k[\epsilon])}k\to End(CC^-_\bullet(A)).$$

A composition of morphisms of complexes:

$$\overline{C}^\lambda_\bullet(A[\eta])[[u]]\to C^{Lie}_\bullet(\mathfrak{gl}(A[\eta]),\mathfrak{gl}(k);k)[[u]]\to k\otimes_{\mathfrak{gl}(k[\eta])}U(\mathfrak{gl}(A[\eta,\epsilon]))[[u]]\otimes_{\mathfrak{gl}(k[\epsilon])}k$$

completes the costruction. Here the first morphism comes from the theorem 4.0.2 while the second one can be constructed using the observation that the quotient morphism

$$k\otimes_{\mathfrak{gl}(k[\eta])}U(\mathfrak{gl}(A[\eta,\epsilon]))\otimes_{\mathfrak{gl}(k[\epsilon])}k\to k\otimes_{\mathfrak{gl}(A[\eta])}U(\mathfrak{gl}(A[\eta,\epsilon]))\otimes_{\mathfrak{gl}(k[\epsilon])}k$$

is in fact a quasiisomorphism (this follows from the fact that $k[\eta]/k\to A[\eta]/k$ is a quis). □

Remark 5.0.2. Note that the formula (4) above defines the map from $\overline{C}^\lambda_\bullet(A)$ to $Ker(B : C_\bullet(A)\to C_{\bullet+1}(A))$ (one can show that this map is a quasi-isomorphism). Clearly, the kernel above embeds into $CC^-_\bullet(A)$. The above theorem shows that this embedding extends to a pairing $\bullet$.

The complex $\overline{C}^\lambda_{\bullet-1}(A[\eta])[[u]]$ is very simple at the level of homology; it is quasi-isomorphic to $k[[u]]$. Therefore the pairing $\bullet$ does not define any new homological operations. It is, however, very important at the level of chains, as one sees in [**53**]. To give some applications, let us suppose that τ is a trace on the algebra A and hence the operations on the cyclic periodic complex constructed in the theorem 5.0.1 produce a map $\#\tau = (\tau \otimes id) \circ \bullet$:

$$\#\tau : \overline{C}^{\lambda}_{\bullet}(A[\eta])[[u]] \to HC^{\bullet}_{per}(A).$$

A few of the examples:

(1) $\#\tau(\eta^{\otimes m}) = \frac{1}{(m-1)!}\tau$;

(2) Suppose that $\sum x_1 \otimes \cdots \otimes x_p$ is a reduced cyclic cycle. then

$$\#\tau(\sum x_1 \otimes \cdots \otimes x_p)(a_0, \dots, a_p) = \frac{1}{p!} \sum \sum_{i=1}^{p} (-1)^{i(p-1)} \tau(a_0[x_{i+1}, a_1][x_{i+2}, a_2] \cdots [x_i, a_p])$$

is a cyclic periodic cocycle in the same class a a multiple of τ determined by the class of $\sum x_1 \otimes \cdots \otimes x_p$.

(3) Suppose that F is an odd element of A satisfying $F^2 = 1$. Then $F^{\wedge(n+1)}$ is a reduced cyclic cycle and

$$\#\tau(F^{\wedge(n+1)})(a_0, \dots, a_n) = \tau(Fa_0[F, a_1] \dots [F, a_n])$$

is a cyclic cocycle representing τ in the periodic cyclic cohomology.

REMARK 5.0.3. Let $\mathcal{A}$ be a graded associative algebra. $\mathcal{A}[1] \oplus \mathrm{Der}(\mathcal{A})$ is a (DG-) Lie subalgebra of $\mathfrak{g}^{\bullet}_{A}$. Let $\mathcal{L}$ be a DG Lie subalgebra of $\mathcal{A}[1] \oplus \mathrm{Der}(\mathcal{A})$ and K a graded space on which $\mathcal{L}$ acts so that elements of $\mathcal{A}[1]$ act by zero. Let $\tau : \mathcal{A} \to K$ is an $\mathcal{L}$-equivariant trace. We extend it by zero to the entire Hochschild complex $C_{-\bullet}(\mathcal{A})$. Given he $X_1, \dots, X_n \in \mathcal{L}$, and for a Hochschild chain c, we set

$$\chi(X_1, \dots, X_n)(c) = \sum_{\sigma \in S_n} \pm \tau(\iota_{X_{\sigma(1)}} \dots \iota_{X_{\sigma(n)}} c); \tag{5.3}$$

The operations ι_D are the ones from the definition 5.0.1 and the sign is computed as follows: a permutation of X_i and X_j introduces a sign $(-1)^{(|X_i|+1)(|X_j|+1)}$.

PROPOSITION 5.0.4. *(cf. [?], [?]) χ defines a cocycle of the complex*

$$C^{\bullet}(\mathcal{L}, \mathrm{Hom}(C_{-\bullet}(A), K)[[u]]$$

with the differential $b + uB + \delta + u\partial_{\mathrm{Lie}}$; the action of L on $\mathrm{Hom}(C_{-\bullet}, K)$ is induced by the action on K. In other words,

$$\chi(X_1, \dots, X_n)((b + uB)(c)) = \frac{1}{n!}(\sum \pm \chi(X_1, \dots, \delta X_i, \dots, X_n) +$$
$$u \sum \pm \chi([X_i, X_j], \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_n) +$$
$$u \sum \pm X_i \chi(X_1 \dots, \widehat{X_i}, \dots, X_n))(c)$$

.

PROOF. We observe directly that $[b, \iota_X] = \iota_{\delta X}$. This implies immediately that the formula in the statement of the Proposition is true modulo u, since τ is zero on the image of b. We denote $\chi(X_1, \dots, X_n)$ by $\chi(X_1 \dots X_n)$. Let $c = a_0 \otimes \dots \otimes a_n$. Let D be an odd derivation and x an even element of $\mathcal{A}$. By definition,

$$\chi(D^n x^N)(c) = \frac{n!N!}{(N+n)!} \sum_{N_0+N_1+\dots+N_n=N} \tau(a_0 x^{N_0} D(a_1) x^{N_1} \dots a_n x^{N_n})$$

therefore

$$\chi(D^{n+1} x^N)(Bc) = \frac{(n+1)!N!}{(N+n+1)!} \times$$

$$\times \sum_{\sum_0^{n+1} N_k=N} \sum_{i=0}^{n} \pm\tau(x^{N_0}D(a_{i+1})x^{N_1}D(a_{i+2})x^{N_2}\dots D(a_i)x^{N_{n+1}});$$

using the fact that τ is a trace and re-indexing the N_k, we see that the latter is equal to

$$\frac{(n+1)!N!}{(N+n+1)!}\sum_{\sum_0^n N_k=N}\sum_{i=0}^{n}\sum_{N_i'+N_i''=N_i} \pm\tau(D(a_0)x^{N_0}\dots D(a_n)x^{N_n}) =$$

$$(N+n+1)\frac{(n+1)!N!}{(N+n+1)!}\sum_{\sum_0^n N_k=N} \pm\tau(D(a_0)x^{N_0}\dots D(a_n)x^{N_n}) =$$

$$\frac{(n+1)!N!}{(N+n)!}(\sum_{\sum_0^n N_k=N} \pm D\tau(a_0x^{N_0}\dots D(a_n)x^{N_n})+$$

$$+\sum_{\sum_0^n N_k=N}\sum_{i=1}^{n}\pm\tau(a_0x^{N_0}\dots[D,D](a_i)x^{N_i}\dots D(a_n)x^{N_n})+$$

$$+\sum_{\sum_0^n N_k=N}\sum_{i=0}^{n}\sum_{N_i'+N_i''=N_i} \pm\tau(a_0x^{N_0}\dots x^{N_i'}D(x)x^{N_i''}\dots D(a_n)x^{N_n})) =$$

$$(n+1)D\chi(D^nx^N)(c)+\frac{n(n+1)}{2}\chi([D,D]D^{n-1}x^N)(c)+$$
$$(n+1)N\chi(D^nD(x)x^{N-1}(c).$$

We see that

$$\chi(D^{n+1}x^N)(Bc) = (n+1)D\chi(D^nx^N)(c)+$$
$$\frac{n(n+1)}{2}\chi([D,D]D^{n-1}x^N)(c)+(n+1)N\chi(D^nD(x)x^{N-1})(c).$$

This is exactly the equality of the terms of the formula in the statement of the Proposition that contain u, in the special case $X_1 = \dots = X_{n+1} = D$ and $X_{n+2} = \dots = X_{N+n+1} = x$ (and with n replaced by $N+n$). To prove the general case, tensor our algebra by $k[t_1,\dots,t_n]$ where $|t_i| = -|X_i|+1$, put $X = t_1X_1+\dots t_nX_n$, apply the special case to $\chi(X,\dots,X)$, and look at the coefficient at $t_1\dots t_n$. $\square$

6. Excision in periodic cyclic homology

Recall the following notion of smoothness for non-commutative algebras, due to Cuntz and Quillen (see [**?**]).

Definition 6.0.1. A unital algebra A is called quasi-free if $H^2(A,M) = 0$ for all A bimodules M.

Remark 6.0.2. For future reference note that free algebras are quasifree.

Proposition 6.0.3. *Suppose that A is a unital quasifree algebra.*

- *$\Omega^1(A)$, the kernel of the multiplication map $A\otimes A \to A$, is a projective A-bimodule;*
- *A is hereditary, i. e. any right (resp. left) submodule of a projective A-module is right (resp. left) projective.*

PROOF. To prove the first statement, look at the exact sequence of A-bimodules:

$$0 \longrightarrow \Omega^1(A) \longrightarrow A \otimes_k A \longrightarrow A \longrightarrow 0. \tag{6.1}$$

Since $A \otimes_k A$ is free A-bimodule, applying the functor $\mathbb{R}Hom^{A^e}(\cdot, M)$ to it shows that

$$Ext^i_{A^e}(\Omega^1(A), M) = Ext^{i+1}_{A^e}(A, M).$$

Since A is quasifree, $Ext^1_{A^e}(\Omega^1(A), M) = 0$ for every A-bimodule M and $\Omega^1(A)$ is projective as an A-bimodule.

For the second claim, suppose that M is a, say, left A module. Tensoring the split exact sequence 6.1 with M from the right, we get an exact sequence of left A-modules

$$0 \longrightarrow \Omega^1(A) \otimes_A M \longrightarrow A \otimes_k M \longrightarrow M \longrightarrow 0. \tag{6.2}$$

Since $\Omega^1(A)$ is projective as an A-bimodule, $\Omega^1(A) \otimes_A M$ is projective as a left A-module and the claim follows. □

Let us start by defining H-unitality in the context of pro-algebras.

DEFINITION 6.0.4. *Let A be an algebra over a field of characteristic zero and let $\cdots \subset A^n \subset A^{n-1} \subset \cdots \subset A^2 \subset A$ be the filtration of A by its increasing powers. A is* approximately H-unital *if the complex*

$$(\varinjlim_k C_\bullet(A^k), b')$$

is acyclic. In the following we set $C_\bullet(A^\infty) = \varinjlim_k C_\bullet(A^k))$.

The following lemma gives a useful criterion for approximate H-unitality.

LEMMA 6.0.5. *Let A be an algebra and let us denote by $m : A \otimes A \to A$ the multiplication map on A. Assume that there is a $k \geq 1$ and a left A-module map map $\phi : A^k \to A \otimes A$ which is a section of m, i. e.*

$$m \circ \phi(x) = x \text{ for all } x \in A^k.$$

Then A is approximately H-unital.

PROOF. For every q there is a $p \geq q$ and a A-linear splitting $\psi : A^p \to A^q \otimes A^q$ of the multiplication $m : A^q \otimes A^q \to A^{2q}. \supset J^p$ which is obtained by iterating ϕ and then multiplying the last q variables together. Set

$$\Psi(a_0, \ldots, a_n) = (a_0, \ldots, \psi(a_n)).$$

Ψ is a contracting homotopy of the complex $(C_\bullet(A^\infty), b^{'})$. □

COROLLARY 6.0.6. *A left ideal J in a unital quasi-free algebra P is approximately H-unital.*

PROOF. By the second clause in the proposition 6.0.3, J is a projective left R-module and hence there exists an R-linear lift $\phi : J \to R \otimes J$ for the multiplication map $m : R \otimes J \to J$. The restriction of ϕ to J^2 satisfies the conditions of the above lemma. □

The basic property of the approximate H-unitality is the following result.

THEOREM 6.0.7. *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be an exact sequence of algebras with A unital and J approximately H-unital. Set

$$K^n_\bullet = \; Ker \; \left\{ CC^{per}_\bullet(A) \overset{\pi_n}{\to} CC^{per}_\bullet(A/J^n) \right\},$$

where π_n is induced by the quotient map $A \to A/J^n$. The morphism of complexes

$$\mathbb{R} \varprojlim_n CC^{per}_\bullet(J^n) \to \mathbb{R} \varprojlim_n K^n_\bullet$$

induced by the inclusion $CC^{per}_\bullet(J^n) \to K^n_\bullet$ is a quasi-isomorphism.

PROOF. Set

$$\tilde{K}^n_\bullet = \; \mathrm{Ker} \; \left\{ (C_\bullet(A), b) \overset{\pi_n}{\to} (C_\bullet(A/J^n), b) \right\},$$

Using the approximate H-unitality of it is straightforward to adapt the proof of the theorem 2.0.2 to the proof of the following statement.

LEMMA 6.0.8. *Given k and n, there exists an $m > n$ such that the following holds.*

$$Im \left\{ H_k(\tilde{K}^m, b) \to H_k(\tilde{K}^n, b) \right\} \subset Im \left\{ H_k((C_\bullet(J^n), b) \to H_k(\tilde{K}^n, b) \right\}$$

COROLLARY 6.0.9.

$$\mathbb{R} \varprojlim_n Cone \left\{ C_\bullet(J^n) \to (\tilde{K}^n_\bullet, b) \right\} = 0.$$

PROOF. Let C^* denote the cone of the morphism $\left\{ C_\bullet(J^n) \to (\tilde{K}^n_\bullet, b) \right\}$. The statement of the corollary is equivalent to the acyclicity of the following complex:

$$\begin{tikzcd}
\ldots \arrow[r] & C^{*+1}_\bullet \arrow[r] \arrow[d, "="] \arrow[dr] & C^*_\bullet \arrow[r] \arrow[d, "="] & \ldots\ldots \arrow[r] & C^3_\bullet \arrow[r] \arrow[d, "="] \arrow[dr] & C^2_\bullet \arrow[r] \arrow[d, "="] \arrow[dr] & C^1_\bullet \arrow[d, "="] \\
\ldots \arrow[r] & C^{*+1}_\bullet \arrow[r] & C^*_\bullet \arrow[r] & \ldots\ldots \arrow[r] & C^3_\bullet \arrow[r] & C^2_\bullet \arrow[r] & C^1_\bullet.
\end{tikzcd}$$

But this follows from the above lemma by a straightforward diagram chasing. □

To complete the proof of the theorem, filter the mapping cone $\mathcal{C}$ of $\mathbb{R} \varprojlim_n CC^{per}_\bullet(J^n) \to \mathbb{R} \varprojlim_n K^n_\bullet$ by the powers of u. The associated spectral sequence has the E^1-term equal to zero, hence $\mathcal{C}$ is quasiisomorphic to zero. □

As the corollary, we get the following result.

THEOREM 6.0.10. *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be an exact sequence of algebras with A unital and J approximately H-unital. Then

$$CC^{per}_\bullet(J) \to CC^{per}_\bullet(A) \to CC^{per}_\bullet(A/J)$$

is an exact triangle in the derived category of complexes of vector spaces.

PROOF. By the theorem 6.0.7, the following triangle is exact:

$$\mathbb{R}\varprojlim_n CC_\bullet^{per}(J^n) \to \varprojlim_n CC_\bullet^{per}(A) \to \varprojlim_n CC_\bullet^{per}(A/J^n).$$

Using Goodwilli theorem 3.1.1,

$$\varprojlim_n CC_\bullet^{per}(A/J^n) \simeq CC_\bullet^{per}(A/J) \text{ and } \varprojlim_n CC_\bullet^{per}(J^n) \simeq CC_\bullet^{per}(J).$$

The claimed result follows. □

THEOREM 6.0.11. *(Cuntz-Quillen) Suppose that*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0\ .$$

is a short exact sequence of algebras over a field k of characteristic cero. Then there is an induced exact triangle

$$\begin{tikzcd} HC_\bullet^{per}(I) \arrow[rr, "\iota_\bullet"] & & HC_\bullet^{per}(A) \arrow[dl, "\pi_\bullet"] \\ & HC_\bullet^{per}(A/I) \arrow[ul, "{[1]}"] & \end{tikzcd} \quad . \tag{6.3}$$

PROOF. Let $T(A)$ denote the unital tensor algebra of A and let $I(A)$ denote the kernel of the natural quotient map $T(A) \to A$. and let $I(A/J))$ denote the kernel of the composition $T(A) \to A \to A/J$. We have the following commuting diagram of short exact sequences.

$$\begin{tikzcd} I(A) \arrow[d, hook] \arrow[dr, hook] & & \\ I(A/J) \arrow[r, hook] \arrow[d] & T(A) \arrow[r, two heads] \arrow[dr, two heads] & A/J \\ J & & A \end{tikzcd}$$

Since both $I(A)$ and $I(A/J)$ are ideals in the free algebra $T(A)$, the theorem 6.0.10 produces the following commuting diagram of morphisms of complexes with rows given by exact triangles.

$$\begin{tikzcd} CC_\bullet^{per}(J) \arrow[r, "{[1]}"] \arrow[d] & CC_\bullet^{per}(I(A)) \arrow[r] \arrow[d, "="] & CC_\bullet^{per}(I(A/J)) \arrow[d] \\ CC_\bullet^{per}(A) \arrow[r, "{[1]}"] \arrow[d] & CC_\bullet^{per}(I(A)) \arrow[r] \arrow[d] & CC_\bullet^{per}(T(A)) \arrow[d, "="] \\ CC_\bullet^{per}(A/J) \arrow[r, "{[1]}"] & CC_\bullet^{per}(I(A/J)) \arrow[r] & CC_\bullet^{per}(T(A)) \end{tikzcd}$$

It follows immediately that the rightmost column is an exact triangle, and the claim of the theorem is the long exact homology sequence associated to it. □

7. Bibliographical notes

CHAPTER 6

Cyclic objects

1. Introduction

2. The simplicial and cyclic categories

For any monoid X with a neutral element 1 one can define the following operations on powers of X.

1) $d_j : X^{n+1} \to X^n$, $n \geq 0$, $0 \leq j < n$;
$d_j(x_0, \dots, x_n) = (x_0, \dots, x_j x_{j+1}, \dots, x_n)$
2) $d_n : X^{n+1} \to X^n$, $n \geq 0$;
$d_n(x_0, \dots, x_n) = (x_n x_0, x_1, \dots, x_{n-1})$
3) $s_j : X^n \to X^{n+1}$, $n \geq 0$, $0 \leq j \leq n$;
$s_j(x_0, \dots, x_n) = (x_0, \dots, x_j, 1, \dots, x_n)$

By Δ^{op} we denote the category with objects $[n]$, $n \geq 0$, whose set of morphisms $\Delta^{\mathrm{op}}([n], [m])$ is the set of natural operations $X^{n+1} \to X^{m+1}$ that can be obtained from d_j and s_j by composing them.

Define also
4) $t : X^{n+1} \to X^{n+1}, n \geq 0$; $t(x_0, x_1, \dots, x_n) = (x_n, x_0, \dots, x_{n-1})$

By Λ we denote the category with objects $[n]$, $n \geq 0$, whose set of morphisms $\Lambda([n], [m])$ is the set of natural operations $X^{n+1} \to X^{m+1}$ that can be obtained from d_j, s_j, and t by composing them.

2.1. Categories Λ_ℓ. More generally, let ℓ be a natural number or infinity. Consider all monoids X with a neutral element 1 and an automorphism α such that $\alpha^\ell = \mathrm{id}_X$. Define natural operations s_j, d_j, and t as in 1)-4) above but with 2) and 4) replaced by
2') $d_n : X^{n+1} \to X^n$, $n \geq 0$;
$d_n(x_0, \dots, x_n) = (\alpha(x_n) x_0, x_1, \dots, x_{n-1})$
and
4') $t : X^{n+1} \to X^{n+1}, n \geq 0$; $t(x_0, \dots, x_n) = (\alpha(x_n), x_0, \dots, x_{n-1})$ Let Δ^{op} and Λ_ℓ be the categories defined as above but with d_j, s_j, and t being as in 1), 2'), 3), 4'). We keep the notation Δ^{op} because the new category is isomorphic to the one we had before. One has $\Lambda = \Lambda_1$.

More explicitly, $\Lambda([n], [m])$ is the set of maps $X^{n+1} \to X^{m+1}$ of the form

$$(x_0, \dots, x_m) \mapsto (x_{J_0}, \dots, x_{J_m}) \tag{2.1}$$

where each x_{J_q} is the product $x_j \dots x_p$ for some j and p so that any x_i enters exactly one x_{J_q} and the cyclic order of the factors x_j is preserved. The product of zero factors x_j is by definition equal to 1. More generally, $\Lambda_\ell([n], [m])$ is defined the same way but now $x_{J_q} = \widetilde{x}_j \widetilde{x}_p$ where, for some integer r, $\widetilde{x}_j = \alpha^{r+1}(x_j)$ if $\widetilde{x}_j$ is strictly to the right from $\widetilde{x}_0$ and $\widetilde{x}_j = \alpha^r(x_j)$ otherwise.

The subcategory Δ^{op} consists of those morphisms for which x_{J_0} contains x_0 as a factor.

EXAMPLE 2.1.1. $(x_0, \ldots, x_6) \mapsto (\alpha^2(x_5), \alpha^2(x_6)\alpha(x_0)\alpha(x_1), 1, \alpha(x_2)\alpha(x_3), 1, \alpha(x_4))$ defines a morphism in $\Lambda_\ell([6],[5])$ (but not in $\Delta^{\mathrm{op}}([6],[5])$).

Define also $\Delta'([n],[m])$ to be the set of all morphisms $[n] \to [m]$ for which the order of $x_0, \ldots, x_n$ is preserved. In other words, a morphism is in Δ' if, when we denote by r the smallest index for which $x_{J_r} \neq 1$, then $\widetilde{x}_0$ is the leftmost factor in x_{J_r}.

Let $C([n],[n])$ be the group of automorphisms of $[n]$ in Λ_ℓ. It is easy to see that it is generated by t. Define

$$\sigma = t^{n+1} \in \Lambda_\ell([n],[n]). \tag{2.2}$$

The following is clear from the above description.

LEMMA 2.1.2. *a)* $C([n],[n])$ *is the group* $\mathrm{Aut}_{\Lambda_\ell}([n])$; *it is cyclic of order* $\ell(n+1)$.

b) The elements σ *are central in* Λ_ℓ.

c) For any morphism $\lambda \in \Lambda_\ell([n],[m])$ *there exist unique* $c \in C([n],[n])$ *and* $\delta' \in \Delta'([n],[m])$ *such that*

$$\lambda = c\delta'.$$

d) For any morphism $\lambda \in \Lambda_\ell([n],[m])$ *there exist unique* $c \in C([n],[n])$ *and* $\delta' \in \Delta'([n],[m])$ *such that*

$$\lambda = \delta c.$$

3. Simplicial objects

Denote by Δ the category whose objects are sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$ with their standard order, and morphisms are nondecreasing maps. For a category C define a simplicial object of C an a functor from Δ^0 to C where Δ^{op} is the category opposite to Δ. Any morphism in Δ can be written as a composition of the following:

1. Embeddings $d_i : [n] \to [n+1]$, $0 \leq i \leq n+1$ (i is not in the image of d_i);
2. Surjections $s_i : [n+1] \to [n]$, $0 \leq i \leq n$ (i is the image of two succesive points under s_i).

A simplicial object of C can be equivalently described as a collection of objects X_n of C, $n \geq 0$, together with morphisms

$$d_i : X_{n+1} \to X_n,\ 0 \leq i \leq n+1 \tag{3.1}$$

$$s_i : X_n \to X_{n+1},\ 0 \leq i \leq n \tag{3.2}$$

subject to

$$d_i d_j = d_{j-1} d_i,\ i < j \tag{3.3}$$

$$s_i s_j = s_{j+1} s_i,\ i \leq j \tag{3.4}$$

$$d_i s_j = s_{j-1} d_j,\ i < j;\ d_i s_i = d_{i+1} s_i = \mathrm{id};\ d_i s_j = s_j d_{i-1},\ i > j+1 \tag{3.5}$$

we see that the versions of the category Δ^{op} that are defined here and in 2 are isomorphic.

EXAMPLE 3.0.1. For a topological space X define $\mathrm{Sing}_n(X)$ to be the set of singular simplices of X. Then $\mathrm{Sing}(X)$ has a natural structure of a simplicial set.

EXAMPLE 3.0.2. Let A be a graded algebra. Put $A_n^\sharp = A^{\otimes(n+1)}$. Define

$$d_i(a_0 \otimes \ldots \otimes a_{n+1}) = (-1)^{\sum_{p\leq i}|a_p|} a_0 \otimes \ldots a_i a_{i+1} \otimes \ldots \otimes a_{n+1} \tag{3.6}$$

for $i \leq n$

$$d_{n+1}(a_0 \otimes \ldots \otimes a_{n+1}) = (-1)^{|a_{n+1}|\sum_{p\leq n}|a_p|} a_{n+1}a_0 \otimes a_1 \otimes \ldots \otimes a_n \tag{3.7}$$

$$s_i(a_0 \otimes \ldots \otimes a_n) = (-1)^{\sum_{p\leq i}|a_p|} a_0 \otimes \ldots a_i \otimes 1 \otimes \ldots \otimes a_n \tag{3.8}$$

for $i \leq n$.

Let us record the following simple observation.

PROPOSITION 3.0.3. *The above formulas make $A^\sharp$ a simplicial graded vector space.*

4. Cyclic objects

Let $\ell \geq 1$. A n ℓ-cyclic object of a category C is a functor $\Lambda_\ell \to C$. Explicitly, it is a simplicial object X in C together with morphisms $t : X_n \to X_n$ for all $n \geq 0$ such that

$$d_1 t = t d_0;\ d_2 t = t d_1;\ \ldots; d_n t = t d_{n-1};\ d_0 t = d_n \tag{4.1}$$

as morphisms $X_n \to X_{n-1}$;

$$s_1 t = t s_0;\ s_2 t = t s_1\ \ldots; s_n t = t s_{n-1};\ s_0 t = t^2 s_n \tag{4.2}$$

as morphisms $X_n \to X_{n+1}$;

$$t^{\ell(n+1)} = \mathrm{id} \tag{4.3}$$

on X_n. *A cyclic object* is a 1-cyclic object.

For a graded algebra A with an automorphism α such that $\alpha^\ell = \mathrm{id}$, put

$$t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^{|a_n|\sum_{i<n} a_i}(a_n \otimes \alpha(a_0) \otimes \ldots \otimes a_{n-1})$$

PROPOSITION 4.0.1. *Put*

$$({}_\alpha A^\sharp)_n = A^{\otimes(n+1)}$$

The above formula for t, together with (2.10), (2.11), (2.12), *makes $A^\sharp$ a cyclic graded k-module.*

For $\ell = 1$ and $\alpha = \mathrm{id}$ we write $A^\sharp$ instead of ${}_\alpha A^\sharp$.

For an ℓ-cyclic k-module M define

$$b : M_n \to M_{n-1};\ b = \sum_{i=0}^{n}(-1)^i d_i;\ t = (-1)^{n+1} t : M_n \to M_n; \tag{4.4}$$

$$N = \sum_{i=0}^{\ell(n+1)-1} t^i;\ B : M_n \to M_{n+1};\ B = (1-t)sN \tag{4.5}$$

where

$$s = t s_n : M_n \to M_{n+1} \tag{4.6}$$

(s represents the morphism $(x_0, \ldots, x_n) \mapsto (1, x_0, \ldots, x_n)$). As in the case of algebras, the following holds.

PROPOSITION 4.0.2.

$$b^2 = bB + Bb = B^2 = 0$$

DEFINITION 4.0.3. *For an ℓ-cyclic k-module put*

$$C_\bullet(M) = (M_\bullet, b)$$

$$CC^-_\bullet(M) = (M_\bullet[[u]], b + uB)$$

$$CC_\bullet(M) = (M_\bullet[u^{-1}, u]]/uM_\bullet[[u]], b + uB)$$

$$CC^{\mathrm{per}}_\bullet(M) = (M_\bullet[u^{-1}, u]], b + uB)$$

where u is a formal parameter of degree -2.

4.1. Generalities on functors. For a small category Γ, a Γ-module is by definition a functor $\Gamma \to \mathrm{mod}(k)$. In other words, it is a collection of k-modules M_j, $j \in \mathrm{Ob}(\Gamma)$, and morphisms $\gamma : M_j \to M_k$ for any $\gamma \in \Gamma(j,k)$, such that $(fg)m = f(gm)$.

4.1.1. *The functor f^*.* Let $f : B \to \Gamma$ be a functor between small categories. For a Γ-module N let f^*N be its restriction to B, *i.e.* the composition $N \circ f$: $\Gamma_1 \to \mathrm{mod}(k)$. We have $(f^*N)_j = N_{f(j)}$.

4.1.2. *The functor $f_!$.* For a B-module M, define a Γ-module $f_!M$ by

$$(f_!M)_n = (\oplus_{j\in \mathrm{Ob}(B)} k\Gamma(f(j), n) \otimes_k M_n)/\mathcal{K} \tag{4.7}$$

where $\mathcal{K}$ is the k-submodule generated by $\gamma f(\beta) \otimes m - \gamma \otimes \beta m$ for all $\beta \in B(j_1, j_2)$, $\gamma \in \Gamma(f(j_2), n)$, and $m \in M_{j_2}$.

The Γ-module structure is defined by

$$\gamma_0(\gamma \otimes m) = (\gamma_0\gamma) \otimes m$$

for $\gamma_0 \in \Gamma(n, n_0)$, $\gamma \in \Gamma(f(j), n)$, $m \in M_n$.

For $p : B \to *$ and a B-module B, one has $p_!M \xrightarrow{\sim} \mathrm{colim}_B(M)$.

4.1.3. *The tensor product.* For a B-module M and a B^{op}-module N, define

$$N \otimes_B M = (\oplus_{j\in \mathrm{Ob}(B)} M_j \otimes_k N_j)/\mathcal{K} \tag{4.8}$$

where $\mathcal{K}$ is the k-submodule generated by all $nf \otimes m - n \otimes fm$ where $n \in N_k$, $f \in B(j,k)$, and $m \in M_j$.

Let $k_\sharp$ be the constant B^{op}-module. Then

$$k_\sharp \otimes_B M \xrightarrow{\sim} \mathrm{colim}_B(M) \tag{4.9}$$

4.1.4. *The functor f_*.* Consider a functor $B : B \to \Gamma$ and a B-module M as above. For $n \in \mathrm{Ob}(B)$ define $(f_*(M))_n$

$$\varphi = (\varphi_j) \in \prod_{j\in\mathrm{Ob}(B)} \mathrm{hom}_k(k\Gamma(n, f(j)), M_n) \tag{4.10}$$

such that

$$\varphi_{j_2}(f(\beta)\gamma) = \beta\varphi_{j_1}(\gamma)\}$$

for all $j_1, j_2 \in \mathrm{Ob}(B)$, $\gamma \in \Gamma(n, j_1)$, and $\beta \in B(j_1, j_2)$. The B-module structure is defined by

$$(\gamma_0\varphi)_j(\gamma) = \varphi_j(\gamma\gamma_0)$$

for $\gamma_0 \in \Gamma(n, n_0)$, $\varphi_j : k\Gamma(n, f(j)) \to M_n$, and $\gamma \in \Gamma(n_0, f(j))$.

LEMMA 4.1.1. *a) The functor $f_!$ is left adjoint to f^*.*
b) The functor f_* is right adjoint to f^*.
c) The functor $f_!$ is right exact.
d) The functors $\_\otimes_B M$ and $N \otimes_B M$ are right exact for any M and N.

As usual, $\mathbb{L}f_!$ stands for the left derived functor.

DEFINITION 4.1.2. *The homology of $\mathbb{L}\mathrm{colim}_B(M)$ is denoted by $H_\bullet(B, M)$.*

4.2. Hochschild and cyclic homology and derived functors.

THEOREM 4.2.1. *One has, for an integer $\ell \geq 1$ and an ℓ-cyclic k-module X,*

$$HH_\bullet(M) = H_\bullet(\Delta^{\mathrm{op}}, M)$$
$$HC_\bullet(M) = H_\bullet(\Lambda_\ell, M)$$

PROOF. As above, let $k_\sharp$ be the constant Δ- or Λ_ℓ-module. Note that the right hand side is the homology of $k_\sharp \otimes^{\mathbb{L}}_B M$ where B is either Δ^{op} or Λ_ℓ.We will construct a projective resolution of the B^{op}-module $k_\sharp$ in both cases. Define the complex of Λ^{op}-modules $\mathcal{P}_*$ by

$$\mathcal{P}_n([m]) = k\Lambda([m],[n]);\ b : \mathcal{P}_n \to \mathcal{P}_{n-1};\ b\lambda = \sum_{j=0}^{n} (-1)^j d_j \lambda \tag{4.11}$$

We claim that the homology of $\mathcal{P}_*$ is as follows: for every $m \geq 0$,

$$H_0(\mathcal{P}([m]) \xrightarrow{\sim} k;\ H_1(\mathcal{P}([m]) \xrightarrow{\sim} k; H_p(\mathcal{P}([m]) \xrightarrow{\sim} 0$$

for $p > 1$. The free generators of the k-modules H_0 and H_1 are as follows. For H_0, consider the operation

$$d_0^m : (x_0, \ldots, x_m) \mapsto (x_0 \ldots x_m) \tag{4.12}$$

in $\Lambda_\ell([n],[0])$. For H_1, consider the operation

$$d_0^{m-1} : (x_0, \ldots, x_m) \mapsto (x_0 \ldots x_{m-1}, x_m) \tag{4.13}$$

in $\Lambda_\ell([m],[1])$. We claim that d_0^m is a free generator of H_0 and

$$\kappa_m = \sum_{j=0}^{(m+1)\ell-1} d_0^{m-1} t^j \tag{4.14}$$

is a free generator of H_1.

Indeed, consider the free algebra $\mathcal{F}$ with generators $\alpha^j(x_i)$ where $0 \leq i \leq m$ and $0 \leq j < \ell$. Define the automorphism α of this algebra on generators by $\alpha^j(x_k) \mapsto \alpha^{j+1}(x_i)$ where $\alpha^\ell(x_i) = \alpha^0(x_i) = x_i$. The complex $\mathcal{P}_*([m])$ is the direct summand of the Hochschild complex $C_\bullet(\mathcal{F},_\alpha \mathcal{F})$. It is k-linearly generated by monomials where for each i there is exactly one factor $\alpha^j(x_i)$ (for some j). In fact, if we compute $HH_\bullet$ using the Koszul resolution of $\mathcal{F}$ as in Proposition **??**, we obtain the quasi-isomorphic subcomplex of the full Hochschild complex. When intersected with the direct summand $\mathcal{P}_*([m])$, this subcomplex is quasi-isomorphic to the subcomplex is as follows: the basis of zero-chains is $\{d_0^m t^j | 0 \leq j < j \leq \ell(n+1)\}$; the basis of one-chains is $\{d_0^{m-1} t^j | 0 \leq j < j \leq \ell(n+1)\}$; the differential b acts by

$$d_0^m t^j \mapsto d_0^{m-1} t^j (1-t).$$

Therefore b has kernel and cokernel both of rank one, with free generators d_0^m and κ_m as in (4.12), (4.14).

Next we claim that

$$\mathcal{P}_*((u))/u\mathcal{P}_*[[u]], b + uB \tag{4.15}$$

where

$$B : \mathcal{P}_n([m]) \to \mathcal{P}_{n+1}([m]) \tag{4.16}$$

sends λ to λB is a resolution of $k_\sharp$. Indeed, $(b + uB)^2 = 0$ (***WHY??)** and, at the level of homology, B sends the generator d_0^m to κ_m

Applying the functor $k_\sharp \otimes_\Lambda$ _ to a cyclic object, we get the statement of the theorem. □

COROLLARY 4.2.2. *One has, for an associative algebra A,*

$$HH_\bullet(A) \xrightarrow{\sim} H_\bullet(\Delta^{\mathrm{op}}, A^\sharp)$$
$$HC_\bullet(A) = H_\bullet(\Lambda, A^\sharp)$$

4.2.1. *Other versions of the cyclic complex.* For an ℓ-cyclic object M, define

$$b' : M_n \to M_{n-1};\ b = \sum_{j=0}^{n-1} (-1)^j d_j \tag{4.17}$$

Note that the complex $(M_\bullet, b')$ is contractible, s from (4.6) being the homotopy). Define also

$$\tau = (-1)^n : M_n \to M_n;\ N = \sum_{j=0}^{\ell(n+1)-1} \tau^j \tag{4.18}$$

LEMMA 4.2.3.

$$b(1-\tau) = (1-\tau)b';\ b'N = Nb; (1-\tau)N = N(1-\tau) = 0$$

Therefore the sequence of complexes

$$\ldots \xrightarrow{N} (M_\bullet, -b') \xrightarrow{1-\tau} (M_\bullet, b) \xrightarrow{N} (M_\bullet, -b') \xrightarrow{1-\tau} (M_\bullet, b) \tag{4.19}$$

is a double complex. We will denote it by $\widetilde{\mathrm{CC}}_\bullet(M)$; it is a generalization of $\widetilde{\mathrm{CC}}_\bullet(A)$ from (**??**).

LEMMA 4.2.4. *The homology of the total complex of $\widetilde{\mathrm{CC}}_\bullet(M)$ computes $HC_\bullet(M)$.*

PROOF. The same argument as in the proof of Theorem 4.2.1 shows that

$$[n] \mapsto (\ldots \xrightarrow{N} (\mathcal{P}_*([n]), -b') \xrightarrow{1-\tau} (\mathcal{P}_*([n]), b))$$

is a projective resolution of $k_\sharp$. Alternatively, one can compare the two double complexes directly.

To do that, denote by $\widetilde{C}_\bullet(M)$ the total complex of the double complex

$$(C_\bullet(M), -b') \xrightarrow{1-\tau} (C_\bullet(M), b) \tag{4.20}$$

Define the map

$$p : \widetilde{C}_\bullet(M) \to C_\bullet(M) \tag{4.21}$$

as follows:

$$p(m) = m$$

if m is in the b column, and

$$p(m) = (1-\tau)sm$$

if m is in the $-b'$ column. Define also

$$\widetilde{B} : \widetilde{C}_\bullet(M) \to \widetilde{C}_{\bullet+1}(M) \tag{4.22}$$

as follows: if m is in the $-b'$ column then $\widetilde{B}m = 0$; if m is in the b column then $\widetilde{B}m = Nm$ which is located in the $-b'$ column. A straightforward check shows that p is a morphism of complexes and $\widetilde{B}$ anti-commutes with the differential. The complex $\widetilde{\mathrm{CC}}_\bullet(M)$ is isomorphic to $\widetilde{C}_\bullet(M)[[u, u^{-1}]/u^{-1}\widetilde{C}_\bullet(M)[[u]]$, and p defines a morphism

$$\widetilde{\mathrm{CC}}_\bullet(M) \to CC_\bullet(M) \tag{4.23}$$

This morphism is a quasi-isomorphism for columns, therefore a quasi-isomorphism of total complexes. $\square$

DEFINITION 4.2.5. *For an integer $\ell \geq 1$ and for an ℓ-cyclic k-module M put*

$$C^\lambda_\bullet(M) = (M_\bullet/\mathrm{im}(1-\tau), b)$$

COROLLARY 4.2.6. *Assume that either $\mathbb{Q} \subset k$ or each M_n is free as a $\mathbb{Z}/\ell(n+1)\mathbb{Z}$-module. Then the projection to the rightmost column induces a quasi-isomorphism*

$$\widetilde{\mathrm{CC}}_\bullet(M) \to C^\lambda_\bullet(M)$$

Indeed, it is a quasi-isomorphism of row complexes. (See also Proposition 2.0.1).

5. Self-duality of Λ_ℓ

PROPOSITION 5.0.1. *There is an isomorphism of categories $\Lambda_\ell \xrightarrow{\sim} \Lambda_\ell^{\mathrm{op}}$ for all ℓ.*

PROOF. Consider all unital monoids X with an automorphism α such that $\alpha^\ell = \mathrm{id}$ and with an α-trace with values in a set K, i.e. with a map $\mathrm{tr} : X \to K$ such that $\mathrm{tr}(xy) = \mathrm{tr}(\alpha(y)x)$ for all x, y. For every $n \geq 0$ define a pairing

$$X^{n+1} \times X^{n+1} \to K;\ (\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$$

by

$$\langle (x_0, \ldots, x_n), (y_0, \ldots, y_n) \rangle = \mathrm{tr}(x_0 y_0 \ldots x_n y_n) \tag{5.1}$$

For every $\lambda \in \Lambda_\ell([n],[m])$ there is unique $\lambda^R \in \Lambda_\ell^{\mathrm{op}}([n],[m])$ such that for all X, α, tr and for all λ, $\mathbf{x} \in X^{n+1}$, $\mathbf{y} \in X^{m+1}$

$$\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \lambda^R \mathbf{y} \rangle \tag{5.2}$$

The map $\lambda \mapsto \lambda^R$ defines as isomorphism as in the statement of the Proposition. $\square$

REMARK 5.0.2. The above isomorphism $\Lambda_\ell \xrightarrow{\sim} \Lambda_\ell^{\mathrm{op}}$ identifies $\Delta^{\mathrm{op}} \subset \Lambda_\ell$ with $\Delta'^{\mathrm{op}} \subset \Lambda_\ell^{\mathrm{op}}$. Note also that, since the pairing $\langle\, ,\, \rangle$ is not symmetric, the isomorphism $\lambda \mapsto \lambda^R$ is not an involution. For example, the isomorphism sends $\Delta' \subset \Lambda_\ell$ to another subcategory of Λ^{op} isomorphic to $(\Delta^{\mathrm{op}})^{\mathrm{op}}$, the one for which

$$d_0(y_0, \ldots, y_n) = (y_1, \ldots, y_n \alpha^{-1}(y_0));$$

$$d_j(y_0, \ldots, y_n) = (y_0, \ldots, y_{j-1} y_j, \ldots, y_n), j \geq 1;$$

$$s_j(y_0, \ldots, y_n) = (y_0, \ldots, 1, y_j, \ldots, y_n)$$

The two embeddings of Δ^{op} into Λ_ℓ can be explained in terms of Hochschild complexes. Recall that $\mathcal{B}_\bullet$ is the standard bar resolution of A which is a simplicial bimodule. The Hochschild complex is by definition $\mathcal{B}_\bullet \otimes_{A\otimes A^{\mathrm{op}}} A$. There are two ways to identify this with $A^{\otimes(n+1)}$: $(1\otimes a_1\otimes\ldots\otimes a_n\otimes 1)\otimes a$ may go to $a\otimes a_1\otimes\ldots\otimes a_n$ or to $a \otimes a_1 \otimes \ldots \otimes a_n$.

6. Functors between various cyclic and simplicial categories

6.1. The functors j_ℓ and j'_ℓ. For any ℓ, let

$$j_\ell : \Delta^{\mathrm{op}} \to \Lambda_\ell \tag{6.1}$$

and

$$j'_\ell : \Delta' \to \Lambda_\ell \tag{6.2}$$

be the embeddings of the subcategories from 2.

6.2. The functors π_ℓ. Let

$$\pi_\ell : \Lambda_\ell \to \Lambda \tag{6.3}$$

be the functor which is identical on objects, sends d_j to d_j, s_j to s_j, and t to t (and therefore $\sigma = t^{n+1} \in \Lambda_\ell([n],[n])$ to the identity).

6.3. The functors i_ℓ. Observe first that for any monoid X and any ℓ there is a monoid X^ℓ with an automorphism

$$\alpha(x_1, \ldots, x_\ell) = (x_\ell, x_1, \ldots, x_{\ell-1}) \tag{6.4}$$

Identify $(X^l)^{n+1}$ with $X^{l(n+1)}$ via

$$((x_0^{(1)}, \ldots, x_0^{(\ell)}), \ldots, (x_n^{(1)}, \ldots, x_n^{(\ell)}) \mapsto (x_0^{(1)}, \ldots, x_n^{(1)}, \ldots, x_0^{(\ell)}, \ldots, x_n^{(\ell)}) \tag{6.5}$$

Under this identification, any morphism λ from $\Lambda_\ell([n],[m]$ defines a map $i_\ell(\lambda)$: $X^{l(n+1)} \to X^{\ell(m+1)}$. Let us observe that $i_\ell(\lambda)$ is defined by a unique morphism in $\Lambda(i_\ell[n], i_\ell[m])$ where

$$i_\ell[n] = \ell(n+1) - 1 \tag{6.6}$$

We have constructed a functor

$$i_\ell : \Lambda_\ell \to \Lambda \tag{6.7}$$

Example: the morphism $d_0 : [4] \to [3]$ in Λ_2 is mapped by i_2 to $d_0 d_5 : [9] \to [7]$ in Λ.

Indeed:

$$i_2(d_0) : (x_0^{(1)}, x_1^{(1)}, \ldots, x_4^{(1)}, x_0^{(2)}, \ldots, x_4^{(2)}) \mapsto ((x_0^{(1)}, x_0^{(2)}), \ldots, (x_4^{(1)}, x_4^{(2)})) \mapsto$$
$$((x_0^{(1)}x_1^{(1)}, x_0^{(2)}x_1^{(2)}), \ldots, (x_4^{(1)}, x_4^{(2)})) \mapsto ((x_0^{(1)}x_1^{(1)}, \ldots, x_4^{(1)}, x_0^{(2)}x_1^{(2)}, \ldots, x_4^{(2)}) =$$
$$d_0 d_5((x_0^{(1)}, x_1^{(1)}, \ldots, x_4^{(1)}, x_0^{(2)}, \ldots, x_4^{(2)}))$$

More generally, the restriction of i_ℓ to Δ^{op} defines a functor

$$i_\ell | \Delta^{\mathrm{op}} = r_\ell : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \tag{6.8}$$

If we identify Δ^{op} with the opposite of the standard Δ, then r_ℓ is the subdivision functor: it sends the totally ordered set $[n] = \{0, \ldots, n\}$ to the set $[n] \times \{1, \ldots, \ell\}$ with the lexicographic order.

The functor i_ℓ also preserves the subcategory Δ'.

6.4. Cyclic homology of $(A^{\otimes \ell}, \alpha)$**.** For any algebra A, consider the algebra $A^{\otimes \ell}$ with the automorphism $\alpha(a_1 \otimes \ldots \otimes a_\ell) = a_\ell \otimes a_1 \otimes \ldots a_{l-1}$. By definition,

$$CC_\bullet(A^{\otimes \ell}, \alpha) \xrightarrow{\sim} CC_\bullet(i_\ell^* A^\sharp) \tag{6.9}$$

PROPOSITION 6.4.1. *For any cyclic k-module M, there are natural quasi-isomorphisms of complexes*

$$C_\bullet(i_\ell^* M) \xrightarrow{\sim} C_\bullet(M);\ CC_\bullet(i_\ell^* M) \xrightarrow{\sim} CC_\bullet(M)$$

PROOF. Let us start with the Hochschild complex $C_\bullet$ and the case $M = A^\sharp$ for an asociative unital algebra A. Let $\mathcal{B}_\bullet$ be the bar resolution of the bimodule A. We view it as a simplicial A-bimodule. Consider the tensor product of simplicial k-modules with the diagonal simplicial structure

$$\mathcal{B}_\bullet^{(\ell)} = \mathcal{B}_\bullet \boxtimes_k \ldots \boxtimes_k \mathcal{B}_\bullet$$

(ℓ times). (We used the symbol $\otimes$ in the proof above but are using $\boxtimes$ here to avoid confusion with the tensor product of complexes). Note that $\mathcal{B}_\bullet^{(\ell)}$ is the bar resolution of $A^{\otimes \ell}$. One has

$$_\alpha(A^{\otimes \ell}) \otimes_{A^{\boxtimes \ell} \otimes (A^{\boxtimes \ell})^{\mathrm{op}}} \mathcal{B}_\bullet^{(\ell)} \xrightarrow{\sim} A \otimes_{A \otimes A^{\mathrm{op}}} (\mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet)$$

But $\mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet$ is a free A-bimodule resolution of A. Therefore the right hand side is quasi-isomorphic to the Hochschild complex $C_\bullet(A)$. To construct the quasi-isomorphism of the right hand side and $C_\bullet(A, A)$, all is needed is a bimodule morphism of complexes

$$F : \mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet \to \mathcal{B}_\bullet \tag{6.10}$$

(together with a morphism G in the opposite direction and homotopies for $\mathrm{id} - FG$ and $\mathrm{id} - GF$). We will get a homotopy equivalence

$$(i_\ell^* A_\bullet^\sharp, i_\ell^*(b)) \xrightarrow{\sim} (A_\bullet^\sharp, b) \tag{6.11}$$

Choose, for example, the composition of the Alexander-Whitney morphism

$$\mathrm{AW} : \mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet \to \mathcal{B}_\bullet \otimes_A \ldots \otimes_A \mathcal{B}_\bullet$$

with

$$\epsilon \otimes_A \ldots \epsilon \ldots \otimes_A \mathrm{id} : \mathcal{B}_\bullet \otimes_A \ldots \otimes_A \mathcal{B}_\bullet \to \mathcal{B}_\bullet \otimes_A A \ldots \otimes_A A = \mathcal{B}_\bullet$$

where $\epsilon : \mathcal{B}_\bullet \to A$ is the augmentation. We get

$$a_0^{(1)} \otimes \ldots \otimes a_n^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \otimes \ldots \otimes a_n^{(\ell)} \mapsto a_0^{(1)} \ldots a_n^{(\ell-1)} a_0^{(\ell)} \otimes a_1^{(\ell)} \otimes \ldots \otimes a_n^{(\ell)}$$

This is clearly given by a morphism

$$F_0 \in \Delta^{\mathrm{op}}(i_\ell[n], [n]). \tag{6.12}$$

Furthermore, the morphism F, as well as the homotopies, are also given by ***MORE*** This implies the homotopy equivalence$(i_\ell^*(M)_\bullet, i_\ell(b)) \to (M_\bullet, b)$ for an arbitrary cyclic object M.

Now we need to pass from Hochschild to cyclic complexes. For this, consider complexes of Λ^{op}-modules (cf. also (4.15))

$$\mathcal{R}_\bullet = (\mathcal{P}_\bullet((u))/u\mathcal{P}_\bullet[[u]], b + uB);\ \mathcal{R}_\bullet^{(\ell)} = (\mathcal{P}_\bullet^{(\ell)}((u))/u\mathcal{P}_\bullet^{(\ell)}[[u]], i_\ell b + u i_\ell B) \tag{6.13}$$

$$\mathcal{P}_\bullet : [m] \mapsto \Lambda([m], [\bullet]), b;\ \mathcal{P}_\bullet^{(\ell)} : [m] \mapsto \Lambda([m], i_\ell[\bullet]) \tag{6.14}$$

We have

$$\mathcal{R}_\bullet \otimes_\Lambda M \xrightarrow{\sim} CC_\bullet(M);\ \mathcal{R}_\bullet^{(\ell)} \otimes_\Lambda M \xrightarrow{\sim} CC_\bullet(i_\ell M)$$

It remains to construct a morphism of complexes of Λ^{op}-modules

$$F = F_0 + u^{-1}F_1 + \ldots : \mathcal{R}_\bullet^{(\ell)} \to \mathcal{R}_\bullet \tag{6.15}$$

where F_0 is the composition from the left with the morphism from (6.12).

There is precisely one obstruction for constructing F_1, F_2, etc. Indeed, we know that the homology of $\mathcal{P}_\bullet$ is nonzero only in degrees 0 and 1. So, as soon as we check that $BF_0 - F_0 i_\ell B : i_\ell[0] \to [1]$ is zero in $H_1(\mathcal{P}_\bullet)$, we will have no non-zero obstructions: we will know that $BF_0 - F_0 i_\ell(B)$ is homotopic to zero, F_1 will be a homotopy, etc. But

$$BF_0 : a_0^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \mapsto B(a_0^{(1)} \ldots a_0^{(\ell)}) = 1 \otimes a_0^{(1)} \ldots a_0^{(\ell)} - a_0^{(1)} \ldots a_0^{(\ell)} \otimes 1,$$

whereas

$$F_0 i_\ell(B) : a_0^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \mapsto F_0(-a_0^{(1)} \otimes 1 \otimes \ldots \otimes a_0^{(\ell)} \otimes 1 + 1 \otimes a_0^{(1)} \otimes \ldots \otimes 1 \otimes a_0^{(\ell)}) =$$
$$1 \otimes a_0^{(1)} \ldots a_0^{(\ell)} - a_0^{(1)} \ldots a_0^{(\ell)} \otimes 1$$

and therefore

$$BF_0 - F_0 i_\ell(B) : a_0^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \mapsto 0.$$

□

7. The Kaledin resolution of a cyclic object

Here we will construct a resolution of any cyclic module M. It will not be projective but its value at any object $[n]$ will be free as a $C_n = \mathbb{Z}/(n+1)\mathbb{Z}$-module. As a result, we can compute $HC_\bullet(M)$ by applying to this resolution the generalized construction of the cyclic complex C^λ from 4. The result is the double complex $\widetilde{\mathrm{CC}}_\bullet(M)$ from 4.2.1

Let us start with the case $M = k^\sharp$. Note that the category Λ can be interpreted as follows. Objects are homotopy classes of triangulations of the circle S^1; the object $[n]$ corresponds to the triangulation by points $0, 1, \ldots, n$ located counterclockwise. Morphisms are homotopy classes of maps $S^1 \to S^1$ that are of degree one and and map triangulation to triangulation. Explicitly, the morphism

$$(x_0, \ldots, x_n) \mapsto (x_{J_0}, \ldots, x_{J_m}) \tag{7.1}$$

in $\Lambda([n], [m]$ corresponds to the counterclockwise-nondecreasing continuous map that sends the vertex i to the vertex k if x_i is a factor in x_{J_k}. Let $C_1(S^1, [n]) \xrightarrow{\partial} C_0(S^1, [n])$ be the chain complex of the triangulation corresponding to $[n]$. We get a cyclic object in the category of chain complexes, as well as a length two extension of cyclic modules

$$0 \to k^\sharp \to C_1(S^1, [\_]) \xrightarrow{\partial} C_0(S^1, [\_]) \to k^\sharp \to 0 \tag{7.2}$$

which we also denote by

$$0 \to k^\sharp \to \mathbb{K}_1 \xrightarrow{\partial} \mathbb{K}_0 \to k^\sharp \to 0 \tag{7.3}$$

We also define the map $\mathbb{K}_0 \xrightarrow{N} \mathbb{K}_1$ to be the composition $\mathbb{K}_1 \to k^\sharp \to \mathbb{K}_0$. We get a resolution of $k^\sharp$:

$$\ldots \xrightarrow{N} \mathbb{K}_1 \xrightarrow{\partial} \mathbb{K}_0 \xrightarrow{N} \mathbb{K}_1 \xrightarrow{\partial} \mathbb{K}_0 \to k^\sharp \to 0 \tag{7.4}$$

Now define for any cyclic module M

$$\mathbb{K}_i(M)_n = (\mathbb{K}_i)_n \otimes M_n$$

with the diagonal action of morphisms in Λ. We get the resolution of M

$$\dots \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \to M \to 0 \tag{7.5}$$

LEMMA 7.0.1. *One has*

$$\mathbb{K}_0(M) \overset{\sim}{\to} j_! j^* M; \;\; \mathbb{K}_1(M) \overset{\sim}{\to} j'_* j'^* M$$

PROOF. Start with $M = k^\sharp$. Identify $j_! j^* k^\sharp$ with $C^0(S^1, [n])$ as follows: $t^j \otimes 1$ corresponds to the vertex j for all j. Now look at the unique decomposition of any morphism in Λ into $\lambda = t^i \delta$ where $\delta \in \Delta^{\mathrm{op}}$, cf. Lemma 2.1.2. Observe that, when λ is identified with the corresponding triangulated map $S^1 \to S^1$, $j = \lambda(0)$. Consequently, if $\lambda t^k = t^j \delta$ for $\lambda \in \Lambda([n_1],[n])$ and $\delta \in \Delta^{\mathrm{op}}([n_1],[n])$, then $j = \lambda(k)$. Therefore the action of λ on $t^k \otimes 1 \in (j_! j^* k^\sharp)_{n_1}$ agrees with the action of the corresponding map on the kth vertex of the triangulation $[n_1]$.

Now identify $C_1(S^1,[n])$ with $(j'_* j^* k^\sharp)_n$ as follows. By definition, the edge e_p from $p-1$ to p will correspond to the collection $\varphi^{(p)} = \{\varphi_j^{(p)}\}$ defined by

$$\varphi_j^{(p)} : (\delta' t^{-k}) \mapsto \delta_p^k$$

for any k and any $\delta' \in \Delta'([j],[n])$. (Here $n+1 = 0$, and δ_p^k is the Kronecker symbol). Let us look at the decomposition $\lambda = \delta' t^{-p}$ from Lemma 2.1.2. Let λ correspond to the map $(x_0, \dots, x_n) \mapsto (x_{J_0}, \dots, x_{J_m})$. Let r be the smallest index for which $x_{J_r} \neq 1$. Then the leftmost factor in x_{J_r} is x_p. More generally, assume

$$t^{-k}\lambda = \delta' t^{-p} \tag{7.6}$$

Let k' be the smallest index $k' \geq k$ (in the cyclic order) for which $x_{J_{k'}} \neq 1$. Then the leftmost factor in $x_{J_{k'}}$ is x_p.

Consequently, the action of λ on $j'_* j^* k^\sharp$ sends $\varphi^{(p)}$ to the sum of all $\varphi^{(k)}$ with given k' as above (this sum may be empty). In the language of triangulated maps, λ sends the edge e_p to the sum of all e_k that are contained in $\lambda(e_p)$.

For a general M, note that $j_! j^* M \overset{\sim}{\to} M \otimes j_! j^* k^\sharp$ and $M \otimes j'_* j'^* k^\sharp \overset{\sim}{\to} j'_* j'^* M$ The first isomorphism sends $\sigma\delta \otimes m = \sigma \otimes \delta m$ to $\sigma \otimes \sigma\delta m$ where δ is a morphism in Δ^0 and σ is a power of t. The second sends $m \otimes (\varphi_j)$ to $(\widetilde{\varphi}_j)$ defined by $\widetilde{\varphi}_j(\delta'\sigma) = \varphi(\sigma)\delta'\sigma^{-1} m$. Here δ' is a morphism in Δ' and σ is a power of t. **ELABORATE*** $\square$

LEMMA 7.0.2. *There are isomorphisms of double complexes*

$$(C^\lambda_\bullet(\mathbb{K}_1(M)) \xrightarrow{\partial} C^\lambda_\bullet(\mathbb{K}_0(M))) \overset{\sim}{\to} ((M_\bullet, b') \xrightarrow{1-t} (M_\bullet, b))$$
$$(C^\lambda_\bullet(\mathbb{K}_0(M)) \xrightarrow{N} C^\lambda_\bullet(\mathbb{K}_1(M))) \overset{\sim}{\to} ((M_\bullet, b) \xrightarrow{N} (M_\bullet, b'))$$

PROOF. Identify $C^\lambda_\bullet(\mathbb{K}_0(M))$, resp. $C^\lambda_\bullet(\mathbb{K}_1(M))$, with $M_\bullet$ by choosing $\mathbb{Z}/(n+1)\mathbb{Z}$-free generators of the nth component of $\mathbb{K}_0(M)$, resp. of $\mathbb{K}_1(M)$, to be $t^0 \otimes M_n$, resp. $(\varphi_j^{(0)}) \otimes M_n$ (cf. the proof above). As we saw in this proof, d_j act on t^0 by identity for all j; on $(\varphi^{(0)}$, d_j act by identity if $j < n$ and by zero for $j = n$. Indeed, $d_n : (x_0, \dots, x_n) \mapsto (x_n x_0, \dots, x_{n-1})$ and therefore x_0 is not a leftmost factor of any monomial x_{J_k}. Now, ∂ sends $(\varphi^{(0)}$ to $(t^0 - t^{-1}) \otimes m = t^0 \otimes (1-\tau)m$ in the quotient by the image of $1-\tau$ (here τ acts diagonally). As for N, the map $\mathbb{K}_0(k)_n \to k$ sends all $\tau^j \otimes 1$ to 1, and the map $k \to \mathbb{K}_1(k)_n$ sends 1 to the sum of all $\varphi^{(0)}$. After tensoring with M the composition of the two maps becomes the following: for every q,$t^q \otimes m \mapsto \sum_p t^p(\varphi^{(0)}) \otimes m = \varphi^{(0)} \otimes \sum_p \tau^p m$ modulo the image of $1-\tau$. $\square$

REMARK 7.0.3. It is straightforward that both factors in the composition $j_!j^*M \to M \to j'_*j'^*M$ are the standard adjunction maps.

7.1. The case of an ℓ-cyclic module. The above can be easily generalized to the following

PROPOSITION 7.1.1. *Let ℓ be an integer ≥ 1. For an ℓ-cyclic k-module M define*

$$\mathbb{K}_0(M) = j_{\ell !}j_\ell^* M;\ \mathbb{K}_1(M) = {j_\ell}'_* {j'_\ell}^* M \tag{7.7}$$

There are morphisms p, ∂, and N of ℓ-cyclic objects such that

$$\ldots \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \to M \to 0 \tag{7.8}$$

is an acyclic complex of ℓ-modules. For every object $[n]$, both $\mathbb{K}_0(M)_n$ and $\mathbb{K}_1(M)_n$ are free $\mathbb{Z}/\ell(n+1)\mathbb{Z}$-modules. The sequence of complexes

$$\ldots \xrightarrow{N} C^\lambda_\bullet(\mathbb{K}_1(M)) \xrightarrow{\partial} C^\lambda_\bullet(\mathbb{K}_0(M)) \xrightarrow{N} C^\lambda_\bullet(\mathbb{K}_1(M)) \xrightarrow{\partial} C^\lambda_\bullet(\mathbb{K}_0(M)) \tag{7.9}$$

is isomorphic to

$$\ldots \xrightarrow{N} (M_\bullet, b') \xrightarrow{1-\tau} (M_\bullet, b) \xrightarrow{N} (M_\bullet, b') \xrightarrow{1-\tau} (M_\bullet, b)$$

PROOF. Let us start with interpreting Λ_ℓ in terms of triangulations. Let $[n]_\ell$ be the triangulation of S^1 with vertices $j^{(p)} = j + p(n+1)$, $0 \leq j \leq n$, $0 \leq p < \ell$ going counterclockwise. In other words, $[n]_\ell = i_\ell[n] = [\ell(n+1) - 1]$. Then $\Lambda_\ell([n],[m])$ can be identified with homotopy classes of triangulated maps $(S^1, [n]_\ell) \to (S^1, [m]_\ell)$ which are:

(1) non-decreasing in counterclockwise order;
(2) of degree one;
(3) commuting with the shift $\sigma : [j]^{(p)} \mapsto [j]^{(p+1)}$ for all j and p (where $j^{(\ell)} = j^{(0)}$).

The identification is as follows. Start with a triangulated map $(S^1, [n]_\ell) \to (S^1, [m]_\ell)$ and construct a morphism in Λ_ℓ represented by $(x_0, \ldots, x_n) \mapsto (x_{J_0}, \ldots, x_{J_m})$ as in (2.1). First, for every $0 \leq j \leq n$, $\widetilde{x}_j$ will be a factor in x_{J_k} if the map sends $j^{(0)}$ to some $k^{(p)}$. Second, if the map sends $0^{(0)}$ to $i^{(r)}$, then $\widetilde{x}_0 = \alpha^r(x_0)$. As for other $\widetilde{x}_j$: we put $\widetilde{x}_j = \alpha^{r+1}(x_j)$ for all $j \leq n$ such that $j^{(\ell-1)}$ is mapped to the same point $i^{(r)}$ as $0^{(0)}$, and $\widetilde{x}_j = \alpha^r(x_j)$ otherwise. This determines λ uniquely.

For example, there are four morphisms in $\Lambda_2([1],[0])$. They are represented my maps sending (x_0, x_1) to:

$$1)\, (x_0x_1);\ 2)\, (\alpha(x_1), x_0);\ 3)\, (\alpha(x_0), \alpha(x_1));\ 4)\, (x_1, \alpha(x_0)).$$

The triangulations $[1]_2$ and $[0]_2$ are, respectively, the four points $0^{(0)}, 1^{(0)}, 0^{(1)}, 1^{(1)}$ and $0^{(0)}, 0^{(1)}$ located counterclockwise. The four morphisms above correspond to the four triangulated maps $(S^1, [1]_2) \to (S^1, [0]_2)$ as follows:

$$1)\, 0^{(0)}, 1^{(0)} \mapsto 0^{(0)};\ 0^{(1)}, 1^{(1)} \mapsto 0^{(1)}$$
$$2)\, 1^{(1)}, 0^{(0)} \mapsto 0^{(0)};\ 0^{(1)}, 1^{(0)} \mapsto 0^{(1)}$$
$$3)\, 0^{(0)}, 1^{(0)} \mapsto 0^{(1)};\ 0^{(1)}, 1^{(1)} \mapsto 0^{(0)}$$
$$4)\, 1^{(1)}, 0^{(0)} \mapsto 0^{(1)};\ 0^{(1)}, 1^{(0)} \mapsto 0^{(0)}$$

After these identifications, the above proof for Λ works for Λ_ℓ without any change. $\square$

REMARK 7.1.2. We have identified $\Lambda_\ell([n],[m])$ with those morphisms in $\Lambda(i_\ell[n], i_\ell[m])$ that commute with $\sigma = \tau^{n+1}$. This identification is the functor i_ℓ from (6.7).

8. Cyclic categories and the circle

8.1. Geometric realization of a cyclic space, I. ***Loday's construction

8.2. Geometric realization of a cyclic space, II. ***Drinfeld's construction

8.3. Connes-Konsani?*.**

9. The Frobenius map

In this section k is a perfect field of characteristic $p > 0$. As usual, for $a \in k$, $Fa = a^p$ is the Frobenius morphism. For any vector space V over k we will consider $V^{\otimes p}$ as a $\mathbb{Z}/p\mathbf{Z}$-module (with the action by cyclic permutations). As usual, for every module M over a finite group G, its Tate cohomology in degree zero is defined by

$$\check{H}^0(G, M) = M^G/\mathrm{im}(N) \tag{9.1}$$

where

$$N = \sum_{g\in G} g : M \to M.$$

9.1. Frobenius map for vector spaces.

LEMMA 9.1.1. *For a vector space V over p, the map $x \mapsto x^{\otimes p}$ induces an F-linear isomorphism*

$$V \xrightarrow{\sim} \check{H}^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \tag{9.2}$$

PROOF. Consider a basis $\mathbf{B}$ of V over k. Then the following two sets of vectors form a basis of $V^{\otimes p}$: a) $v^{p\otimes}$, $v \in \mathbf{B}$ and b) $v_1 \otimes \ldots \otimes v_p$ where v_j are all in $\mathbf{B}$ and not all the same. The subset a) generates a constant $\mathbb{Z}/p\mathbb{Z}$-module that coincides with its degree zero Tate cohomology. The subset b) generates a free $\mathbb{Z}/p\mathbb{Z}$-module, therefore its Tate cohomology vanishes. Furthermore, the map $x \mapsto x^p$ is additive because

$$(x+y)^p = x^p + y^p + Nz$$

for some z. □

9.2. Frobenius map for cyclic objects.

PROPOSITION 9.2.1. *Let A be an algebra over k. There is a natural isomorphism of cyclic objects*

$$\varphi_p : A^\sharp \xrightarrow{\sim} \check{H}^0(\mathbb{Z}/p\mathbb{Z}, i_p^*(A^\sharp)) \tag{9.3}$$

where the action of $\mathbb{Z}/p\mathbb{Z}$ on the p-cyclic vector space $i_p^*(A^\sharp)$ is via the group generated by σ from (2.2).

PROOF. Note that $i_p^*(A^\sharp)_n \xrightarrow{\sim} (A_n^\sharp)^{\otimes p}$ ((6.5)); it is straightforward that the map $x \mapsto x^{\otimes p}$ is a morphism of cyclic vector spaces. □

10. Topological Hochschild and cyclic homology in characteristic zero

We will assume here that A is a ring. Set

$$A_n = A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} A = A^{\otimes_{\mathbb{Z}} n}.$$

The formulas from the example 3.0.2 give us a simplicial object

$$TTH_\bullet(A) : n \to A_n$$

in the category of sets.

Definition 10.0.1. *The topological Hochschild homology of A is given by the homotopy groups of the geometric realisation of the simplicial set $TTH_\bullet(A)$, i. e.*

$$TTH_k(A) = \pi_k(|TTH_\bullet(A)|)$$

The following construction is due to Drinfeld.

Theorem 10.0.2. *$TTH(A)$ has a geometric realisation with a natural action of the unit circle induced from the cyclic module structure on $A^\sharp$ described in the proposition 3.0.3.*

Proof. Let $\mathcal{F}$ be the family of finite subsets of the unit circle. We will choose an orientation of the unit circle. Given an ordered (n+1)-tuple $F = \{x_0 \ldots x_n\}$ of points in $\mathbb{T}$, set

$$A(F) = A^{\otimes n+1}.$$

For notational simplicity we will set,

$$\otimes_{i \in F}^{F} a_i = a_{x_0} \otimes \ldots \otimes a_{x_n}$$

For an inclusion $F \subset F^{'} \in \mathcal{F}$, we define a map by $\iota : A(F) \to A(F^{'})$

$$\iota_* : \otimes_{i \in F}^{F} a_i = \otimes_{i \in F'}^{F} b_i,$$

where, on the right hand side, $b_i = a_i$ if $i \in F$ and $b_i = 1$ otherwise. We filter $\mathcal{F}$ by inclusion and set

$$B^\lambda(A) = \operatorname{colim}_{\mathcal{F}} A(F).$$

$B^\lambda(A)$ is topologised as follows. Two points a and b of A_F are close to each other if there are two points

$$x_k, x_{k+1} \in F, k \in \mathbb{Z} \mod (\sharp F)$$

close to each other and such that

$$a_{x_k} a_{x_{k+1}} = b_{x_k} b_{x_{k+1}}.$$

We will leave it to the reader to check that this defines a metric topology on $B^\lambda(A)$. It is immediate that the unit circle acts by rotations on $B^\lambda(A$. □

Remark 10.0.3. Suppose that the algebrsa A is commutative. Then we can replace the filtered set $\mathfrak{F}$ by the analogous filtered set $\mathfrak{F}_M$ of finite subsets of a, say, metric space M and the construction above defines the simplicial space $TTH_M(A)$.

Remark 10.0.4. In the terms of the above construction, negative cyclic homology coincides with the equivariant cohomology

$$HC^-_{-*}(A) = \mathbb{H}^*(\mathbb{T}, THH(A)),$$

cyclic homology coincides with the equivariant homology

$$H^\lambda_*(A) = \mathbb{H}_*(\mathbb{T}, THH(A))$$

while cyclic periodic homology coincides with the Tate cohomology

$$CC^{per}(A) = \widehat{\mathbb{H}}^*(\mathbb{T}, THH(A)).$$

11. Cyclotomic objects

11.1. **Naive? Toy? cyclotomic modules.** As we saw in 9.2, the cyclic vector space $A^\sharp$ of an algebra over a perfect field of characteristic $p > 0$ carries an additional structure. Namely, the zeroth Tate cohomology of $C_p = \mathbb{Z}/p\mathbb{Z}$ with values in $i_p^* A^\sharp$ is F-linearly isomorphic to $A^\sharp$.

**Give a definition for iterations of i_p^*, and also over $\mathbb{Z}$ for all integer p?

To what extent can one replace the zeroth cohomology $\check{H}^0$ by the full complex $\check{C}^*$?

In many respects the answer turns out to be easier if one passes from algebras to *ring spectras*. In fact, C_p-equivariant spectra happen to admit both an analog of the full Cech complex (denoted by $X \mapsto X^{tC_p}$ together with a diagonal morphism $X \to (\wedge^p X)^{tC_p}$. As soon as basic properties if these two constructions are established, we get a full analog of what was defined in 11.1. We do this below in 11.2. The category of algebras, esecially differential graded algebras, makes carrying out such a construction more complicated (for example, the Frobenius map $x \mapsto x^{\otimes p}$ does not commute with the differential). Nevertheless, a theory of cyclotomic modules exists; it is due to Kaledin. We outline in partially in 11.3.

11.2. Cyclotomic spectra.

11.3. Cyclotomic modules.

12. Bibliographical notes

CHAPTER 7

Hochschild and cyclic homology as non-Abelian derived functors

1. Homology of free algebras

Let V be a free k-module and $A = T(V)$ the free algebra over k generated by V.

PROPOSITION 1.0.1. *The embedding of the subcomplex*

$$T(V) \otimes V \xrightarrow{b} T(V) \tag{1.1}$$

located in degrees 1 *and* 0 *into* $C_\bullet(V)$ *is a homotopy equivalence.*

PROOF. Indeed, the subcomplex $T(V) \otimes V \otimes T(V) \to T(V) \otimes T(V)$ of the bar resolution $\mathcal{B}_\bullet(T(V)) = T(V) \otimes \overline{T}(V)^{\otimes \bullet} \otimes T(V)$ is a free bimodule resolution of V. The proof follows from applying the functor $\otimes_{T(V)\otimes T(V)^{\mathrm{op}}} T(V)$ to this resolution. □

The subcomplex V can be defined more invariantly as a quotient rather that a subcomplex: for any algebra A, put

$$C_\bullet(A)^{\mathrm{sh}} = (C_1(A)/bC_2(A) \xrightarrow{b} C_0(A)) \tag{1.2}$$

To see that (1.1) and (1.2) are the same for $A = T(V)$, observe that the former maps to the latter; denote this map by i. Now construct the map P in the opposite direction as follows: in degree zero it is the identity; in degree one,

$$r \otimes v_1 \dots v_n \mapsto \sum_{j=1}^{n} v_{j+1} \dots v_n r v_1 \dots v_{j-1} \otimes v_j \tag{1.3}$$

for $r \in T(V)$ and $v_i \in V$. We have $P \circ i = \mathrm{id}$ whereas

$$(\mathrm{id} - i \circ P)(r \otimes v_1 \dots v_n) = b \sum_{j=1}^{n-1} v_1 \dots v_{j-1} \otimes v_j \otimes v_{j+1} \dots v_n \tag{1.4}$$

COROLLARY 1.0.2. *For* $A = T(V)$, *the projection* $C_\bullet(A) \to C_\bullet(A)^{\mathrm{sh}}$ *is a homotopy equivalence.*

PROOF. It is immediate that for $A = T(V)$ the above projection comes from a map of bimodule resolutions, and the statement follows from standard homological algebra. It is easy to write an explicit homotopy, and we will do so in order to use it later in various cases, instead of referring to more general statements of homological algebra. In fact our homotopy directly generalizes (1.4).

Put

(1.5)

$$h(r_0 \otimes v_1 \dots v_n \otimes r_2 \otimes \dots \otimes r_m) = r_0 \sum_{j=1}^{n-1} v_1 \dots v_{j-1} \otimes v_j \otimes v_{j+1} \dots v_n \otimes r_2 \otimes \dots \otimes r_m$$

for $r_k \in T(V)$ and $v_i \in V$.

LEMMA 1.0.3. *Let* $P : C_\bullet(T(V)) \to C_\bullet(T(V))^{\mathrm{sh}}$ *be the projection; let* $i : C_\bullet(T(V))^{\mathrm{sh}} \to C_\bullet(T(V))$ *be the embedding equal to the identity on* $T(V)$ *and sending* $r \otimes v$ *to* $r \otimes v$ *for* $r \in T(V)$ *and* $v \in V$. *Then*

$$P \circ i = \mathrm{id};\ \mathrm{id} - i \circ P = [b, h]$$

where h *is as in* (1.5).

The proof is straightforward. □

REMARK 1.0.4. Formula (1.5) for h can be generalized to the graded case. The sign of the jth term in the sum becomes $(-1)^{|r_0| + \sum_{p<j} |v_p|}$.

1.1. Cyclic complexes of a free algebra. For any algebra we have well-defined

$$C_0(A)^{\mathrm{sh}} \xrightarrow{B} C_1(A)^{\mathrm{sh}} \xrightarrow{b} C_0(A)^{\mathrm{sh}} \tag{1.6}$$

satisfying $bB = 0$; $Bb = 0$. We can therefore form short versions of the negative and other cyclic complexes; for example, put

$$\mathrm{CC}^-_\bullet(A)^{\mathrm{sh}} = (C_\bullet(A)^{\mathrm{sh}}[[u]], b + uB) \tag{1.7}$$

PROPOSITION 1.1.1. *The projection* $\mathrm{CC}^-_\bullet(T(V)) \to \mathrm{CC}^-_\bullet(T(V))^{\mathrm{sh}}$ *is a homotopy equivalence, and similarly for* $\mathrm{CC}_\bullet$ *and for* $\mathrm{CC}^{\mathrm{per}}_\bullet$.

PROOF. Follows from the fact that the projection preserves the filtration by powers of u and is a homotopy equivalence on associated graded quotients. □

REMARK 1.1.2. For any algebra A one has $C_1(A)^{\mathrm{sh}} \xrightarrow{\sim} \Omega^1(A)/[A, \Omega^1(A)] = \mathrm{DR}^1(A)$ in the language of 18. Under this identification, (1.6) becomes

$$A \xrightarrow{d} \mathrm{DR}^1(A) \xrightarrow{b} A \tag{1.8}$$

This justifies calling the short cyclic complexes like (1.7) *two-periodic De Rham complexes* of A.

COROLLARY 1.1.3. *The reduced cyclic complex* $\mathrm{CC}_\bullet(T(V))/\mathrm{CC}_\bullet(k)$ *is homotopy equivalent to*

$$\bigoplus_{n \geq 1} (\dots \xrightarrow{N} V^{\otimes n} \xrightarrow{1-t} \dots \xrightarrow{N} V^{\otimes n} \xrightarrow{1-t} V^{\otimes n})$$

where t *is the cyclic permutation of cyclic factors and* $N = 1 + t + \dots + t^{n-1}$.

When k *contains* $\mathbb{Q}$, *then*

$$\overline{\mathrm{HC}}_m(T(V)) \xrightarrow{\sim} 0$$

for $m > 0$;

$$\overline{\mathrm{HC}}_0(T(V)) \xrightarrow{\sim} T(V)/([T(V), T(V)] + k).$$

PROOF. The embedding i identifies both $C_0(T(V))^{\mathrm{sh}}$ and $C_1(T(V))^{\mathrm{sh}}$ with $\oplus_{n \geq 1} V^{\otimes n}$. Under this identification, b becomes $1 - t$ and B becomes N. □

2. Semi-free algebras

DEFINITION 2.0.1. *A differential graded algebra R is semi-free over k if*

(1) *as a graded algebra, it is equal to $T(V)$ where V is a free graded k-module;*
(2) *V has a filtration $0 = V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset V_n \subset \ldots$ such that dV_n is contained in the subalgebra generated by V_{n-1} for all n.*

If R is concentrated in non-positive degrees then the second condition is redundant as we can take $V_n = \oplus_{j \leq n} V^{-j}$.

PROPOSITION 2.0.2. *For any DG algebra A there exists a semi-free DG algebra R together with a surjective quasi-isomorphism $R \to A$.*

PROOF. Choose a k-module V_0 that generates A as an algebra. Let $R_0 = TV_0$ with zero differential. Consider the epimorphism ********** □

A DG algebra R such as in Proposition 2.0.2 is called a semi-free resolution of A.

PROPOSITION 2.0.3. *Let R be a semi-free resolution of $R \xrightarrow{\pi_A} A$ and $S \xrightarrow{\pi_B} B$ a semi-free resolution of B. For a morphism $f : A \to B$ there exists a morphism $F : R \to S$ such that $\pi_B F = f \pi_A$. Any two such morphisms F are homotopic.*

$$\begin{tikzcd} R \arrow[r, "F"] \arrow[d, "\pi_A"'] & Q \arrow[d, "\pi_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Any two semi-free resolutions of a DG algebra A are homotopy equivalent.

PROOF. We construct F, as well as D, on V_n inductively in n. ***A bit more?*** □

LEMMA 2.0.4. *Let A be semi-free. Then being homotopic is an equivalence relation on morphisms $A \to B$.*

PROOF. As shown in 8, being homotopic is an equivalence relation on A_∞ morphisms $A \to B$. But such an A_∞ morphism is a DG algebra morphism

$$\mathrm{CobarBar}(A) \to B.$$

Let π_A be the projection of $\mathrm{CobarBar}(A)$ to A. If A is semi-free then there is a morphism of DG algebras q suct tat $q\pi_A = \mathrm{id}_A$. For any $f_0, f_1 : A \to B$, a homotopy $\widetilde{f}$ between $f_0\pi_A$ and $f_1\pi_A$ leads to a homotopy $f = \widetilde{f}q$ between f_0 and f_1.

$$\begin{tikzcd} \mathrm{CobarBar}(A) \arrow[d, shift left, "\pi_A"] \arrow[dr, "\widetilde{f}"] & \\ A \arrow[u, bend left, "q"] \arrow[r, "f"] & B \otimes C^*(\Delta^1) \end{tikzcd}$$

□

REMARK 2.0.5. If we replace morphisms of DG algebras by morphisms of complexes then we arrive at the usual definition of chain homotopic maps.

3. Hochschild and cyclic homology and semi-free resolutions

3.1. Hochschild and cyclic complexes of semi-free algebras. Let us start by observing that complexes $C_\bullet(A)^{\mathrm{sh}}$ are defined for DG algebras (now they are double complexes with two columns). Also, the complexes $\mathrm{CC}^-_\bullet(R)^{\mathrm{sh}}$, etc. are defined.

LEMMA 3.1.1. *For a semi-free DG algebra R, the projections*

$$C_\bullet(R) \to C_\bullet(R)^{\mathrm{sh}};\ \mathrm{CC}^-_\bullet(R) \to \mathrm{CC}^-_\bullet(R)^{\mathrm{sh}};$$

$$\mathrm{CC}_\bullet(R) \to \mathrm{CC}_\bullet(R)^{\mathrm{sh}};\ \mathrm{CC}^{\mathrm{per}}_\bullet(R) \to \mathrm{CC}^{\mathrm{per}}_\bullet(R)^{\mathrm{sh}}$$

are homotopy equivalences of complexes.

PROOF. Let h be the homotopy as in Remark 1.0.4. Define

$$H = h + \sum_{n\geq 1}(-1)^n(hd)^n h;\ I = \sum_{n\geq 0}(-1)^n(hd)^n i \tag{3.1}$$

These are infinite sums but $[h,d]$ is locally nilpotent because the algebra is semi-free. We have

$$P \circ I = \mathrm{id};\ \mathrm{id} - I \circ P = [b+d, H] \tag{3.2}$$

Indeed,

$$[b, (hd)^n h] = (hd)^n + (dh)^n - (hd)^n iP$$

(which follows from $Ph = 0$);

$$[d, (hd)^n h] = (hd)^{n+1} + (dh)^{n+1}$$

Formulas (3.2) show that the projection of the long Hochschild complex to the short is a homotopy equivalence. Therefore the same is true for all the cyclic complexes. Explicitly, one can modify H and I from (3.1) replacing d by $d+uB$. □

PROPOSITION 3.1.2. *Let R be a semi-free resolution of a DG algebra A. Then*

1. *the complex $C_\bullet(R)^{\mathrm{sh}}$ computes the Hochschild homology of A;*
2. *the complex $\mathrm{CC}^-_\bullet(R)^{\mathrm{sh}}$ computes the negative cyclic homology of A;*
3. *similarly for cyclic and periodic cyclic homologies.*

PROOF. In fact both morphisms

$$C_\bullet(R)^{\mathrm{sh}} \longleftarrow C_\bullet(R) \longrightarrow C_\bullet(A) \tag{3.3}$$

are quasi-isomorphisms. Same for cyclic complexes of all types. □

PROPOSITION 3.1.3. *Let k contain $\mathbb{Q}$. Let R be a semi-free resolution of A.The complex $R/([R,R]+k)$ computes the reduced cyclic homology $\overline{\mathrm{HC}}_\bullet(A)$.*

PROOF. Indeed, both morphisms

$$R/([R,R]+k) \longleftarrow \overline{\mathrm{CC}}_\bullet(R) \longrightarrow \overline{\mathrm{CC}}_\bullet(A) \tag{3.4}$$

are quasi-isomorphisms. □

REMARK 3.1.4. Because of Remark 2.0.5 , it is clear that all complexes defined above in terms of a quasi-free resolution are well-defined up to chain homotopy equivalence of complexes.

3.2. The relative version. Consider two DG algebras A and R. We say that R is semi-free over A if R is freely generated over A as a graded algebra, V has a filtration $0 = V_{-1} \subset V_0 \subset V_1 \subset \ldots$, dV_n is inside the subalgebra generated by A and V_{n-1}, and $d|A$ is the differential of the DGA A.

Let $A \xrightarrow{f} B$ is a morphism of DG algebras. Let R be a DG algebra semi-free over A. A morphism $R \to B$ is a morphism over A if its restriction to A is f. A homotopy between two such morphisms is a homotopy over A if its restriction to A is the composition $A \to B \to B \otimes C^*(\Delta^1)$.

LEMMA 3.2.1. *Being homotopic over A is an equivalence relation on morphisms $R \to A$ iver A.*

PROOF. The proof is exactly as in the absolute case, except we use the relative Hochschild cochain complex

$$\widetilde{C}^\bullet(R/A, B) = \prod_{n=0}^{\infty} \underline{\mathrm{Hom}}_{A \otimes A^{\mathrm{op}}}(R \otimes_A \ldots \otimes_A R, B) \tag{3.5}$$

□

Let $f : A \to B$ be a morphism of DG algebras. A semi-free resolution of B over A is a semi-free DG algebra over A together with a surjective quasi-isomorphism $\pi : R \to B$ whose restriction to A is f.

$$\begin{tikzcd} & R \arrow[d, "\pi"] \\ A \arrow[ur] \arrow[r, "f"] & B \end{tikzcd}$$

Any two such resolutions of the same B are homotopy equivalent over A.

Now define $\Omega^1_{R/A}$ to be the DG bimodule generated by symbols dr, $r \in R$, that are k-linear in r and of degree $j+1$ for $r \in R^j$, subject to relations

$$d(r_1 r_2) = dr_1 r_2 + (-1)^{|r_1|} r_1 dr_2;\ da = 0, a \in A. \tag{3.6}$$

Put

$$\mathrm{DR}^1(R/A) = \Omega^1_{R/A}/[R, \Omega^1_{R/A}]; \tag{3.7}$$

define

$$\mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A] \xrightarrow{B} \mathrm{DR}^1(R/A) \tag{3.8}$$

by

$$b(r_0 dr_1 r_2) = (-1)^{|r_0|}(|r_1| + |r_2| + 1)[r_1, r_2 r_0];\ Br = dr$$

PROPOSITION 3.2.2. *Let $f : A \to B$ be a morphism of DGA. Let R be a resolution of B which is quasi-free over A. Then the complex*

$$\mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A] \tag{3.9}$$

is quasi-isomorphic to $\mathrm{Cone}(C_\bullet(A) \xrightarrow{f} C_\bullet(B))$; *the complex*

$$\ldots \xrightarrow{B} \mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A] \xrightarrow{B} \mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A]$$

is quasi-isomorphic to $\mathrm{Cone}(\mathrm{CC}_\bullet(A) \xrightarrow{f} \mathrm{CC}_\bullet(B))$; *and similarly for the negative and periodic cyclic complexes.*

PROOF. Consider resolutions Q of A and $\mathbf{R}$ of B such that the following diagram is commutative and $\mathbf{R}$ is semi-free over Q.

$$\begin{tikzcd} Q \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_A"] & \mathbf{R} \arrow[d, "\pi_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

LEMMA 3.2.3. *Proposition 3.2.2 is true with* (3.9) *replaced by*

$$\mathrm{DR}^1(\mathbf{R}/Q) \xrightarrow{b} (\mathbf{R}/Q)/[Q, \mathbf{R}/Q] \tag{3.10}$$

where Q and $\mathbf{R}$ are as above.

PROOF. For a DG algebra D and a DG bimodule M we will write

$$M_{\sharp,D} = M/[D, M] = M \otimes_{D\otimes D^{\mathrm{op}}} D \tag{3.11}$$

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & (\Omega^1_Q \otimes_Q \mathbf{R})_{\sharp,Q} \arrow[r] \arrow[d, "b"] & (\Omega^1_{\mathbf{R}})_{\sharp,\mathbf{R}} \arrow[r] \arrow[d, "b"] & (\Omega^1_{\mathbf{R}/Q})_{\sharp,\mathbf{R}} \arrow[r] \arrow[d, "b"] & 0 \\ 0 \arrow[r] & Q + [Q, \mathbf{R}] \arrow[r] & \mathbf{R} \arrow[r] & (\mathbf{R}/Q)_{\sharp,Q} \arrow[r] & 0 \end{tikzcd}$$

We observe that its rows are exact. Now, the left column of this diagram fits into its own diagram with short exact rows:

$$\begin{tikzcd} 0 \arrow[r] & (\Omega^1_Q)_{\sharp,Q} \arrow[r] \arrow[d] & (\Omega^1_Q \otimes_Q \mathbf{R})_{\sharp,Q} \arrow[r] \arrow[d] & (\Omega^1_Q \otimes_Q (\mathbf{R}/Q))_{\sharp,Q} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Q \arrow[r] & Q + [Q, \mathbf{R}] \arrow[r] & [Q, \mathbf{R}/Q] \arrow[r] & 0 \end{tikzcd}$$

We claim that the right column is an acyclic complex. In fact, the complex

$$(\Omega^1_Q \otimes_Q (R/Q))_{\sharp,Q} \to R$$

is quasi-isomorphic to the Hochschild complex $C_\bullet(Q, R/Q)$. Because R/Q is a semi-free bimodule over Q, the projection

$$C_\bullet(Q, R/Q) \to (R/Q)/[Q, R/Q]$$

is a quasi-isomorphism, whence the claim. □

Now consider a diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "\pi_A"] & \mathbf{R} \arrow[d] \\ A \arrow[r] \arrow[dr, "f"] & R \arrow[d, "\pi_B"] \\ & B \end{tikzcd}$$

It remains to compare the column complexes

$$
\begin{tikzcd}
(\Omega^1_{\mathbf{R}/Q})_{\sharp,\mathbf{R}} \arrow[r] \arrow[d] & (\Omega^1_{R/A})_{\sharp,R} \arrow[d] \\
(\mathbf{R}/Q)_{\sharp,Q} \arrow[r] & (R/A)_{\sharp,A}
\end{tikzcd}
\tag{3.12}
$$

We claim that the horizontal maps induce their quasi-isomorphism. Indeed, for a morphism of algebras $D \to E$ and for an E-bimodule M, define the relative Hochschild complex $C_\bullet(E/D, M)$ by

$$C_n(E/D, M) = (M \otimes_D (E/D) \otimes_D \ldots \otimes_D (E/D))_{\sharp,D}$$

where there are n factors E/D; the Hochschild differential b is given by the usual formula. We claim that the projections of $C_\bullet(R/A, R)$ to the left column and of $C_\bullet(\mathbf{R}/Q, \mathbf{R})$ to the left column of (3.12) are quasi-isomorphisms. This is easily seen, for example, by observing that Lemma 1.0.3 holds for a D-free algebra $E = D{*}T(V)$ for any algebra D, with the identical proof. Consequently, Proposition 3.1.1 also admits generalization to the case of semi-free DGA over a DGA D. Finally, observe that

$$C_\bullet(\mathbf{R}/Q, \mathbf{R}) \to C_\bullet(R/A, R)$$

is a quasi-isomorphism. $\square$

CHAPTER 8

Examples

1. Introduction

2. Polynomial algebras

2.1. Hochschild homology of algebras of polynomials. Let $A = k[x_1, \ldots, x_n]$. Let $\mathcal{B}_\bullet$ be the bar resolution of the A-bimodule A. One has

$$\mathcal{B}_p = A \otimes \overline{A}^{\otimes p} \otimes A \tag{2.1}$$

with the differential b' as in (1.6). Let $\mathcal{K}_\bullet$ be the Koszul resolution. By definition,

$$\mathcal{K}_p = A \otimes \wedge V \otimes A$$

where $V = \oplus_{j=1}^n k \cdot x_j$. The differential, that we also denote by b', acts as follows:

$$b'(a \otimes (v_1 \wedge \ldots \wedge v_p) \otimes b) = \sum_{j-1}^{p} (-1)^{j-1} a v_j \otimes (v_1 \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_p) \otimes b \tag{2.2}$$

$$- \sum_{j-1}^{p} (-1)^{j+p} a \otimes (v_1 \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_p) \otimes v_j b$$

This is a complex of A-bimodules (A acts on it by left and right multiplication). Moreover, it is a free bimodule resolution of A. One can see this, for example, by observing that the complex $\mathcal{K}_\bullet$ is the nth tensor power of $\mathcal{K}_\bullet(1)$, the latter being the Koszul resolution for $n = 1$.

There is an embedding $\mathcal{K}_\bullet \xrightarrow{i} \mathcal{B}_\bullet$ given by

$$i(a \otimes (v_1 \wedge \ldots \wedge v_p) \otimes b) = \sum_{\sigma \in \Sigma_p} (-1)^{\mathrm{sgn}(\sigma)} a \otimes (v_{\sigma 1} \otimes \ldots \otimes v_{\sigma p}) \otimes b \tag{2.3}$$

For any commutative k-algebra A, let $\Omega^1_{A/k}$ be the module of Kähler differentials of A. Denote

$$\Omega^p_{A/k} = \wedge^p_A \Omega^1_{A/k} \tag{2.4}$$

The exterior product makes $\Omega^\bullet_{A/k}$ a graded commutative algebra. There is unique graded derivation d of degree one that sends a to da and da to zero for every a in A.

Proposition 2.1.1. *The embedding* (2.3) *induces an isomorphism*

$$\Omega^p_{k[x_1,\ldots,x_n]/k} \xrightarrow{\sim} \mathrm{HH}_p(k[x_1, \ldots, x_n])$$

Proof. Follows from the isomorphism $\mathcal{K} \otimes_{A \otimes A^{\mathrm{op}}} A \xrightarrow{\sim} \Omega^\bullet_{A/k}$. $\square$

2.2. The HKR map. For any commutative algebra A, define the map [?]

$$I_{\mathrm{HKR}} : C_\bullet(A) \to \Omega^\bullet_{A/k}$$

by

$$I_{\mathrm{HKR}} : a_0 \otimes \ldots \otimes a_p \mapsto \frac{1}{p!} a_0 da_1 \ldots \otimes da_p \tag{2.5}$$

LEMMA 2.2.1. *The above map is a morphism of complexes $(C_\bullet(A), b) \to (\Omega^\bullet_{A/k}, 0)$. One has $I_{\mathrm{HKR}} \circ B = d \circ I_{\mathrm{HKR}}$.*

This is verified by a direct computation. For $A = k[x_1, \ldots, x_n]$, the HKR map is a left inverse to the map induced by i as in (2.3). Therefore, when A is a polynomial algebra, the HKR map is a is a quasi-isomorphism of complexes. We will later specified by which morphism of free resolutions it is induced.

2.3. More details on the Hochschild homology of polynomial algebras. The standard procedure of homological algebra provides morphisms of resolutions $\mathcal{B}_\bullet \overset{\longrightarrow}{\longleftarrow} \mathcal{K}_\bullet$ over A which are homotopy inverse. In this subsection we will construct them explicitly, together with the homotopies and with the maps induced by them on the Hochschild complex and on Kähler differentials. This will be used later to establish analogues of Proposition (2.1.1).

Start with observing that $A = k[x_1, \ldots, x_n] \overset{\sim}{\to} A_1^{\otimes n}$ where $A_1 = k[x]$. So start with the case $n = 1$. Let $\mathcal{B}_\bullet(1)$ and $\mathcal{K}_\bullet(1)$ be the bar and Koszul resolutions for $n = 1$. We have the map $j : \mathcal{B}_\bullet(1) \to \mathcal{K}_\bullet(1)$ given by $j(a_0 \otimes a_1) = a_0 \otimes a_1$;

$$j(a_0 \otimes x^m \otimes a_2) = \sum_{k=0}^{m-1} a_0 x^k \otimes x \otimes x^{m-1-k} a_2; \tag{2.6}$$

$j = 0$ on $\mathcal{B}_p(1)$ for $p = 1$. In other words, if we identify $\mathcal{B}_1(1)$ with $k[x, y, z]$ and $\mathcal{K}_1(1)$ with $k[x, z] \otimes ky$, then

$$j : f(x, y, z) \mapsto \frac{f(x, x, z) - f(x, z, z)}{x - z} \otimes y.$$

We have $j \circ i = \mathrm{id}$, whereas $i \circ j = [b', s]$ where $s : \mathcal{B}_p(1) \to \mathcal{B}_{p+1}(1)$ can be chosen as follows. Let us use the notation

$$\frac{f(x) - f(y)}{x - y} = \sum f^{(1)}(x) f^{(2)}(y) \tag{2.7}$$

Define

$$s(a_0 \otimes \ldots \otimes a_{p+1}) = (-1)^p \sum a_0 \otimes \ldots \otimes a_{p-1} \otimes a_p^{(1)} \otimes x \otimes a_p^{(2)} a_{p+1}$$

for $p > 0$ and $s(a_0 \otimes a_1) = 0$. The fact that s is indeed a homotopy for $i \circ j - \mathrm{id}$ follows from the identities

$$\sum (a^{(1)} x \otimes a^{(2)} - a^{(1)} \otimes x a^{(2)}) = a \otimes 1 - 1 \otimes a;$$

$$\sum (a_1 a_2)^{(1)} \otimes (a_1 a_2)^{(2)} = \sum a_1 a_2^{(1)} \otimes a_2^{(2)} + \sum a_1^{(1)} \otimes a_1^{(2)} a_2$$

For general n, as we mentioned before, $\mathcal{K}_\bullet \overset{\sim}{\to} \mathcal{K}_\bullet(1)^{\otimes n}$. There are two standard morphisms of resolutions (0.4), (2.2)

$$\mathrm{EZ} : \mathcal{B}_\bullet(1)^{\otimes n} \longrightarrow \mathcal{B}_\bullet; \ \mathrm{AW} : \mathcal{B}_\bullet(1)^{\otimes n} \longleftarrow \mathcal{B}_\bullet.$$

Both morphisms EZ and AW are associative in the obvious sense. This allows us to define

$$\mathrm{EZ} : \otimes_{j=1}^{n} \mathcal{B}_\bullet(A_j) \longrightarrow \mathcal{B}_\bullet(\otimes_{j=1}^{n} A_j)$$

and

$$\mathrm{AW} : \otimes_{j=1}^{n} \mathcal{B}_\bullet(A_j) \longleftarrow \mathcal{B}_\bullet(\otimes_{j=1}^{n} A_j)$$

One has $\mathrm{AW} \circ \mathrm{EZ} = \mathrm{id}$; in Lemma 2.0.2 we constructed an explicit homotopy t for $\mathrm{id} - \mathrm{EZ} \circ \mathrm{AW}$ for $n = 2$. We can easily extend it to the case of any n. All that we need to know here is that the element

$$t[a_1 \otimes \ldots \otimes a_n | a'_1 \otimes \ldots \otimes a'_n | \ldots]$$

is given by an algebraic expression involving taking elements a_j, a'_j, *etc.* from one position to another and multiplying them with some other elements.

Note that while EZ is commutative in the obvious sense, AW is not. For $\sigma \in \Sigma_n$, let AW^σ be the map AW constructed for the product $A_{\sigma 1} \otimes \ldots \otimes A_{\sigma_n}$ Put

$$\mathrm{AW}^{\mathrm{sym}} = \frac{1}{n!} \sum_\sigma \mathrm{AW}^\sigma \tag{2.8}$$

The same argument as above allows to construct a homotopy t for $\mathrm{AW}^{\mathrm{sym}}$, of the same algebraic nature as discussed above. Now apply this to the case when $A_1 = \ldots = A_n = k[x]$. We have morphisms

$$\mathrm{EZ} : \mathcal{B}_\bullet(1)^{\otimes n} \longrightarrow \mathcal{B}_\bullet;\ \mathrm{AW}^{\mathrm{sym}} : \mathcal{B}_\bullet(1)^{\otimes n} \longleftarrow \mathcal{B}_\bullet,$$

as well as

$$i^{\otimes n} : \mathcal{K}_\bullet \longrightarrow \mathcal{B}_\bullet(1)^{\otimes n};\ j^{\otimes n} : \mathcal{K}_\bullet \longleftarrow \mathcal{B}_\bullet(1)^{\otimes n}.$$

The homotopy for $\mathrm{id} - i^{\otimes n} \circ j^{\otimes n}$ can be easily constructed from the one for $\mathrm{id} - ij$ for $n = 1$, for example one can take

$$s^{\otimes n} = \sum_{k=1}^{n} (\mathrm{id} - ij)^{\otimes(k-1)} \otimes s \otimes \mathrm{id}^{n-1-k}$$

Observe that

$$i = i^{\otimes n} \circ \mathrm{EZ};$$

define

$$j = \mathrm{AW}^{\mathrm{sym}} \circ j^{\otimes n};$$

we have

$$ij = i^{\otimes n} \mathrm{EZ} \circ \mathrm{AW}^{\mathrm{sym}} j^{\otimes n} = i^{\otimes n} j^{\otimes n} - i^{\otimes n}[b', t] j^{\otimes n} = -[b', s^{\otimes n}] - i^{\otimes n}[b', t] j^{\otimes n}$$

therefore we can chose the homotopy for $\mathrm{id} - ij$ to be

$$h = -s^{\otimes n} - i^{\otimes n} t j^{\otimes n}$$

Note also that the map $C_\bullet(A) \to \Omega^\bullet_{A/k}$ induced by j is I_{HKR}.

DEFINITION 2.3.1. *Set*

$$C_p(n) = k[x_j^{(k)} | 1 \le j \le n;\ 0 \le k \le p]$$

A generalized differential operator is a linear map $C_p(n) \to C_q(n)$ that is a linear combination of compositions of the following maps:

1) $T_j(k, l)$ that substitutes $x_j^{(k)}$ in place of $x_j^{(l)}$

2) The map

$$D_j(k;l,m)f = \frac{T_j(k,l)f - T_j(k,m)f}{x_j^{(l)} - x_j^{(m)}};$$

3) partial derivatives.

More generally, if a generalized differential operator sends a subspace L to a subspace L', the induced operator on quotients will be also called a generalized differential operator.

Let us identify $C_p(k[x_1, \ldots, x_n])$ with the quotient of $C_p(n)$. Recall the HKR map (2.5). Put

$$i(f dx_{j_1} \ldots dx_{j_p}) = (-1)^{\mathrm{sign}\sigma} \frac{1}{p!} \sum_{\sigma \in \Sigma_p} f \otimes x_{j_{\sigma 1}} \otimes \ldots \otimes x_{j_{\sigma p}} \tag{2.9}$$

PROPOSITION 2.3.2. *There is a generalized differential operator $h : C_\bullet(k[x_1, \ldots, k_n]) \to C_{\bullet+1}(k[x_1, \ldots, k_n])$ such that*

$$\mathrm{id} - i \circ I_{\mathrm{HKR}} = [b, h].$$

2.4. Completed Hochschild complexes of commutative algebras. Now consider an ideal I in any commutative algebra P. Consider the embeddings $i_k : P \to P^{\otimes(m+1)}$, $0 \le k \le m$, given by $a \mapsto 1 \otimes \ldots \otimes a \otimes \ldots \otimes 1$. For every $m \ge 0$, let I_Δ be the ideal in $P^{\otimes m+1}$ generated by $i_k(a) - i_l(a), a \in P$, and by $i_k(a)$, $a \in I$, for all possible k and l. Denote by $\widehat{C}^{\mathrm{un}}_m(P)_{\Delta,I}$ the completion of $P^{\otimes m+1}$ with respect to I_Δ.

We write

$$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots a_i a_{i+1} \otimes \ldots \otimes a_{n+1} \tag{2.10}$$

for $i < n$;

$$d_n(a_0 \otimes \ldots \otimes a_{n+1}) = a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}; \tag{2.11}$$

$$s_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots a_i \otimes 1 \otimes \ldots \otimes a_n \tag{2.12}$$

for $i \le n$. One has

$$b = \sum_{j=0}^{n} (-1)^j d_j$$

Note that all d_j and s_j are algebra homomorphisms preserving I_Δ. Therefore they all extend to $\widehat{C}^{\mathrm{un}}_m(P)_{\Delta,I}$. We denote the quotient by the sum of images of all s_j by $\widehat{C}_m(P)_{\Delta,I}$.

PROPOSITION 2.4.1. *Let $P = k[x_1, \ldots, x_n]$. Then I_{HKR} induces a quasi-isomorphism*

$$(\widehat{C}_m(P)_{\Delta,I}, b) \to (\widehat{\Omega}^\bullet_{P/k}, 0)$$

where the right hand side stands for the I-adic completion.

PROOF. In fact, any map from 1), Definition 2.3.1, is a ring homomorphism preserving I_Δ. Any map D from 2) satisfies $D(fg) = S(f)D(g) + D(f)T(g)$ where S and T are as in 1). Therefore D sends I_Δ^{N+1} to I_Δ^N. We see that any generalized differential operator extends to the completed complex, and the statement follows from Proposition 2.3.2. □

For a commutative algebra A, denote by $\widehat{C}_\bullet(A)$ the complex $\widehat{C}_\bullet(A)_{\Delta,0}$ defined before Proposition 2.4.1 (i.e. the case $I = 0$).

PROPOSITION 2.4.2. *For a Noetherian commutative algebra A, the inclusion*

$$C_\bullet(A) \to \widehat{C}_\bullet(A)$$

is a quasi-isomorphism.

PROOF. For any algebra P, any P-module M, and any ideal I of P, denote by $\widehat{M}_I$ the I-adic completion of M.

LEMMA 2.4.3. *Let P a Noetherian algebra and let $J_0 \subset J$ be two ideals of P. Then the map*

$$\widehat{P}_J/J_0\widehat{P}_J \longrightarrow (\widehat{P/J_0})_{J/J_0}$$

is an isomorphism.

PROOF. To prove that the map is injective, note that the right hand side is the same as $(\widehat{P/J_0})_J$ and that completion is right exact (actually exact) on finitely generated modules. To prove injectivity, let $(p_N \in P/J^{N+1})|N \geq 0$ be an element of the kernel, such that $p_0 = 0$. Then $p_N \in J_0/J_0 \cap J^{N+1}$. Lift p_N to elements $\widetilde{p}_N$ of J_0. Then $\widetilde{p}_N - \widetilde{p}_{N+1} \in J_0 \cap J^N$. By Artin-Rees lemma [?], there exists $d \geq 0$ such that $J_0 \cap J^N = J^{N-d}(J_0 \cap J^d)$. Let $x_1, \ldots, x_m$ be generators of J_0. Then

$$\widetilde{p}_{N+1} - \widetilde{p}_N = \sum_{j=1}^{m} a_j^{(N)} x_j$$

where $a_j^{(N)}$ is in J^{N-d}. Put

$$a_j = \sum_{N=1}^{\infty} a_j^{(N)}.$$

Then

$$\sum_{N=1}^{\infty} (p_{N+1} - p_N) = \sum_{j=1}^{m} a_j x_j$$

□

Let $\widehat{\mathcal{B}}_m$ be the completion of $A^{\otimes(m+2)}$ by the ideal J_Δ generated by all $i_k(a) - i_l(a)$ for $0 \leq k, l \leq m+1$. Let J_δ be the ideal generated by all $i_0(a) - i_{m+1}(a)$. Apply the lemma to J_Δ instead of J_0 and J_Δ instead of J. We see that

$$\mathcal{B}_m \otimes_{A^e} A \xrightarrow{\sim} \widehat{C}^{\mathrm{un}}(A).$$

We have ring morphisms

$$A \otimes A \longrightarrow A^{\otimes(m+2)} \longrightarrow \widehat{\mathcal{B}}_m$$

(the one to the left given by $i_0 \otimes 1^{\otimes m} \otimes i_m$). Each algebra to the right is flat over its neighbor on the left [?]. Therefore $\widehat{\mathcal{B}}_m$ is flat over $A \otimes A$. Endowed with the differential b', it is a flat resolution of A over A^e because the usual homotopy $a_0 \otimes \ldots \mapsto 1 \otimes a_0 \otimes \ldots$ extends to it. We conclude that $\widehat{C}^{\mathrm{un}}_\bullet(A)$, and therefore $\widehat{C}_\bullet(A)$, computes the Hochschild homology of A. □

3. Periodic cyclic homology of finitely generated commutative algebras

For a finitely generated commutative algebra A, choose an algebra of polynomials $P = k[x_1, \dots, x_n]$ and an epimorphism $P \to A$ with the kernel I. Put

$$\widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(P)_{\Delta,I} = (\widehat{C}_{\bullet}(P)_{\Delta,I}((u)), b + uB)$$

$$\widehat{\mathrm{CC}}_{\bullet}(A) = (\widehat{C}_{\bullet}(A)((u)), b + uB)$$

$$(\widehat{\Omega}^{\bullet}_{P/k})_I = \varprojlim \Omega^{\bullet}_{P/k}/I^{N+1}\Omega^{\bullet}_{P/k}$$

THEOREM 3.0.1. *Both morphisms*

$$\mathrm{CC}^{\mathrm{per}}(A) \longrightarrow \widehat{\mathrm{CC}}_{\bullet}(A) \longleftarrow \widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(P)_{\Delta,I} \longrightarrow ((\widehat{\Omega}^{\bullet}_{P/k})_I((u)), ud)$$

are quasi-isomorphisms.

PROOF. The first map is a quasi-isomorphism by Proposition 2.4.2, the second by Theorem 3.2.2, and the third by Proposition 2.4.1. $\square$

4. Smooth Noetherian algebras

By definition, a commutative Noetherian k-algebra A is smooth if, for any k-algebra C and any ideal I of C such that $I^2 = 0$, the map $\mathrm{Hom}(A, C) \to \mathrm{Hom}(A, C/I)$ is surjective. The class of smooth algebras includes the class of coordinate rings of nonsingular affine varieties over k.

THEOREM 4.0.1. *(see* [?], [?]*) The HKR map from 2.2 defines quasi-isomorphisms of complexes*

$$C_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}, 0)$$

$$CC^{-}_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}[[u]], ud)$$

$$CC_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}[u^{-1}, u]]/u\Omega^{\bullet}_{A/k}[[u]], ud)$$

$$CC^{\mathrm{per}}_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}[u^{-1}, u]], ud)$$

PROOF. We will need a few standard results from commutative algebra.

(1) Let $\mathfrak{m}$ be a maximal ideal in A. Since A is smooth, its localization $A_{\mathfrak{m}}$ at $\mathfrak{m}$ is a regular local ring and a basis $x_1, \dots x_n$ for $\mathfrak{m}/\mathfrak{m}^2$ over k is a regular generating sequence for the ideal $\mathfrak{m}A_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$.
(2) A morphism of two A-modules is an isomorphism if its localisations at all maximal ideals are isomorphisms.
(3) Suppose that $x_1, \dots x_n$ is a regular sequence generating an ideal $I \subset A$. The associated Koszul complex is a free A-resolution of A/I of the form

$$(\Lambda^*_A(A^n), d), \tag{4.1}$$

where

$$d = \sum_i \iota_i \otimes x_i,$$

(ι_i is the contraction with the i'th standard basis vector in k^n). In particular,

$$Tor^A_*(A/I, A/I) \simeq \Lambda^*_{A/I}(I/I^2)$$

as algebras over A.

The proof procedes as follows. Let $\mu : A \otimes A \to A$ denote the multiplication in A and suppose that $\mathfrak{m}$ is a maximal ideal in A. Applying the Koszul complex computation to the ideal $\mu^{-1}(\mathfrak{m})(A\otimes A)_{\mu^{-1}(\mathfrak{m})} \subset (A\otimes A)_{\mu^{-1}(\mathfrak{m})}$, we get

$$Tor_*^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_\mathfrak{m}, A_\mathfrak{m}) \simeq \Lambda^*_{A_\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2).$$

Since the Hochshild complex computes the Tor-functor, the fact that the map

$$(C_\bullet(A), b) \to (\Omega^\bullet_{A/k}, 0)$$

given by (2.5) is a quasiisomorphism of complexes follows now from the isomorphisms

$$Tor_*^{A\otimes A}(A,A)_\mathfrak{m} \leftarrow Tor_*^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_\mathfrak{m}, A_\mathfrak{m}) \to \Omega^*_{A_\mathfrak{m}/k} \simeq (\Omega^*_{A/k})_\mathfrak{m}.$$

To deal with cyclic complexes one notices that the B-boundary map becomes the de Rham differential on $\Omega^\bullet_{A/k}$. As the result we get a morphism of double complexes, say for negative cyclic complex,

$$(\Omega^\bullet_{A/k}[[u]], ud) \to (CC^-_\bullet(A)[[u]], b + uB)$$

which, by above Hochschild homology case, is a quasiisomorphism on the rows and hence quasiisomorphism of double complexes. The claimed result follows. □

5. Finitely generated commutative algebras

For a finitely generated commutative algebra A over k, choose a surjective homomorphism $f : P \to A$ where P is a ring of polynomials. Let I be the kernel of f. Consider the complexes

$$(5.1) \qquad 0 \to P/I^{n+1} \xrightarrow{d} \Omega^1_{P/k}/I^n\Omega^1_{P/k} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{P/k}/I\Omega^n_{P/k} \xrightarrow{d} 0$$

for all $n \geq 0$. Denote their cohomologies by $H^*_{\mathrm{cris}}(A/k; n)$ Denote by $H^\bullet_{\mathrm{cris}}(A/k)$ the cohomology of the complex $(\widehat{\Omega}^\bullet_{P/k}, d)$, the I-adic completion of the De Rham complex of P. It is well known that the cohomologies above are independent of a choice of P and f.

THEOREM 5.0.1. *(cf. [**206**]). 1. There exists the canonical morphism*

$$\mu : HC_n(A) \to \bigoplus_{i\geq 0} H^{n-2i}_{\mathrm{cris}}(A/k; n-i);$$

if A is smooth then μ is induced by the map from Theorem 4.0.1.

2. If A *is a locally complete intersection then μ is an isomorphism.*

3. There is the canonical isomorphism

$$\mu : HC^{\mathrm{per}}_\bullet(A) \to H^\bullet_{\mathrm{cris}}(A/k)$$

PROOF. □

5.1. Free commutative resolutions. For a commutative algebra morphism $A \to B$, a free commutative resolution of B over A is a differential graded A-algebra $R_\bullet$ with the differential $\partial : R_\bullet \to R_{\bullet-1}$ together with a morphism of DGAs $R_\bullet \xrightarrow{\epsilon} B$ such that:

1. $R_\bullet$ is concentrated in nonnegative degrees;
2. as a graded algebra, R_δ is free commutative over A;
3. the morphism ϵ is a surjective quasi-isomorphism.

Here we view B as a DGA concentrated in degree zero. A morphism of two resolutions $(R_\bullet, \epsilon)$ and $(R'_\bullet, \epsilon')$ is a morphism f of DGA over A such that $\epsilon' f = \epsilon$. Two morphisms f, $g : (R_\bullet, \epsilon) \to (R'_\bullet, \epsilon')$ are homotopic if **** The following facts about resolutions are standard.

PROPOSITION 5.1.1. (1) *A free resolution of B over A always exists;*
(2) *for every two free resolutions of B over A there is a quasi-isomorphism from one to another;*
(3) *every two morphisms between two resolutions are homotopic.*

PROOF. □

6. Smooth functions

For a compact smooth manifold M one can compute the Hochschild and cyclic homology of the algebra $C^\infty(M)$ where the tensor product in the definition of the Hochschild complex is one of the following three:

$$C^\infty(M)^{\otimes n} = C^\infty(M^n); \tag{6.1}$$

$$C^\infty(M)^{\otimes n} = \mathrm{germs}_\Delta\, C^\infty(M^n); \tag{6.2}$$

$$C^\infty(M)^{\otimes n} = \mathrm{jets}_\Delta\, C^\infty(M^n) \tag{6.3}$$

where Δ is the diagonal.

THEOREM 6.0.1. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes

$$C_\bullet(C^\infty(M)) \to (\Omega^\bullet(M), 0)$$

and a $\mathbb{C}[[u]]$-linear, (u)-adically continuous quasi-isomorphism

$$CC^-_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[[u]], ud)$$

Localizing with respect to u, we also get quasi-isomorphisms

$$CC_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]]/u\Omega^\bullet(M)[[u]], ud)$$
$$CC^{\mathrm{per}}_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]], ud)$$

PROOF. The statement for the Hochschild complex for tensor products (6.2, 6.3), follows from Proposition 2.3.2. Indeed, this proposition implies that the homotopy h extends to these tensor products. For the first tensor product, the following construction, due to Alain Connes (see [**?**]), provides a resolution of $C^\infty(M)$ which can be used to prove that μ is a quasi-isomorphism. Suppose first that $\chi(M)$ is zero and hence there exists an everywhere non-zero vector field V on M. Fix a metric on M and define a vector field W in a geodesic neighbourhood U of the diagonal $\Delta \subset M \times M$ by

$$(\exp_x(tV), y) \to \exp_x(tV)_*(tV) \oplus 0 \in T_{(\exp_x(tV), y)}(M \times M).$$

Let W_1 be a vector field on $M \times M$ which vanishes on a neighbourhood U_1 of the diagonal and such that $||W_1|| \geq 1$ on $M \times M \setminus U$, hence, in particular, $\overline{U}_1 \subset U$. Let

$$\pi : M \times M \to M$$

be the projection onto the first factor. The complex

$$(\Gamma(M \times M, \pi^*(\Lambda^\bullet T^*M \otimes \mathbb{C}), \iota_{W+iW_1})$$

is quasiisomorphic to the complex of $(C^\infty(M \times M \times M^{\times \bullet}), b)$ of $C^\infty(M \times M)$-modules and one easily concludes that μ is a quasi-isomorphism. In the case when $\chi(M) \neq 0$, one replaces M by $M \times \mathbb{T}^1$ and uses Künneth formula.

The claim of the theorem for the cyclic complexes follows from the Hochschild-to cyclic spectral sequence. In fact, the HKR map is a quasi-isomorphism at the level of E_1 and therefore is a quasi-isomorphism. $\square$

6.1. Holomorphic functions. Let M be a complex manifold with the structure sheaf $\mathcal{O}_M$ and the sheaf of holomorphic forms $\Omega^\bullet_M$. If one uses one of the following definitions of the tensor product, then $C_\bullet(\mathcal{O}_M)$, etc. are complexes of sheaves:

$$\mathcal{O}_M^{\otimes n} = \mathrm{germs}_\Delta\, \mathcal{O}_{M^n}; \tag{6.4}$$

$$\mathcal{O}_M^{\otimes n} = \mathrm{jets}_\Delta\, \mathcal{O}_{M^n} \tag{6.5}$$

where Δ is the diagonal.

THEOREM 6.1.1. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes of sheaves

$$C_\bullet(\mathcal{O}_M) \to (\Omega^\bullet_M, 0)$$

and a $\mathbb{C}[[u]]$-linear, (u)-adically quasi-isomorphism of complexes of sheaves

$$CC^-_\bullet(\mathcal{O}_M) \to (\Omega^\bullet_M[[u]], ud)$$

Similarly for the complexes $CC_\bullet$ and CC^{per}.

7. Group rings

Let G be a discrete group.

THEOREM 7.0.1.

$$HH_\bullet(k[G]) \simeq \bigoplus_{<x>} H_\bullet(G_x, k)$$

where the sum is taken over all conjugacy classes of G and G_x is the centralizer of an element x of G.

SKETCH OF THE PROOF. Let $Ad(G)$ be the groupoid with objects $g \in G$ and morphisms

$$Mor_{Ad(G)}(h, k) = \{\alpha \in G \mid \alpha h \alpha^{-1} = k\}$$

with the composition of morphisms given by product of the corresponding group elements. It is easy to check, using the Borel construction of the classifying space of $Ad(G)$, that the homology of $Ad(G)$ with coefficients in k coincides with the Hochschild homology of $k[G]$. But $Ad(G)$ splits into a direct sum over the conjugacy classes of G and, given such a conjugacy class $< x >$, the restriction of $Ad(G)$ to the invariant subset $< x >$ of units is Morita equivalent (as a groupoid) to the centralizer G_x of x. $\square$

In particular we get a component $H_\bullet(G,k) \subset HH_\bullet(k[G])$ corresponding to the conjugacy class of the unit element $e \in G$. The following observation is used in the construction of the chern character on the algebraic K-theory.

PROPOSITION 7.0.2. *The image of $H_\bullet(G,k)$ in $HH_\bullet(k[G])$ is in the kernel of*

$$B : HH_\bullet(k[G]) \to HH_{\bullet+1}(k[G]).$$

PROOF. The above decomposition into the components corresponding to the conjugacy classes is preserved by B. The standard (bar) complex computing group homology is quasiisomorphic to the normalised totaly antisymmetrised bar complex. But in the normalised complex $C_n(k[G]) = k[G] \otimes (k[G]/k)^{\otimes n}$ the composition

$$g_1 \otimes \ldots \otimes g_n \to \text{total antisymmetrization of } (e \otimes g_1 \otimes \ldots \otimes g_n)$$

vanishes. □

The above decomposition into conjugacy classes obviously extends to the cyclic homology. The structure of the corresponding components depends on whether a class is elliptic (i.e. of the form $< x >$ where x is of finite order) or not.

THEOREM 7.0.3. [**73**]

$$HC_n(k[G]) \simeq \bigoplus_{<x>|\,\mathrm{ord}\,x<\infty} \bigoplus_i H_{n-2i}(G_x,k) \oplus \bigoplus_{<x>|\,\mathrm{ord}\,x=\infty} H_{n-2i}(G_x/(x),k)$$

For a detailed proof we will refere the reader to the original paper.

REMARK 7.0.4. The exact sequence of the triangle (1.12) splits into a direct sum over the set of conjugacy classes of G.

(1) For the summands corresponding to elliptic classes, the component of the differential $HH_\bullet \to HC_\bullet$ is injective. The proof follows the lines of the above proposition for the class $< e >$. As the result, the components of the spectral sequence (1.14) which correspond to elliptic classes degenerate at E^1 and the corresponding components in the cyclic homology of $k[G]$ are of the form claimed above.
(2) In the non-elliptic case, the corresponding component of the exact sequence is isomorphic to the exact sequence of the fibration

$$B\mathbb{Z} \to BG_x \to B(G_x/(x)).$$

8. Rings of differential operators

For a C^∞ manifold X of dimension n let $D(X)$ be the ring of differential operators on X. We use the tensor products defined analogously to (6.2), (6.3).

THEOREM 8.0.1. *([**70**], [**?**]). There is a quasi-isomorphism*

$$C_\bullet(D(X)) \to (\Omega^{2n-\bullet}(T^*X), d)$$

which extends to a $\mathbb{C}[[u]]$-linear, (u)-adically continuous quasi-isomorphism

$$CC^-_\bullet(D(X)) \to (\Omega^{2n-\bullet}(T^*X)[[u]], d)$$

As in 6, one also has analogous statements for the cyclic and periodic cyclic complexes.

8.1. Holomorphic differential operators. Let X be a complex manifold of complex dimension n. For the sheaf D_X of holomorphic differential operators, define the Hochschild, cyclic, etc. complexes of sheaves using tensor products analogous to those in 6.1. Let $\pi : T^*X \to X$ be the projection.

THEOREM 8.1.1. [**64**] *There exists an isomorphism*

$$\pi^{-1}C_\bullet(D_X) \to (\Omega^\bullet_{T^*X}[2n], d)$$

in the derived category of the category of sheaves on T^*X, which extends to a $\mathbb{C}[[u]]$-linear, (u)-adically continuous isomorphism in the derived category

$$\pi^{-1}CC^-_\bullet(D_X) \to (\Omega^\bullet_{T^*X}[2n][[u]], d)$$

As in 6.1, similar isomorphisms exist for the cyclic and periodic cyclic complexes.

9. Rings of complete symbols

For a compact smooth manifold X, let $CL(X)$ be the algebra of classical pseudo-differential operators. By $L_\infty(X)$ denote the algebra of smoothing operators (i.e. integral operators with smooth kernel), and put

$$CS(X) = CL(X)/L_\infty(X).$$

We use the projective tensor products.

For any manifold M, denote by $\widehat{\Omega}^*(M \times S^1)$ the space of power series

$$\sum_{\epsilon=0,\ 1;\ i=-\infty}^{N} \alpha_{i,\ \epsilon} z^i dz^\epsilon$$

where α_i are forms on M. Denote by S^*X the cosphere bundle of X.

THEOREM 9.0.1. [**?**] *There exists a quasi-isomorphism*

$$C_\bullet(CS(X)) \to (\widehat{\Omega}^{2n-\bullet}(S^*X \times S^1), d)$$

which extends to a $\mathbb{C}[[u]]$-linear, (u)-adically continuous quasi-isomorphism

$$CC^-_\bullet(CS(X)) \to (\widehat{\Omega}^{2n-\bullet}(S^*X \times S^1)[[u]], d)$$

Similarly for $CC_\bullet$, $CC^{\mathrm{per}}_\bullet$. In particular:

COROLLARY 9.0.2.

$$HH_p(CS(X)) = H^{2n-p}(S^*X \times S^1)$$

$$HC_p(CS(X)) = \bigoplus_{i \geq 0} H^{2n-p+2i}(S^*X \times S^1)$$

Combined with the pairing with the fundamental class of $S^*X \times S^1$, the first of the above isomorphisms gives an isomorphism

$$HH_0(CS(X)) = CS(X)/[CS(X), CS(X)] \to \mathbb{C} \tag{9.1}$$

This isomorphism is given by the Wodzicki-Guillemin residue [**?**]. The above theorem has also a holomorphic version where the ring of complete symbols is replaced by the sheaf of microdifferential operators [**51**].

10. Rings of pseudodifferential operators

THEOREM 10.0.1.

$$HH_0(L_\infty(X)) \simeq \mathbb{C};\ \ HH_p(L_\infty(X)) = 0,\ p > 0;$$

$$HC_{2p}(L_\infty(X)) \simeq \mathbb{C};\quad HC_{2p+1}(L_\infty(X)) = 0$$

THEOREM 10.0.2. [**?**]

$$HH_p(CL(X)) \simeq HH_p(CS(X)) \text{ for } p \neq 1;$$

there is an exact sequence

$$0 \to HH_1(CL(X)) \to HH_1(CS(X)) \to \mathbb{C} \to 0$$

THEOREM 10.0.3. *For all $p \geq 0$*

$$HC_{2p}(CL(X)) \simeq HC_{2p}(CS(X))$$

and there is an exact sequence

$$0 \to HC_{2p+1}(CL(X)) \to HC_{2p+1}(CS(X)) \to HC_{2p}(\mathbb{C}) \to 0$$

Theorems 10.0.2 and 10.0.3 follow from Theorem 10.0.1 and from Wodzicki excision theorem 2.0.2.

The results of the two previous subsections were extended to more general rings of symbols and of pseudodifferential operators by Melrose-Nistor and Benameur-Nistor ([**?**], [**?**]). A survey of these results can be found in [**?**].

11. Noncommutative tori

Let $\{\exp 2\pi i\theta_{ij}\}_{i,j}$ be a $n \times n$ matrix representing a class ω in $H^2(\mathbb{Z}^n, \mathbb{T})$. In particular, $\theta_{ij} = -\theta_{ji} \in \mathbb{R}$.

DEFINITION 11.0.1. $Q(\mathbb{T}^n_\theta)$ denote the $*$-algebra over $\mathbb{C}$ with unitary generators $u_1, \ldots, u_n$ subject to relations relation

$$u_i u_j = \exp 2\pi i\theta_{ij} u_j u_i \tag{11.1}$$

We will call $Q(\mathbb{T}^n_\theta)$ the algebra of *rational functions* on the non-commutative n-torus $(\mathbb{T}^n_\theta)$ and will let $\mathcal{Z}$ denote its center.

THEOREM 11.0.2. [**97, ?**] *Let $(f_1, \ldots, f_n)$ be the standard orthonormal basis of $\mathbb{C}^n$. The formula*

$$u_i \otimes \omega \to u_i \otimes f_i \wedge \omega$$

extends uniquely to a derivation $d : Q(\mathbb{T}^n_\theta) \otimes \Lambda^\bullet\mathbb{C}^n \to Q(\mathbb{T}^n_\theta) \otimes \Lambda^{\bullet+1}\mathbb{C}^n$. The following holds.

(1) $H_\bullet(Q(\mathbb{T}^n_\theta)) \simeq \mathcal{Z} \otimes \Lambda^\bullet\mathbb{C}^n$
(2) $HC_k(Q(\mathbb{T}^n_\theta)) \simeq \begin{cases} \Lambda^k\mathbb{C}^n, & \textit{for } k < n \\ \mathcal{Z} \otimes \Lambda^n\mathbb{C}^n, & \textit{for } k = n \end{cases}$
(3) $HC^-_\bullet(Q(\mathbb{T}^n_\theta)) \simeq (\mathcal{Z} \otimes \Lambda^\bullet\mathbb{C}^n[[u]], ud)$.
(4) $HC^{per}_\bullet(Q(\mathbb{T}^n_\theta)) \simeq (\Lambda^\bullet\mathbb{C}^n[u^{-1}, u]], ud)$.

PROOF. For $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{Z}^n$, we set

$$u^\alpha = u_1^{\alpha_1} \ldots u_n^{\alpha_n} \text{ and } u_i u^\alpha u_i^{-1} =< i \mid \alpha > u^\alpha.$$

For simplicity, we will denote the algebra $Q(\mathbb{T}^n_\theta)$ by $\mathcal{A}$. It is easy to check that a free resolution of $\mathcal{A}$ as an $\mathcal{A}$-bimodule has a form

$$(\mathcal{A}^e \otimes \Lambda^\bullet \mathbb{C}^n, t) \xrightarrow{\epsilon} \mathcal{A}, \tag{11.2}$$

where $\epsilon(a\overset{\circ}{b}) = ab$ and

$$t(1 \otimes \omega) = \sum_{i=1}^{n} (1 - u_i(\overset{\circ}{u}_i)^{-1}) \otimes \iota(f_i)\omega.$$

Here ι_v denotes contraction with the vector $v \in \mathbb{C}^n$. Set $v_\alpha = \sum_i (1 - < i \mid \alpha >) f_i$. Tensoring the resolution (11.2) with $\mathcal{A}$ over $\mathcal{A}^e$ produces a direct sum of complexes parametrised by $\alpha \in \mathbb{Z}^n$:

$$\bigoplus_{\alpha \in \mathbb{Z}^n} (\mathbb{C}u^\alpha \otimes \Lambda^\bullet \mathbb{C}^n, 1 \otimes \iota_{v_\alpha}). \tag{11.3}$$

But, since

$$\iota_v(v \wedge \omega) + v \wedge (\iota_v \omega) = ||v||\omega$$

for all $\omega \in \Lambda \mathbb{C}^n$, the complex (11.3) is contractible precisely when $v_\alpha \neq 0$ or, equivalently, when $u^\alpha \notin \mathcal{Z}$. The first statement, about Hochschild homology of $\mathcal{A}$, follows.

The quasi-isomorphism

$$\bigoplus_{\alpha} (\mathbb{C}u^\alpha \otimes \Lambda^\bullet \mathbb{C}^n, 1 \otimes \iota_{v_\alpha}) \to (C_\bullet(\mathcal{A}), b) \tag{11.4}$$

is given by

$$u^\alpha \otimes f_{i_1} \wedge \ldots \wedge f_{i_k} \mapsto u^\alpha \sum_{\sigma \in \mathfrak{S}_k} (-1)^{sgn(\sigma)} (u_{i_{\sigma(1)}} \ldots u_{i_{\sigma(k)}})^{-1} \otimes u_{i_{\sigma(1)}} \otimes \ldots \otimes u_{i_{\sigma(k)}}.$$

The quasi-isomorphism (11.4) intertwines B and d and the rest of the proof is the same as the proof of the theorem 6.0.1. $\square$

Remark 11.0.3. Let $u_1, \ldots u_n$ satisfy the relations the definition 11.0.1. The smooth version of the non-commutative torus is given by the algebra $C^\infty(\mathbb{T}^n_\theta)$ of sums of the form

$$\sum_{\alpha \in \mathbb{Z}^n} a_\alpha u^\alpha, \ \{a_\alpha\} \in \mathcal{S}(\mathbb{Z}^n).$$

$\mathcal{S}(\mathbb{Z}^n)$ stands for the Schwartz space of rapidly decreasng sequences:

$$\sum_\alpha |\alpha|^N |a_\alpha| < \infty \text{ for all } N \in \mathcal{N},$$

where $|\alpha| = \sum_k |\alpha_k|$.

The definition of the Hochschild and cyclic complexes is the same as before with the proviso that all tensor products $C^\infty(\mathbb{T}^n_\theta)^{\otimes} k$ are completed in the Fréchet topology (from $\mathcal{S}(\mathbb{Z}^n)$). The Hochschild homology of $C^\infty(\mathbb{T}^n_\theta)$ depends on the properties of the sequence

$$\alpha \to (1 - ||v_\alpha||^2)^{-1}, \ u^\alpha \notin \mathcal{Z}.$$

As long as it is a multiplier of $\mathcal{S}(\mathbb{Z}^n)$, for example for generic irrational θ, the results are the same. Otherwise the Hochshild homology is much larger. The periodic cyclic homology does not depend on θ and is the same as the one for $Q(\mathbb{T}^n_\theta)$ (and hence equal to $H^\bullet(\mathbb{T}^n)$).

12. Topological algebras

13. Bibliographical notes

CHAPTER 9

Characteristic classes

1. Introduction

2. Chern character on K_0

Let A be an associative algebra, as usual over a commutative unital ring k. Recall that the abelian group $K_0(A)$ is defined as the universal abelian group generated by the stable isomorphism classes of idempotents in $M_\infty(A)$ under the addition given by direct sum.

Definition 2.0.1. *Let p be an idempotent in $M_n(A)$. The chern character $ch(p)$ of p is the image of the class of 1 in $CC_0^-(k)$ under the composition*

$$CC_0^-(k) \simeq CC_0^-(M_n(k)) \to CC_0^-(M_n(A)) \simeq CC_0^-(A)$$

where the middle map is induced by the homomorphism

$$\phi_p : k \ni \lambda \mapsto \lambda p \in M_n(A). \tag{2.1}$$

It is easy to see that it extends to a homomorphism

$$ch : K_0(A) \to CC_0^-(A).$$

An easy computation gives the following formula

Proposition 2.0.2. *Let $p \in M_n(A)$ be an idempotent. Then*

$$ch(p) = \left(p + \sum_{n>0} (-1)^n \frac{(2n)!}{(n!)^2} u^n (p - \frac{1}{2}) \otimes p^{\otimes 2n} \right)$$

Proof. It is easy to check directly that the above formula does indeed define a class in $CC_0^-(A)$. To see that it is indeed the image of the class of $1 \in CC_0^-(k)$, it is enough to check that our formula is true in the case when $A = kp \oplus k(1-p)$. This is easily seen using the splitting exact sequence of negative cyclic homology associated to the split exact sequence

$$0 \longrightarrow k \xrightarrow{\phi_p} A \longrightarrow k \longrightarrow 0.$$

□

3. Chern character on higher algebraic K-theory of algebras

The starting point is a simple observation.

Lemma 3.0.1. *The map*

$$G^n \ni (g_1, \dots, g_n) \to (g_1 g_2 \dots g_n)^{-1} \otimes g_1 \dots \otimes g_n \in C_n(k[G])$$

extends to a morphism of complexes

$$C_\bullet(BG, \mathbb{Z}) \to C_\bullet(k[G])_{<e>},$$

where the left hand side stands for the singular chain complex of the standard simplicial model of BG and the right hand side stands for the part of the Hochschild complex of the group ring $k[G]$ localised at the cojugacy class of the unit $e \in G$.

Let us apply it to the case of discrete group $G = GL_n(A)$, where A is an associative algebra. Let

$$\tau : k[GL_n(A)] \to M_n(A) \tag{3.1}$$

be the homomorphism of rings induced by the inclusion $GL_n(A) \subset M_n(A)$.

PROPOSITION 3.0.2. *The composition*

$$C_n(BGL_k(A), \mathbb{Z}) \longrightarrow C_n(k[GL_k(A)])_{<e>} \xrightarrow{\tau_*} C_n(M_k(A)) \xrightarrow{\#} C_n(A), \tag{3.2}$$

where # is the trace map (5.2), extends to a morphism of complexes

$$C_\bullet(BGL_k(A), \mathbb{Z}) \to C^-_\bullet(A).$$

PROOF. Recall that, by the remark 7.0.4, B vanishes on the image of $H_n(k[GL_k(A)])_{<e>}$. Another way of formulating this is thet the map

$$C_\bullet(k[GL_k(A)])_{<e>} \to CC_\bullet(k[GL_k(A)])_{<e>}$$

is injective on homology. So suppose that x_n is a cycle representing a class in $H_n(k[GL_k(A)])_{<e>}$. Then uBx_n is zero cycle in cyclic homology, hence it is zero in Hochschild homology, i. e. $uBx_n = buy_{n+1}$ for some uy_{n+1}. Set $x_{n+2} = By_{n+1}$. Then $-u^2x_{n+2}$ is a cycle in Hochschild homology which vanishes in cyclic homology (in fact it is the boundary of $-uy_{n+1} + x_n$), hence $x_{n+2} = by_{n+3}$ for some y_{n+3}. By induction we get a sequence $(x_{n+2k},\ k \geq 0)$ and

$$\tilde{x}_n = \sum_{k \geq 0} u^k x_{n+2k}$$

is a class in negative cyclic homology extending x_n. It is easy to see that the class of $\tilde{x}_n$ in $CC^-_n(k[GL_k(A)])$ is independent of the choices made in this construction. □

Before formulating the definition below, recall that algebraic K-theory of a ring R is defined as follows. One constructs the space $BGl_\infty(A)^+$ by adding a few cells to $BGL_\infty(A)$ - essentially by killing the commutator subgroup of $GL_\infty(A)$ - so that

(1) $H_\bullet(BGL_\infty(A), \mathbb{Z}) \to H_\bullet(BGL_\infty(A)^+, \mathbb{Z})$ is an isomorphism;
(2) $K^{alg}_i(A) = \pi_i(BGL_\infty(A)^+)$.

DEFINITION 3.0.3. *The Chern character*

$$ch_n : K^{alg}_n(A) \to CC^-_n(A)$$

for $n \geq 1$ is given by the composition

$$\begin{aligned} K^{alg}_n(A) = \pi_n(BGL_\infty(A)_+) \to H_n(BGL_\infty(A)^+) \simeq \\ \simeq H_n(BGL_\infty(A)) \to CC^-_n(A). \end{aligned}$$

Here the first arrow is the Hurewicz homomorphism and the second arrow is constructed in the proposition 3.0.2 above.

The particular case of K_1 deserves a separate formulation.

PROPOSITION 3.0.4. *Let $U \in GL_\infty(A)$. Then*

$$ch_1(U) = \#\left(\sum_{k=0}^{\infty}(-1)^k\frac{(k+1)!}{(2k+1)!}(U^{-1}-1)\otimes U\otimes(U^{-1}\otimes U)^{\otimes k}u^k\right).$$

?

PROOF. Given an invertible element U of $GL_\infty(A)$, the associated Hochschild cycle is of the form $U^{-1}\otimes U$ and it has an extension (unique) to a negative cyclic homology class, easily seen to be of the claimed form.

REMARK 3.0.5. An alternative construction of the chern character follows from the following simple observation.

PROPOSITION 3.0.6. *The map*

$$t : (g_0, \ldots, g_n) \mapsto ((g_0 \ldots g_n)^{-1}, g_1 \ldots g_n, g_1 \ldots, g_{n-1})$$

gives the simplicial model of $BGL(A)$ above the structure of a cyclic module and the Dennis map from the lemma 3.0.1 is a morphism of cyclic objects.

□

4. The Chern character in topological and relative K-theory

In this section we will work with a topological algebra A over a field k of characteristic zero. The (mild) assumptions that we make will be listed below.

4.1. The topological and the relative K-theory. Recall that the topological K-theory $K^{top}(A)$ of A is given by the homotopy groups of $BGL_\infty(A)^{top}$, where $GL_\infty(A)^{top}$ is considered as a topological group. We assume that the commutator subgroup of $GL_\infty(A)^{top}$ is in the connected component of identity. Then there is a homotopy fibration

$$BGL_\infty(A)^+ \to BGL_\infty(A)^{top}. \tag{4.1}$$

DEFINITION 4.1.1. *The relative K-theory $K_\bullet^{rel}(A)$ of A is given by the homotopy groups of the fiber of the fibration* (4.1)*. In particular there exists a boundary map*

$$K_{k+1}^{top}(A) \xrightarrow{\delta} K_k^{rel}(A)\ ,\ k \geq 1$$

and the relative K-theory fits into a long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & K_k^{rel}(A) \arrow[r] & K_k^{alg}(A) \arrow[r] & K_k^{top}(A) \arrow[dll, dashed, rounded corners] & \\
 & K_1^{rel}(A) \arrow[r] & K_1^{alg}(A) \arrow[r] & K_1^{top}(A) \arrow[r] & 0.
\end{tikzcd}$$

In what follows, we will define the completed (negative, periodic) cyclic homology $\widehat{\mathrm{HC}}_\bullet(A)$ and the Chern character on the topological K-theory is the map

$$ch : K_\bullet^{top}(A) \to \widehat{\mathrm{HC}}_\bullet^{per}(A)$$

We will prove the following

THEOREM 4.1.2. *The restriction of the Chern character from (3.0.3) to the image of the relative K-theory in the algebraic K-theory factors through a map* $K_\bullet^{rel}(A) \to \mathrm{HC}_{\bullet-1}(A)$ *and the following diagram is commutative*

$$\begin{tikzcd} \dots \arrow[r] & K_k^{rel}(A) \arrow[r] \arrow[d] & K_k^{alg}(A) \arrow[r] \arrow[d] & K_k^{top}(A) \arrow[r] \arrow[d] & K_{k-1}^{rel}(A) \arrow[r] \arrow[d] & \dots . \\ \dots \arrow[r] & \widehat{\mathrm{HC}}_{k-1}(A) \arrow[r] & \widehat{\mathrm{HC}}_k^-(A) \arrow[r] & \widehat{\mathrm{HC}}_k^{\mathrm{per}}(A) \arrow[r] & \widehat{\mathrm{HC}}_{k-2}(A) \arrow[r] & \dots \end{tikzcd}$$

The bottom exact sequence here is induced by (the completed version of) the short exact sequence of complexes (1.13).

The rest of this section is devoted to constructing the Chern characters from K^{top} and from K^{top} and to proving the theorem.

4.2. The Dold-Kan correspondence. One can pass from complexes of Abelian groups to simplicial Abelian groups as follows. Let $\mathbf{D}$ be the category whose objects are $[m]$, $m \in \mathbb{Z}$, and the only morphisms are multiples iof the units and of $d : [m] \to [m-1]$ such that $d^2 = 0$. In other words, $\mathbf{D}$-modules are complexes of Abelian groups. Consider the $(\mathbf{D}, \mathbb{Z}\Delta)$-bimodule $C_\bullet(\Delta^*)$ or, in other words, the cosimplicial object in complexes whose value at $[n]$ is the normalized chain complex of the n-simplex.

This is precisely the complex of cosimplicial Abelian groups

$$\dots \xrightarrow{b} \Delta^{\mathrm{op}}([\bullet],[1]) \xrightarrow{b} \Delta^{\mathrm{op}}([\bullet],[0]) \tag{4.2}$$

with $b = \sum_{j=0}^{n}(-1)^j d_j$ that we used in **??**. One has

$$C_\bullet(\Delta^*) \otimes_{\mathbb{Z}\Delta} A_* = (A_\bullet, b),$$

the standard chain complex of a simplicial Abelian group A_*. Now we would like to use the same bimodule to pass fro complexes to simplicial Abelian groups:

$$|C_\bullet|_{\mathrm{DK}} = \mathrm{Hom}_{\mathbf{D}}(C_\bullet(\Delta^*), C_\bullet) \tag{4.3}$$

We have

$$\mathrm{Hom}_{\mathrm{Complexes}}((A_*, b), C_\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\Delta^{\mathrm{op}}}(A_\bullet, |C_\bullet|_{\mathrm{DK}}) \tag{4.4}$$

4.3. Chern character in topological K theory. For a topological algebra A, let $A(\Delta^n)$ denote ***the space of *appropriate*** functions $\Delta^n \to A$. We get a simplicial algebra $A(\Delta^n)$. The Chern character provides a morphism of bisimplicial Abelian groups

$$\mathbb{Z}BGL_\infty(A(\Delta^*)) \to |\mathrm{CC}_\bullet^-(\mathrm{A}(\Delta^*))|_{\mathrm{DK}} \tag{4.5}$$

and therefore

$$\mathbb{Z}BGL_\infty(A(\Delta^*)) \to |\mathrm{CC}_\bullet^{\mathrm{per}}(\mathrm{A}(\Delta^*))|_{\mathrm{DK}} \tag{4.6}$$

4.3.1. *The morphism* $\mathrm{CC}^-_\bullet(\widehat{\mathrm{A}}(\Delta^*)) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)$. We assume that there are:

1) completions $A^{\otimes(n+1)} \to A^{\widehat{\otimes}(n+1)}$;
2) sheaves $C^{\mathrm{sm}}_{\mathbb{R}^m}(A^{\widehat{\otimes}n+1})$ of $A^{\widehat{\otimes}n+1}$-valued functions on $\mathbb{R}^m$ such that:
3) $C^{\mathrm{sm}}_{\mathbb{R}^m}$ contains all polynomial $A^{\widehat{\otimes}n+1}$-valued functions.

We assume that all the maps between $A^{\otimes(n+1)}[t_1,\ldots,t_m]$ that are induced by a)-f) below extend to C^{sm} :

a) partial derivatives $\frac{\partial}{\partial t_j}$;
b) affine maps $L:\mathbb{R}^m \to \mathbb{R}^{m'}$;
c) maps $A^{\otimes n+1} \to A^{\otimes n'+1}$ that are compositions of :
d) permutations of tensor factors;
e) $a_0 \otimes a_1 \otimes \ldots \mapsto a_0 a_1 \otimes \ldots$; $a_0 \otimes a_1 \otimes \ldots \mapsto 1 \otimes a_0 \otimes a_1 \otimes \ldots$;
f) for a bounded polytope K in $\mathbb{R}^k$, the map

$$f \mapsto \int_K f dt_{n+1}\ldots dt_{n+k} : k[t_1,\ldots,t_{n+k}] \to k[t_1,\ldots,t_n]$$

We denote by $C^{\mathrm{sm}}(K)$ the space of restrictions to k of functions from of C^{sm}; by $\Omega^\bullet_{\mathrm{sm}}(K)$, $C^{\mathrm{sm}}(K)\otimes \wedge^\bullet(dt^1,\ldots,dt_m)$; we denote $d=\sum \frac{\partial}{\partial t_j}dt_j$.

We assume all the usual relations to be satisfied, namely: the map L^* as in b) commutes with d; $d^2=0$; the maps c), d), e) commute with d and with f); the Stokes formula is true for f) and d; the usual relationship between L^* and f) holds; $d(fc) = df\cdot c + fdc$ for a polynomial f.

We will write

$$\widehat{C}_\bullet(A) = (A^{\widehat{\otimes}(\bullet+1)}, b);\ \widehat{CC}^-_\bullet(A) = (\widehat{C}_\bullet(A)[[u]], b+uB, \tag{4.7}$$

etc.

For any algebra A and a commutative algebra B, consider the composition of the comultiplication **??** with the HKR map

$$\mathrm{CC}^-_\bullet(A\otimes B) \to \mathrm{CC}^-_\bullet(A)\otimes_{k[[u]]}\mathrm{CC}^-_\bullet(B) \to \mathrm{CC}^-(A)\otimes_{k[[u]]}(\Omega^\bullet_{B/k}, ud)$$

Under the above assumptions,when $B=k[t_0,\ldots,t_n]/(t_0+\ldots+t_n-1)$, this extends to a morphism

$$\mathrm{CC}^-_\bullet(\widehat{A}(\Delta^*)) \to (\Omega^\bullet_{\mathrm{sm}}(\Delta^*, \widehat{\mathrm{CC}}^-_\bullet(A)), b+uB+ud) \tag{4.8}$$

where

$$\widehat{A}(\Delta^n) = C^{\mathrm{sm}}(\Delta^n, \widehat{A}) \tag{4.9}$$

Now consider the map

$$\int : \Omega^\bullet_{\mathrm{sm}}(\Delta^*, \widehat{\mathrm{CC}}^-_\bullet(A)) \to \widehat{\mathrm{CC}}^-_\bullet(A))$$

sending a form ω on Δ^n to $u^{-n}\int_{\Delta^n}\omega$. This is a map preserving the degree (because an n-form on Δ^n contributes homological degree $2n$ and u is of degree $-2n$. It also preserves the differential because of the Stokes formula. Composing (4.9) with this map, we get a morphism of complexes

$$\mathrm{CC}^-_\bullet(\widehat{\mathrm{A}}(\Delta^*)) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A) \tag{4.10}$$

Now denote

$$\mathbb{K}^{\mathrm{top}}(A) = \mathrm{BGL}_\infty(\widehat{A}(\Delta^*)) \tag{4.11}$$

$$\mathbb{K}^{\mathrm{alg}}(A) = \mathrm{BGL}_\infty(A)^+ \tag{4.12}$$

We get the Chern cahracter in topological K theory given by

$$\mathbb{Z}\mathrm{BGL}_\infty(\widehat{A}(\Delta^*)) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A) \tag{4.13}$$

or

$$\mathrm{ch}\colon \mathbb{K}^{\mathrm{top}}(A) \to |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)|_{\mathrm{DK}} \tag{4.14}$$

4.4. The Karoubi regulator. ***When $\mathbb{K}^{\mathrm{top}}(A)$ is an H-space,*** by universality of the + construction, there is the natural morphism

$$\mathbb{K}^{\mathrm{alg}}(A) = \mathrm{BGL}_\infty(A)^+ \to \mathrm{BGL}_\infty(\widehat{A}(\Delta^*)) = \mathbb{K}^{\mathrm{top}}(A)$$

and the natural commutative diagram

$$\begin{tikzcd} \mathbb{K}^{\mathrm{alg}}(A) \arrow[r] \arrow[d] & \mathbb{K}^{\mathrm{top}}(A) \arrow[d] \\ |\mathrm{CC}^-_\bullet(A)|_{\mathrm{DK}} \arrow[r] & |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)|_{\mathrm{DK}} \end{tikzcd} \tag{4.15}$$

or

$$\mathbb{K}^{\mathrm{alg}}(A) \to |\mathrm{CC}^-_\bullet(A)|_{\mathrm{DK}} \times_{|\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)|_{\mathrm{DK}}} \mathbb{K}^{\mathrm{top}}(A) \tag{4.16}$$

as well as

$$\mathbb{K}^{\mathrm{rel}}(A) \to |\widehat{\mathrm{CC}}_\bullet(A)[1]|_{\mathrm{DK}} \tag{4.17}$$

(indeed, if we replace CC^- by $\widehat{\mathrm{CC}}^-$ in the bottom left corner of (4.15) then the fiber of the bottom line becomes the right hand side of (**??**)

5. Karoubi-Villamayor K theory

Recall the definition of the Karoubi-Villamayor K theory of a ring A. For $n \geq 0$, define

$$A[\Delta^n] = A[t_0, \ldots, t_n]/(t_0 + \ldots + t_n - 1) \tag{5.1}$$

These rings form a simplicial ring in the usual way: informally, the action of morphisms from Δ^{op} is induced from the action Δ on simplices Δ^n. More precisely, all d_j and s_j act by identity on A; on the generators t_k they act by

$$d_j : t_k \mapsto t_k,\ k < j;\ t_j \mapsto 0;\ t_k \mapsto t_{k-1},\ k > j; \tag{5.2}$$

$$s_j : t_k \mapsto t_k,\ k < j;\ t_j \mapsto t_j + t_{j+1};\ t_k \mapsto t_{k+1},\ k > j. \tag{5.3}$$

Define

$$\mathbb{K}^{\mathrm{KV}}(A) = \mathrm{BGL}(A[\Delta^*]) \tag{5.4}$$

Since the above is an H-space, there is no need for the plus construction. By the universal property of the plus construction, there is a natural morphism

$$\mathbb{K}^{\mathrm{alg}}(A) \to \mathbb{K}^{\mathrm{KV}}(A) \tag{5.5}$$

We get the following version of (4.17)

$$\mathrm{fiber}(\mathbb{K}^{\mathrm{alg}}(A) \to \mathbb{K}^{\mathrm{KV}}(A)) \to |\mathrm{CC}_\bullet(A)[1]|_{\mathrm{DK}} \tag{5.6}$$

5.1. Karoubi-Villamayor K theory of a filtered ring. Given a filtered ring A with an decreasing filtration Fkj, $k \geq 0$, one can define a refinement of the above definitions as follows. Choose $0 \leq j \leq n$. Put

$$A[\Delta^n]_F = \{\sum_\alpha t^\alpha F^{|\alpha|}A\} \subset A[\Delta^n]$$

where $\alpha = (k_0, \ldots, \widehat{k_j}, \ldots, k_n)$, $t^\alpha = (t_0^{k_0}, \ldots \widehat{t_j^{k_j}}, \ldots, t_n^{k_n})$, and $|\alpha| = \sum_{l \neq j} k_l$. It is easy to see that $A[\Delta^n]_F$ is a subring of $A[\Delta^n]$ that does not depend on j. We define

$$\mathbb{K}^{\mathrm{KV,F}}(A) = \mathrm{BGL}(A[\Delta^*]_F)^+ \tag{5.7}$$

5.2. Relative Karoubi-Villamayor K theory of an ideal. Let R be a ring with an ideal J, and let $F^m = J^m$ be the filtration by powers of J. Let

$$\mathbb{K}^{\mathrm{KV,J}}(R) = \mathbb{K}^{\mathrm{KV,F}}(R) \tag{5.8}$$

with respect to this filtration. Define

$$\mathbb{K}^{\mathrm{KV}}(R, J) = \mathrm{fiber}(\mathbb{K}^{\mathrm{KV,J}}(R) \to \mathbb{K}^{\mathrm{alg}}(R/J)) \tag{5.9}$$

***Seems true: this is the same as

$$\mathbb{K}^{\mathrm{KV}}(R, J) = \mathrm{fiber}(\mathrm{BGL}(R[\Delta^*]_F) \to \mathrm{BGL}(R/J)) \tag{5.10}$$

where, as above, F is the filtration by powers of J.

6. Relative K theory and relative cyclic homology of a nilpotent ideal

Let J be a *nilpotent* ideal in a ring R.

LEMMA 6.0.1. *The simplicial set $\mathbb{K}^{\mathrm{KV}}(R, J)$ is contractible.*

PROOF. Define simplicial subsets

$$X_m = \mathrm{Matr}(\{\sum_{|\alpha| \leq m} J^{|\alpha|}t^\alpha\}) \cap \mathrm{Ker}(\mathrm{GL}(R[\Delta^*]_F) \to \mathrm{GL}(R/J))$$

We have

$$X_N = \mathrm{Ker}(\mathrm{GL}(R[\Delta^*]_F) \to \mathrm{GL}(R/J))$$

if $J^N = 0$. Also, there are fibrations ***EXPAND*** for $m > 0$

$$X_{m-1} \to X_m \to Y_m$$

where $Y_m = \mathrm{Matr}(\{\sum_{|\alpha|=m} J^m t^\alpha\})$ with the simplicial structure that we will describe next.

Observe that formulas (5.2) and (5.3) define a simplicial structure on $W_n = \mathbb{Z}t_0 + \ldots + \mathbb{Z}t_n$. Let

$$c_n = t_0 + \ldots + t_n$$

All the morphisms in Δ^{op} sent c_* to c_*. Let

$$V_n = W_n/c_n.$$

The W_n, resp. V_n, $n \geq 0$, form a simplicial $\mathbb{Z}$-module W_*, resp.V_*.

Identify $\{\sum_{|\alpha|=m} J^m t^\alpha\}$ with $\mathrm{Sym}^m(W_*) \otimes_{\mathbb{Z}} J^m$. This gives rise to the simplicial structure on Y_m.

It remains to show that all Y_m with $m \geq 2$ are contractible, as is X_1. In fact, W_* is contractible, and V_* has one nonzero homotopy group $\pi_1 \xrightarrow{\sim} \mathbb{Z}$. Therefore by Künneth formula

$$\pi_k(\mathrm{Sym}^m(V_*)) \xrightarrow{\sim} \mathrm{Sym}^m \pi_1(V_*) \xrightarrow{\sim} \mathrm{Sym}^m \mathbb{Z}[1] = 0$$

for $m > 1$. ***Improve*** □

Define

$$\mathbb{K}^{\mathrm{alg}}(R, J) = \mathrm{fiber}(\mathbb{K}^{\mathrm{alg}}(R) \to \mathbb{K}^{\mathrm{alg}}(R/J)) \tag{6.1}$$

$$\mathrm{CC}_\bullet(R, J) = \mathrm{fiber}(\mathrm{CC}_\bullet(R) \to \mathrm{CC}_\bullet(R/J)) \tag{6.2}$$

and similarly for other types of cyclic complexes.

We get a commutative diagram

$$\begin{tikzcd}
\mathbb{K}^{\mathrm{alg}}(R, J) \arrow[r] \arrow[d] & \mathbb{K}^{\mathrm{KV}}(R, J) \arrow[d] \\
|\mathrm{CC}^-_\bullet(R, J)|_{\mathrm{DK}} \arrow[r] & |\mathrm{CC}^-_\bullet(R[\Delta]_F)|_{\mathrm{DK}}
\end{tikzcd}$$

(Here, again, F is the filtration by powers of J).

7. The characteristic classes of Goodwillie and Beilinson

Now, for any ring R with an ideal J, we have the analog of (4.10)

$$\mathrm{CC}^-_\bullet(R[\Delta]_F) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J)_{\mathbb{Q}} \tag{7.1}$$

where the completion on the right is the J-adic completion.

Definition 7.0.1. *Let r be a positive integer. An ideal J of a ring R is a r-pd ideal if*

1) the J-adic completion $\widehat{R}^{\otimes n}$ has no $\mathbb{Z}$-torsion, and

2) for any n there is m such that the J-adic completion $\widehat{J}^{\otimes n}$ is inside $n!^r \widehat{J}^{\otimes m}$, and $m \to \infty$ as $n \to \infty$.

Lemma 7.0.2. *Let J be a 2-pd ideal. The map* (7.1) *factors through*

$$\mathrm{CC}^-_\bullet(R[\Delta]_F) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J) \tag{7.2}$$

Proof. The map (7.1) is the sum of maps of the following form. Fix n and recall that $t = (t_0, \ldots, \widehat{t_j}, \ldots, t_n)$ for some j. Start with a Hochschild chain $a_0 t^{\alpha_0} \otimes \ldots \otimes a_N t^{\alpha_N}$ where $a_k \in J^{|\alpha_k|}$ and at least one a_k is in J. Subdivide the monomials t^{α_k} into $n+1$ groups and then multiply the members of each group. We get monomials $t^{\beta_0}, \ldots, t^{\beta_n}$ where $\sum_{k=0}^N |\alpha_k| = \sum_{l=0}^n |\beta_l|$. Then compute

$$\int_{\Delta^n} t^{\beta_0} dt^{\beta_1} \ldots dt^{\beta_n}. \tag{7.3}$$

Then subdivide the a_k into segments in cyclic order; multiply elements of each segment. We get a new Hochschild chain $b_0 \otimes \ldots \otimes b_m$ with $b_j \in J^{r_j}$ and $\sum r_j \geq \sum |\alpha_k|$. Multiply this chain by (7.3). We obtain the general form of a component of (7.1).

Let us assume that $t = (t_0, \ldots, t_{n-1})$. The integral (7.3) is of the form

$$\int_{\Delta^n} t_0^{m_0} \ldots t_{n-1}^{m_{n-1}} dt_0 \ldots dt_{n-1} = \frac{1}{n!} \frac{m_0! \ldots m_{n-1}!}{(m_0 + \ldots + m_{n-1} + n)!} \tag{7.4}$$

where $\sum_{j=0}^{n-1} (m_j + 1) = \sum_{k=0}^{N} |\alpha_j|$ and $m_j \geq 0$ for all j. Therefore it is an integer times $\frac{1}{(\sum |\alpha_k|)!^2}$. □

As a consequence, we have

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{alg}}(R, J) & \longrightarrow & \varprojlim_N \mathbb{K}^{\mathrm{KV}}(R/J^N, J/J^N) \\ \downarrow & & \downarrow \\ |\mathrm{CC}^-_\bullet(R, J)|_{\mathrm{DK}} & \longrightarrow & |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J)_{\mathbb{Q}}|_{\mathrm{DK}} \end{array}$$

for any R and J, and

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{alg}}(R, J) & \longrightarrow & \varprojlim_N \mathbb{K}^{\mathrm{KV}}(R/J^N, J/J^N) \\ \downarrow & & \downarrow \\ |\mathrm{CC}^-_\bullet(R, J)|_{\mathrm{DK}} & \longrightarrow & |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J)|_{\mathrm{DK}} \end{array}$$

when J is a 2-pd ideal. Note that the top right corners are contractible ***a few more words***. After completing the bottom left corners, we get the characteristic classes of Goodwillie

$$\mathbb{K}^{\mathrm{alg}}(R, J) \to |\widehat{\mathrm{CC}}_{\bullet-1}(R, J)_{\mathbb{Q}}|_{\mathrm{DK}} \tag{7.5}$$

for any R and J, and of Beilinson

$$\mathbb{K}^{\mathrm{alg}}(R, J) \to |\widehat{\mathrm{CC}}_{\bullet-1}(R, J)|_{\mathrm{DK}} \tag{7.6}$$

for any R and any 2-pd ideal J.

8. The Chern character of Fredholm modules

9. The JLO construction

10. Chern character of perfect complexes

The following is contained in [**41**].

10.1. Perfect complexes and twisting cochains. Consider a sheaf of algebras $\mathcal{A}$ on a topological space X. Fix an open cover $\mathfrak{U} = \{U_j | j \in J\}$. We denote

$$\mathcal{A}_{j_0 \ldots j_p} = \mathcal{A}|(U_{j_0} \cap \ldots \cap U_{j_p}) \tag{10.1}$$

Following Toledo and Tong, we define *a twisting cochain* as:

(1) a collection of strictly perfect complexes of $\mathcal{A}_{U_j}$-modules $\mathcal{F}_j$;
(2) a collection of morphisms of degree $1 - p$, $p \geq 0$, of $\mathcal{A}_{j_0 \ldots j_p}$-modules on $U_{j_0} \cap \ldots \cap U_{j_p}$

$$\rho_{j_0 \ldots j_p} \colon \mathcal{F}_{j_0} \longleftarrow \mathcal{F}_{j_p}$$

such that

(3)

$$\check{\partial}\rho + \rho \smile \rho = 0$$

Here, for two collections of homogenous $\rho_{j_0\dots \jmath_p}$ and $\varphi_{j_0\dots \jmath_p}$ of any degree, we put

$$(\rho \smile \varphi)_{j_0\dots j_m} = \sum_{p=0}^{m} (-1)^{|\rho_{j_0\dots j_p}|} \rho_{j_0\dots j_p} \varphi_{j_p\dots j_m} \tag{10.2}$$

$$(\check{\partial}\rho)_{j_0\dots j_{m+1}} = \sum_{p=1}^{m} (-1)^p \rho_{j_0\dots \widehat{j_p}\dots j_m} \tag{10.3}$$

Note that $(\mathcal{F}_j, \rho_j)$ is a complex of $\mathcal{A}_j$-modules.

DEFINITION 10.1.1. *Let* $\mathrm{Tw}(\mathfrak{U}, \mathcal{A})$ *be the following DG category.*

(1) *Objects are twisting cochains* ρ*.*
(2) *A morphism of degree* n *between* ρ *and* rho' *is a collection of morphisms*

$$\varphi_{j_0\dots j_p} \colon \mathcal{F}'_{j_p} \to \mathcal{F}_{j_0}$$

of degree $n - p$*.*
(3) *The differential acts by*

$$(d\varphi) = \check{\partial}\varphi + \rho \smile \varphi - (-1)^{|\varphi|}\varphi\rho.$$

(4) *The composition of* φ *and* ψ *is* $\varphi \smile \psi$*.*

Let $I_{\mathfrak{U}}$ be the category whose set of objects is J and such that there is unique morphism between any two objects. A twisting cochain ρ satisfies the same relations as an A_∞ functor

$$\rho \colon kI_{\mathfrak{U}} \to \mathrm{sPerf}(\mathcal{A}) \tag{10.4}$$

that sends j to $(\mathcal{F}_j, \rho_j)$. Similarly, $\mathrm{Tw}(\mathfrak{U}, \mathcal{A})$ is defined almost exactly in the same way as the DG category $\mathbf{C}(kI_{\mathfrak{U}}, \mathrm{sPerf}(\mathcal{A}))$ as in **??**. The only difference is that the targets of the components vary, namely, objects $\mathcal{F}_j$ lie in different categories, and so do morphisms $\rho_{j_0\dots j_p}$. We will first recall the construction that we would like to carry out, and then outline a minor variation on it that suits our purpose.

10.2. A character map from the category of A_∞ functors. For a category Γ and a DG category $\mathcal{P}$, we have constructed ***REF*** a DG functor

$$\mathrm{CC}^-_\bullet(\mathbf{C}(k\Gamma, \mathcal{P})) \otimes \mathrm{CC}^-_\bullet(k\Gamma) \to \mathrm{CC}^-_\bullet(\mathcal{P}) \tag{10.5}$$

If Γ is a groupoid then we also have

$$C_\bullet(\Gamma, k) \to \mathrm{CC}^-_\bullet(k\Gamma) \tag{10.6}$$

??. Combiming, we get

$$\mathrm{CC}^-_\bullet(\mathbf{C}(k\Gamma, \mathcal{P})) \to \underline{\mathrm{Hom}}(C_\bullet(\Gamma, k), \mathrm{CC}^-_\bullet(\mathcal{P})) \tag{10.7}$$

10.3. Chern character of a twisting cochail. We need a modification of the above. For simplicity, let Γ be the groupoid $I_{\mathfrak{U}}$ where $\mathfrak{U}$ is a set. We assume that there is a presheaf $\mathcal{P}$ of categories on the cyclic nerve of $I_{\mathfrak{U}}$ is given, that is, a category $\mathcal{P}_J$ for any finite subset $J = \{j_0, \dots, j_p\}$ of $\mathfrak{U}$ together with functors $r_{JK} : \mathcal{P}_K \to \mathcal{P}_J$ for $J \subset K$ so that $r_{JJ} = \mathrm{id}$ and $r_{IJ}r_{JK} = r_{IK}$ for any $I \subset J \subset K$.

For example, when $\mathfrak{U}$ is an open cover of X and $\mathcal{A}$ is a sheaf of rings on X then one can put

$$\mathcal{P}_{\{j_0,\dots,j_p\}} = \mathrm{Perf}(\mathcal{A}|U_{j_0} \cap \dots \cap \mathcal{U}_{j_p})$$

Having Γ and $\mathcal{P}$ as above, we can modify the definition of the Hochschild chain complex and of an A_∞ functor as follows. For an object F of $\mathcal{P}_j$ and a subset J containing j, we will write

$$F|J = r_{J\{j\}}F \tag{10.8}$$

Fix two collections $\{F_j,\, G_j \in \mathrm{Ob}(\mathcal{P}_j) | j \in \mathrm{Ob}(\Gamma)\}$. Define the *local Hochschild cochain complex*

$$C^\bullet_{\mathrm{loc}}(k\Gamma,_F \mathcal{P}_G) = \prod_J \underline{\mathrm{Hom}}(k\Gamma(j_0, j_1)[1] \otimes \ldots \otimes k\Gamma(j_{n-1}, j_n)[1], \mathcal{P}_J(F_{j_0}|J, G_{j_n}|J))$$

where the product is over all $J = j_0, \ldots, j_n \in \mathrm{Ob}(\Gamma)$. (The category $\mathcal{P}_J$ depends only on the underlying set. In fact everything we do only requires it to be the same for the underlying cyclically ordered set).

The differential δ on $C^\bullet_{\mathrm{loc}}(k\Gamma,_F \mathcal{P}_G)$ and the product

$$\smile : C^\bullet_{\mathrm{loc}}(k\Gamma,_F \mathcal{P}_G) \otimes C^\bullet_{\mathrm{loc}}(k\Gamma,_G \mathcal{P}_H) \to C^\bullet_{\mathrm{loc}}(k\Gamma,_F \mathcal{P}_H)$$

are defined exactly as in the case of two DGA categories. A local $A\infty$ functor $k\Gamma \to \mathcal{P}$ is a collection $F = \{F_j\}$ and a cochain of degree one in $C^\bullet_{\mathrm{loc}}(k\Gamma,_F \mathcal{P}_F)$ satisfying $\delta\rho + \rho \smile \rho = 0$. We denote the pair of F and ρ simply by F. As in **??**, local A_∞ functors form a DG category that we denote by $\mathbf{C}_{\mathrm{loc}}(k\Gamma, \mathcal{P})$. Exactly as in (10.7) we get

$$\mathrm{CC}^-_\bullet(\mathbf{C}_{\mathrm{loc}}(k\Gamma, \mathcal{P})) \to \underline{\mathrm{Hom}}_{\mathrm{loc}}(C_\bullet(\Gamma, k), \mathrm{CC}^-_\bullet(\mathcal{P})) \tag{10.9}$$

Here $\underline{\mathrm{Hom}}_{\mathrm{loc}}$ in the right hand side stands for

$$\prod_{j_0,\ldots,j_n} \underline{\mathrm{Hom}}(k\Gamma(j_0, j_1)[1] \otimes \ldots \otimes k\Gamma(j_{n-1}, j_n)[1], \mathrm{CC}^-_\bullet(\mathcal{P}_{\{j_0,\ldots,j_n\}}))$$

Composing with REF?***

$$\mathrm{CC}^-_\bullet(\mathrm{Perf}(\mathcal{A})) \to \mathrm{CC}^-_\bullet(\mathbf{C}_{\mathrm{loc}}(k\Gamma, \mathcal{P})) \tag{10.10}$$

and observing that

$$\underline{\mathrm{Hom}}_{\mathrm{loc}}(C_\bullet(\Gamma, k), \mathrm{CC}^-_\bullet(\mathcal{P})) \xrightarrow{\sim} \check{C}^*(\mathfrak{U}, \mathrm{CC}^-_\bullet(\mathcal{A})), \tag{10.11}$$

and passing to the direct limit in $\mathfrak{U}$, we get

$$\mathrm{ch} : \mathrm{CC}^-_\bullet(\mathrm{Perf}(\mathcal{A})) \to \check{C}^*(X, \mathrm{CC}^-_\bullet(\mathcal{A})) \tag{10.12}$$

REMARK 10.3.1. For a twisting cochain ρ, the value of Čech-negative cyclic cochain $\mathrm{ch}(\rho)$ on $U_{j_0} \cap \ldots \cap U_{j_p}$ is the sum of terms as follows. Let

$$(\ell_0, \ldots, \ell_N, \ell_0) = (j_k, j_{k+1}, \ldots, j_k, \ldots, j_k, j_{k+1}, \ldots, j_k) \tag{10.13}$$

which is $(j_0, \ldots, j_p)$ shifted cyclically by k and then repeated m times for some m (so $N + 1 = m(p + 1)$). Let $(i_0, i_1, ..., i_M)$ be an ordered subset of the ordered set $(\ell_0, \ldots, \ell_N)$, and choose an ordered subset $(i_k, q_1, \ldots, q_{r_k}, i_{k+1})$ in the segment $i_k \le p \le i_{k+1}$ in (10.13), $0 \le k \le M$ (we put $i_{M+1} = i_0$). Define

$$\rho_(i_k, i_{k+1}) = \rho_{i_k \ldots q_1} \ldots \rho_{q_{r_k} \ldots i_{k+1}} \tag{10.14}$$

$$\mathrm{ch}(\rho)_{j_0 \ldots j_p} = \sum_{m \ge 0} \sum_{i_0,\ldots,i_{p+2m}} c(i_0, \ldots, i_{p+2m}) u^m \rho(i_0, i_1) \otimes \ldots \otimes \rho(i_{p+2m}, i_0)$$

Only terms $\rho_{j_q \ldots j_{q+r}}$ with $r > 0$ participate.

For example, if the twisting cochain consists of the transition isomorphisms $\rho_j k$ (i.e. when our sheaf of modules $\mathcal{M}$ is locally free, then

$$\mathrm{ch}(\rho)_{j_0 \ldots j_p} =$$

*****TO BE CORRECTED****

11. Bibliographical notes

CHAPTER 10

Algebroid stacks

1. Introduction

2. Definition and basic properties

Let M be a smooth manifold (C^∞ or complex). In by a descent datum for an algebroid stack on M we will mean the following data:

1) an open cover $M = \cup U_i$;

2) a sheaf of rings $\mathcal{A}_i^\bullet$ on every U_i;

3) an isomorphism of sheaves of rings $G_{ij} : \mathcal{A}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{A}_i|(U_i \cap U_j)$ for every i, j;

4) an invertible element $c_{ijk} \in \mathcal{A}_i(U_i \cap U_j \cap U_k)$ for every i, j, k satisfying

$$G_{ij}G_{jk} = \mathrm{Ad}(c_{ijk})G_{ik} \tag{2.1}$$

such that, for every i, j, k, l,

$$c_{ijk}c_{ikl} = G_{ij}(c_{jkl})c_{ijl} \tag{2.2}$$

If two such descent data $(U_i',\ \mathcal{A}_i',\ G_{ij}',\ c_{ijk}')$ and $(U_i'',\ \mathcal{A}_i'',\ G_{ij}'',\ c_{ijk}'')$ are given on M, an isomorphism between them is an open cover $M = \cup U_i$ refining both $\{U_i'\}$ and $\{U_i''\}$ together with isomorphisms $H_i : \mathcal{A}_i' \xrightarrow{\sim} \mathcal{A}_i''$ on U_i and invertible elements b_{ij} of $\mathcal{A}_i'(U_i \cap U_j)$ such that

$$G_{ij}'' = H_i \mathrm{Ad}(b_{ij}) G_{ij}' H_j^{-1} \tag{2.3}$$

and

$$H_i^{-1}(c_{ijk}'') = b_{ij} G_{ij}'(b_{jk}) c_{ijk}' b_{ik}^{-1} \tag{2.4}$$

A descent datum for a gerbe is a descent datum for an algebroid stack for which $\mathcal{A}_i = \mathcal{O}_{U_i}$ and $G_{ij} = \mathrm{id}$. In this case c_{ijk} form a two-cocycle in $Z^2(M, \mathcal{O}_M^*)$.

2.1. Categorical interpretation.

A datum defined as above gives rise to the following categorical data:

(1) A sheaf of categories $\mathcal{C}_i$ on U_i for every i;

(2) an invertible functor $G_{ij} : \mathcal{C}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{C}_i|(U_i \cap U_j)$ for every i, j;

(3) an invertible natural transformation

$$c_{ijk} : G_{ij}G_{jk}|(U_i \cap U_j \cap U_k) \xrightarrow{\sim} G_{ik}|(U_i \cap U_j \cap U_k)$$

such that, for any i, j, k, l, the two natural transformations from $G_{ij}G_{jk}G_{kl}$ to G_{il} that one can obtain from the c_{ijk}'s are the same on $U_i \cap U_j \cap U_k \cap U_l$.

If two such categorical data $(U_i',\ \mathcal{C}_i',\ G_{ij}',\ c_{ijk}')$ and $(U_i'',\ \mathcal{C}_i'',\ G_{ij}'',\ c_{ijk}'')$ are given on M, an isomorphism between them is an open cover $M = \cup U_i$ refining both $\{U_i'\}$ and $\{U_i''\}$, together with invertible functors $H_i : \mathcal{C}_i' \xrightarrow{\sim} \mathcal{C}_i''$ on U_i and invertible natural transformations $b_{ij} : H_i G'_{ij}|(U_i \cap U_j) \xrightarrow{\sim} G''_{ij} H_j|(U_i \cap U_j)$ such that, on any $U_i \cap U_j \cap U_k$, the two natural transformations $H_i G'_{ij} G'_{jk} \xrightarrow{\sim} G''_{ij} G''_{jk} H_k$ that can be obtained using H_i's, b_{ij}'s, and c_{ijk}'s are the same. More precisely:

$$((c''_{ijk})^{-1} H_k)(b_{ik})(H_i c'_{ijk}) = (G''_{ij} b_{jk})(b_{ij} G'_{jk}) \tag{2.5}$$

The above categorical data are defined from $(\mathcal{A}_i^\bullet, G_{ij}, c_{ijk})$ as follows:

1) $\mathcal{C}_i$ is the sheaf of categories of $\mathcal{A}_i^\bullet$-modules;

2) given an $\mathcal{A}_i^\bullet$-module $\mathcal{M}$, the $\mathcal{A}_j^\bullet$-module $G_{ij}(\mathcal{M})$ is the sheaf $\mathcal{M}$ on which $a \in \mathcal{A}_i^\bullet$ acts via $G_{ij}^{-1}(a)$;

3) the natural transformation c_{ijk} between $G_{ij} G_{jk}(\mathcal{M})$ and $G_{jk}(\mathcal{M})$ is given by multiplication by $G_{ik}^{-1}(c_{ijk}^{-1})$.

From the categorical data defined above, one defines a sheaf of categories on M as follows. For an open V in M, an object of $\mathcal{C}(V)$ is a collection of objects X_i of $\mathcal{C}_i(U_i \cap V)$, together with isomorphisms $g_{ij} : G_{ij}(X_j) \xrightarrow{\sim} X_i$ on every $U_i \cap U_j \cap V$, such that

$$g_{ij} G_{ij}(g_{jk}) = g_{ik} c_{ijk}$$

on every $U_i \cap U_j \cap U_k \cap V$. A morphism between objects (X'_i, g'_{ij}) and (X''_i, g''_{ij}) is a collection of morphisms $f_i : X'_i \to X''_i$ (defined for some common refinement of the covers), such that $f_i g'_{ij} = g''_{ij} G_{ij}(f_j)$.

3. Hochschild and cyclic complexes of algebroid stacks

We define the Hochschild (co)chain complex and the negative and periodic cyclic complexes associasted to an algebroid stack.

CHAPTER 11

Hochschild and cyclic homology and the bar construction

1. Hochschild and cyclic (co)homology of coalgebras

Just as we constructed the Hochschild and cyclic complexes for differential graded algebras, we can in a dual way construct analogous complexes for differential graded coalgebras. For a DG coalgebra C with coproduct Δ, differential d and counit ϵ, let

$$\widetilde{C}^{\bullet}(C) = \prod_{n \geq 0} C \otimes C^{\otimes n} \tag{1.1}$$

$$C^{\bullet}(C) = \prod_{n \geq 0} C \otimes \overline{C}^{\otimes n} \tag{1.2}$$

where $\overline{C} = \mathrm{ker}\epsilon$. The following is a construction dual to the one for algebras. Put

$$d^j(c_0 \otimes \ldots c_n) = (-1)^{\sum_{p<j} |c_p| + |c_j^{(1)}|} c_0 \otimes \ldots \otimes c_{j-1} \otimes c_j^{(1)} \otimes c_j^{(2)} \otimes \ldots \otimes c_n,$$

$0 \leq j < n+1$;

$$d^{n+1}(c_0 \otimes \ldots \otimes c_n) = (-1)^{(\sum_{p=1}^{n} |c_p| + |c^{(1)}|)|c_0^{(1)}} c_0^{(2)} \otimes c_1 \otimes \ldots \otimes c_n \otimes c_0^{(1)};$$

$$s^j(c_0 \otimes \ldots \otimes c_n) = (-1)^{\sum_{p \leq j} |c_j|} c_0 \otimes \ldots c_j \epsilon(c_{j+1}) \otimes c_{j+2} \otimes \ldots \otimes c_n,$$

$-1 \leq j \leq n$;

$$t(c_0 \otimes \ldots \otimes c_n) = (-1)^{|c_0| \sum_{p>0} |c_p|} c_1 \otimes \ldots \otimes c_n \otimes c_0$$

Note that d^j and s^j with $j \geq 0$ define a cosimplicial structure module structure $[n] \mapsto C^{\otimes(n+1)}$; together with t they define a cocyclic module structure. We put

$$b = \sum_{j=0}^{n+1} (-1)^j d^j; \ \tau = (-1)^n t; \ N = \sum_{j=0}^{n} \tau^j \tag{1.3}$$

on $C \otimes C^{\otimes n}$;

$$B = N s_{-1}(1 - \tau) \tag{1.4}$$

As in the case of algebras, b and B descend to $C^{\bullet}(B)$ and satisfy

$$b^2 = B^2 = bB + Bb = 0$$

Now define the Hochschild complex of C to be

$$(C^{\bullet}(C), d + b) \tag{1.5}$$

Define also

$$\mathrm{CC}^\bullet(C) = (C^\bullet(C)[[v]], b + d + vB) \tag{1.6}$$

where v is a formal parameter of cohomological degree -2;

$$\mathrm{CC}^\bullet_{\mathrm{per}}(C) = (C^\bullet(C)[[v, v^{-1}], b + d + vB) \tag{1.7}$$

and

$$\mathrm{CC}^\bullet_{-}(C) = (C^\bullet(C)[[v, v^{-1}]/vC^\bullet(C)[[v]], b + d + vB) \tag{1.8}$$

1.1. Complexes of the second kind. The above definition is dual to the one we used for DG algebras. An important feature of both is that they are invariant with respect to quasi-isomorphisms of DG (co)algebras. In this chapter, however, we are going to consider the example $C = \mathrm{Bar}(A)$ where A is a DG algebra. Since C is contractible when A has a unit, we cannot get anything meaningful using the complexes above. We can, however, define *the Hochschild complex of the second kind*

$$C^\bullet_{II}(C) = \bigoplus_{n \geq 0} C \otimes \overline{C}^{\otimes n} \tag{1.9}$$

with the differential $b + d$. Define also

$$\mathrm{CC}^\bullet_{II}(C) = (C^\bullet_{II}(C)[[v]], b + d + vB) \tag{1.10}$$

and similarly for the negative and periodic complexes as in (1.14), (1.15).

1.2. Two-periodic De Rham complex. Define for a DG counital coalgebra C

$$\mathrm{DR}_1(C) = \ker(b : C \otimes \overline{C} \to C \otimes \overline{C}^{\otimes 2}); \tag{1.11}$$

Let $C^\bullet(C)_{\mathrm{sh}}$ be the total complex

$$C \xrightarrow{b} \mathrm{DR}_1(C) \tag{1.12}$$

(with the differential $d + b$). Define also

$$\mathrm{CC}^\bullet(C)_{\mathrm{sh}} = (C^\bullet(C)_{\mathrm{sh}}[[v]], b + vB) \tag{1.13}$$

where v is a formal parameter of cohomological degree -2;

$$\mathrm{CC}^\bullet_{\mathrm{per}}(C)_{\mathrm{sh}} = (C^\bullet(C)_{\mathrm{sh}}[[v, v^{-1}], b + vB) \tag{1.14}$$

and

$$\mathrm{CC}^\bullet_{-}(C)_{\mathrm{sh}} = (C^\bullet(C)_{\mathrm{sh}}[[v, v^{-1}]/vC^\bullet(C)_{\mathrm{sh}}[[v]], b + vB) \tag{1.15}$$

PROPOSITION 1.2.1. *For any DG algebra A, the embedding*

$$C^\bullet(\mathrm{Bar}(A))_{\mathrm{sh}} \longrightarrow C^\bullet_{II}(\mathrm{Bar}(A))$$

is a homotopy equivalence. Same if one replaces $C^\bullet$ by $\mathrm{CC}^\bullet$, $\mathrm{CC}^\bullet_{-}$, or $\mathrm{CC}^\bullet_{\mathrm{per}}$.

PROOF. Since $\mathrm{Bar}(A)$ is cofree as a graded coalgebra, we cal construct P, I, and H by formulas dual to (3.1). □

2. Homology of an algebra in terms of the homology of its bar construction

We will now apply the above to the DG coalgebra $C = \mathrm{Bar}(A)$ for a DG algebra A. To avoid confusion, we will use boldface for $C \xrightarrow{\mathbf{B}} \mathrm{DR}_1(C) \xrightarrow{\mathbf{b}} C$, while reserving the symbols b and B for the differentials on the Hochschild complex of A.

PROPOSITION 2.0.1. *There are isomorphisms of complexes*

$$\mathrm{Bar}(A)) \xrightarrow{\sim} (A^{\otimes \bullet}, b'); \ \mathrm{DR}_1(\mathrm{Bar}(A)) \xrightarrow{\sim} C_\bullet(A, A)$$

These isomorphisms intertwine $\mathbf{B}$ *with* N *and* $\mathbf{b}$ *with* $\mathrm{id} - \tau$.

PROOF. ****** $\square$

Therefore we can express the $(b, b', \mathrm{id} - \tau, N)$ double complex computing the cyclic homology of A as

$$\ldots \xrightarrow{\mathbf{B}} \mathrm{Bar}(A) \xrightarrow{\mathbf{b}} \mathrm{DR}_1(\mathrm{Bar}(A)) \tag{2.1}$$

Because of Proposition 1.2.1, the above computes the *negative* cyclic homology of $\mathrm{Bar}(A)$. Similarly, the negative cyclic complex of A gets identified with

$$\mathrm{Bar}(A) \xrightarrow{B} \mathrm{DR}_1(\mathrm{Bar}(A)) \xrightarrow{\mathbf{b}} \ldots \tag{2.2}$$

which computes the *cyclic* homology of $\mathrm{Bar}(A)$. We obtain

THEOREM 2.0.2.

$$\mathrm{HC}_\bullet(A) \xrightarrow{\sim} \mathrm{HC}^{-\bullet}_{-,II}(\mathrm{Bar}(A)); \ \mathrm{HC}^-_\bullet(A) \xrightarrow{\sim} \mathrm{HC}^{-\bullet}_{II}(\mathrm{Bar}(A))$$

PROOF. $\square$

2.1. Action of A_∞ morphisms on Hochschild and cyclic complexes. Since an A_∞ morphism is by definition a morphism of DG coalgebras $\mathrm{Bar}(A) \to \mathrm{Bar}(B)$ and the short complexes $C^\bullet(C)_{\mathrm{sh}}$, $\mathrm{CC}^\bullet(C)_{\mathrm{sh}}$, etc. are functorial in C, Theorem 2.0.2 implies that an A_∞ morphism induces morphisms of Hochschild and cyclic complexes. It is easy tho see that these are the same morphisms as in (7.3).

CHAPTER 12

Operations on Hochschild and cyclic complexes, II

1. The A_∞ action of $\mathrm{CC}^\bullet_{\mathrm{II}}(U(\mathfrak{g}_A))$

Recall that $\mathfrak{g}_A$ denotes the DG Lie algebra $C^{\bullet+1}(A,A)$ with the Gerstenhaber braket which we identified with $\mathrm{Coder}(\mathrm{Bar}(A))$. The latter defines the action of $U(\mathfrak{g}_A)$ on $\mathrm{Bar}(A)$ as well as linear maps

$$\mu_N \colon U(\mathfrak{g}_A)^{\otimes N} \otimes \mathrm{Bar}(A) \to \mathrm{Bar}(A) \tag{1.1}$$

(composition of the above action with the n-fold product on $U(\mathfrak{g}_A)$).

Lemma 1.0.1. *The above are morphisms of DG coalgebras.*

Proof. Clear. $\square$

Recall the definition of the cyclic complex of the second kind from 11. (It uses the definition of the Hochschild complex as the standard complex associated to a bicomplex, i.e. the complex comprised of direct sums over diagonals, as opposed to direct products as would seem more natural if we were doing a dual construction to the case of algebras).

Corollary 1.0.2. *The compositions of*

$$\mathrm{CC}^\bullet_{\mathrm{II}}(U(\mathfrak{g}_A))^{\otimes N} \otimes \mathrm{CC}^\bullet_{\mathrm{II}}(\mathrm{Bar}(A)) \longrightarrow \mathrm{CC}^\bullet_{\mathrm{II}}(U(\mathfrak{g}_A)^{\otimes N} \otimes \mathrm{Bar}(A))$$

(given by $m(U(\mathfrak{g}_A), \ldots, U(\mathfrak{g}_A), \mathrm{Bar}(A))$ (cf. (4.1)) with the morphism induced by μ_N define on $\mathrm{CC}^\bullet_{\mathrm{II}}(\mathrm{Bar}(A))$ a structure of an A_∞ module over the A_∞ algebra $\mathrm{CC}^\bullet_{\mathrm{II}}(U(\mathfrak{g}_A))$.

Let ϵ be a variable of degree -1 such that $\epsilon^2 = 0$. and u a variable of (cohomological) degree two. For any DG Lie algebra $(\mathfrak{g}, \delta)$ construct a DG Lie algebra $\mathfrak{g}[[u]][\epsilon]$, the differential being $\delta + u\frac{\partial}{\partial \epsilon}$. Consider its universal enveloping algebra over $k[[u]]$. It is a DG algebra over $k[[u]]$.

Theorem 1.0.3. *Assume that $\mathbb{Q} \subset k$. There is a natural $k[[u]]$-linear (u)-adically continuous A_∞ action of the DGA $(U(\mathfrak{g}_A[\epsilon])[[u]], d + \delta + u\frac{\partial}{\partial \epsilon})$ on $\mathrm{CC}^-_\bullet(A)$ whose components*

$$\phi_n : U(\mathfrak{g}_A[\epsilon])[[u]]^{\otimes n} \to \mathrm{End}^{1-n}(\mathrm{CC}^-_\bullet(A))$$

satisfy the following.

1. *$\phi_1(D) = (-1)^{|D|} L_D$ and $\phi_1(\epsilon D) = (-1)^{|D|-1} I_D$ for $D \in \mathfrak{g}_A$.*
2. *$\phi_n(a_1, \ldots, a_j, D, a_{j+1}, \ldots, a_{n-1}) = 0$ for $n > 1$, for any j, and $D \in \mathfrak{g}$.*

Proof. To simplify the notation, we will consider an arbitrary DG Lie algebra $(\mathfrak{g}, \delta)$ over $K = k[[u]]$ (so in our example $\mathfrak{g} = \mathfrak{g}_A[[u]]$). Everything will be linear over K. Construct an A_∞ morphism

$$\phi : (U(\mathfrak{g}[\epsilon]), \delta + u\frac{\partial}{\partial \epsilon}) \mapsto (\mathrm{Cobar}(\overline{U}(\mathfrak{g})), \delta + b + uB) \tag{1.2}$$

as follows. Notice that both $\mathrm{CobarBar}(U(\epsilon\mathfrak{g}))$ and $\mathrm{Cobar}(\overline{S}(\mathfrak{g}))$ are resolutions of $U(\epsilon\mathfrak{g})$ (where $\overline{S}(\mathfrak{g})$ is the positive part of the symmetric algebra, viewed as a coalgebra). Moreover,

$$\mathrm{Cobar}(\overline{S}(\mathfrak{g})) \to U(\epsilon\mathfrak{g}) \tag{1.3}$$

admits a contracting homotopy that is invariant under the adjoint action of $\mathfrak{g}$ (in fact under all linear endomorphisms of $\mathfrak{g}$). Therefore we can construct an $\mathrm{ad}(\mathfrak{g})$-equivariant morphism of DG algebras

$$\mathrm{CobarBar}(U(\mathfrak{g}\epsilon)) \to \mathrm{Cobar}(\overline{S}(\mathfrak{g})) \tag{1.4}$$

As usual, for any Lie algebra $\mathfrak{g}$ acting on an associative algebra C by derivations, denote by $U(\mathfrak{g}) \ltimes C$ the algebra generated by two subalgebras $U(\mathfrak{g})$ and C subject to relations $Xc - cX = X(c)$ for $X \in \mathfrak{g}$ and $c \in C$. The multiplication map $U(\mathfrak{g}) \otimes C \to U(\mathfrak{g}) \ltimes C$ is a bijection. Note that

$$U(\mathfrak{g}[\epsilon]) \xrightarrow{\sim} U(\mathfrak{g}) \ltimes U(\epsilon\mathfrak{g})$$

Because (1.4) is $\mathrm{ad}(\mathfrak{g})$-equivariant, we extend it to

$$U(\mathfrak{g}) \ltimes \mathrm{CobarBar}(U(\mathfrak{g}\epsilon)) \to U(\mathfrak{g}) \ltimes \mathrm{Cobar}(\overline{S}(\mathfrak{g})) \tag{1.5}$$

We deform this morphism as follows. Denote by $U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}))$ the same graded algebra with changed differential. Namely, for any $D_1 \dots D_n \in S^n(\mathfrak{g})$, put

$$\mathrm{PBW}(D_1 \dots D_n) = \frac{1}{n!} \sum \pm D_{\sigma 1} \dots D_{\sigma n} \in U(\mathfrak{g})$$

(note that, as usual, PBW is an $\mathrm{ad}(\mathfrak{g})$-equivariant morphism of coalgebras). Define the new differential a free generators of CobarBar to be

$$(D_1 \dots D_n) \mapsto (\delta + \partial_{\mathrm{Cobar}} + u\mathrm{PBW})(D_1 \dots D_n).$$

Note that $U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}))$ is still a resolution of $U(\mathfrak{g}[\epsilon])$. In fact, the morphism

$$U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \to U(\mathfrak{g}[\epsilon]) \tag{1.6}$$

defined by

$$(D_1 \dots D_n) \mapsto \frac{1}{n!} \sum \pm \epsilon D_{\sigma 1} \cdot D_{\sigma 2} \dots D_{\sigma n} \tag{1.7}$$

admits an $\mathrm{ad}(\mathfrak{g})$-equivariant contracting homotopy (that we construct recursively using the one for (1.3)). We use this homotopy to construct a morphism

$$U(\mathfrak{g}) \ltimes \mathrm{CobarBar}(U(\mathfrak{g}\epsilon)) \to U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \tag{1.8}$$

over $U(\mathfrak{g}[\epsilon])$. This is the same as an A_∞ morphism

$$U(\mathfrak{g}[\epsilon]) \to U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \tag{1.9}$$

satisfying

$$\phi_n(\dots, D, \dots) = 0 \tag{1.10}$$

for any $n > 1$. Because of Proposition 4.2.2 and Corollary 1.0.2, we get the morphism as in Theorem 1.0.3. It satisfies property (2). As for (1),

$$\phi_1(D) = D;\ \phi_1(\epsilon D) = (D)$$

Elaborate a bit?* ***FINISH*** □

2. A_∞ structure on $C_\bullet(C^\bullet(A))$

The algebra of operations on the negative cyclic complex that was described in Corollary 1.0.2 can be extended as follows. Start by noting that

$$U(\mathfrak{g}_A) \to \mathrm{Bar}(C^\bullet(A)), \qquad (2.1)$$
$$D \mapsto (D),\ D \in \mathfrak{g},$$

extends to a morphism of bialgebras. The bialgebra morphisms ***REF

$$\mathrm{Bar}(C^\bullet(A)) \otimes \mathrm{Bar}(C^\bullet(A)) \to \mathrm{Bar}(C^\bullet(A));\ \mathrm{Bar}(C^\bullet(A)) \otimes \mathrm{Bar}(A) \to \mathrm{Bar}(A)$$

induce (because of ***REF) an A_∞ algebra structure on $\mathrm{CC}^-_\bullet(C^\bullet(A))$ and an A_∞ module structure on $\mathrm{CC}^-_\bullet(A)$.

Below we will construct such a structure explicitly. All the pairings described in 1, 5 are in fact different parts of this structure. *We do not know whether it is the same as described above. ***Seems that we can prove it directly for the Hochschild complex.*

2.1. Explicit construction.

Let A be a differential graded algebra. The complex $C_\bullet(C^\bullet(A))$ contains the Hochschild cochain complex $C^\bullet(A)$ as the subcomplex of zero-chains

$$C^\bullet(A) = C_0(C^\bullet(A)) \xrightarrow{\iota} C_\bullet(C^\bullet(A))$$

and has the Hochschild chain complex $C_\bullet(A)$ as a quotient complex induced by the projection on the zero Hochschild cochains $C^\bullet(A) \to C^0(A)$

$$C_\bullet(C^\bullet(A)) \xrightarrow{\pi} C_\bullet(C^0(A)) = C_\bullet(A).$$

The projection π splits if A is commutative. In general $C_\bullet(A)$ is naturally a graded subspace but not a subcomplex.

THEOREM 2.1.1. *There is an A_∞ structure $\mathbf{m}$ on $C_\bullet(C^\bullet(A))[[u]]$ such that:*

(1) *All m_n are $k[[u]]$-linear, (u)-adically continuous*
(2) $m_1 = b + \delta + uB$
For $x,\ y \in C_\bullet(A)$:
(3) $(-1)^{|x|} m_2(x,y) = (\mathrm{sh} + u\,\mathrm{sh}')(x,y)$
For $D,\ E \in C^\bullet(A)$:
(4) $(-1)^{|D|} m_2(D,E) = D \smile E$
(5) $m_2(1\otimes D,\ 1\otimes E) + (-1)^{|D||E|} m_2(1\otimes E,\ 1\otimes D) = (-1)^{|D|} 1 \otimes [D,\ E]$
(6) $m_2(D,\ 1\otimes E) + (-1)^{(|D|+1)|E|} m_2(1\otimes E,\ D) = (-1)^{|D|+1}[D,\ E]$

THEOREM 2.1.2. *On $C_\bullet(A)[[u]]$, there exists a structure of an A_∞ module over the A_∞ algebra $C_\bullet(C^\bullet(A))[[u]]$ such that:*

(1) *All μ_n are $k[[u]]$-linear, (u)-adically continuous*
(2) $\mu_1 = b + uB$ *on* $C_\bullet(A)[[u]]$
For $a \in C_\bullet(A)[[u]]$:
(3) $\mu_2(a, D) = (-1)^{|a||D|+|a|}(i_D + uS_D)a$
(4) $\mu_2(a, 1\otimes D) = (-1)^{|a||D|} L_D a$
For $a,\ x \in C_\bullet(A)[[u]]$: $(-1)^{|a|}\mu_2(a,x) = (\mathrm{sh} + u\,\mathrm{sh}')(a,x)$

CONSTRUCTION 2.1.3. *The explicit description of the A_∞ structure on $C_\bullet(C^\bullet(A))$. We define for $n \geq 2$*

$$m_n = m_n^{(1)} + u m_n^{(2)}$$

where, for

$$a^{(k)} = D_0^{(k)} \otimes \ldots \otimes D_{N_k}^{(k)},$$

$$m_n^{(1)}(a^{(1)}, \ldots, a^{(n)}) = \sum \pm m_k\{\underline{\ldots}, D_0^{(0)}\{\underline{\ldots}\}, \underline{\ldots}, D_0^{(n)}\{\underline{\ldots}\}\underline{\ldots}\} \otimes \underline{\ldots}$$

The space designated by $\_$ *is filled with* $D_i^{(j)}$, $i > 0$, *in such a way that:*

- *the cyclic order of each group* $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ *is preserved;*
- *any cochain* $D_j^{(i)}$ *may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form* $D_q^{(p)}$ *with* $p < i$. *The sign convention: any permutation contributes to the sign; the parity of* $D_j^{(i)}$ *is always* $|D_j^{(i)}| + 1$.

$$m_n^{(2)}(a^{(1)}, \ldots, a^{(n)}) = \sum \pm 1 \otimes \underline{\ldots} \otimes D_0^{(0)}\{\underline{\ldots}\} \otimes \underline{\ldots} \otimes D_0^{(n)}\{\underline{\ldots}\} \otimes \underline{\ldots}$$

The space designated by $\_$ *is filled with* $D_i^{(j)}$, $i > 0$, *in such a way that:*

- *the cyclic order of each group* $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ *is preserved;*
- *any cochain* $D_j^{(i)}$ *may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form* $D_q^{(p)}$ *with* $p < i$. *The sign convention: any permutation contributes to the sign; the parity of* $D_j^{(i)}$ *is always* $|D_j^{(i)}| + 1$.

To obtain a structure of an A_∞ *module from Theorem 2.1.2, one has to assume that all* $D_j^{(1)}$ *are elements of* A *and to replace braces* $\{\ \}$ *by the usual parentheses* $(\ )$ *symbolizing evaluation of a multi-linear map at elements of* A.

Proof of the theorem 2.1.1. First let us prove that $m^{(1)}$ is an A_∞ structure on $C_\bullet(C^\bullet(A))$. Decompose it into the sum $\delta + \widetilde{m}^{(1)}$ where δ is the differential induced by the differential on $C^\bullet(A)$. We want to prove that $[\delta, \widetilde{m}^{(1)}] + \frac{1}{2}[\widetilde{m}^{(1)}, \widetilde{m}^{(1)}] = 0$. We first compute $\frac{1}{2}[\widetilde{m}^{(1)}, \widetilde{m}^{(1)}]$. It consists of the following terms:

(1) $m\{\ldots D_0^{(1)} \ldots m\{\ldots D_0^{(i+1)} \ldots D_0^{(j)} \ldots\} \ldots D_0^{(n)} \ldots\} \otimes \ldots$

where the only elements allowed inside the inner $m\{\ldots\}$ are $D_p^{(q)}$ with $i + 1 \leq q \leq j$;

(2) $m\{\ldots D_0^{(1)} \ldots m\{\ldots\} \ldots D_0^{(n)} \ldots\} \otimes \ldots$

where the only elements allowed inside the inner $m\{\ldots\}$ are $D_p^{(q)}$ for one and only q (these are the contributions of the term $\widetilde{m}^{(1)}(a^{(1)}, \ldots, ba^{(q)}, \ldots, a^{(n)})$;

(3) $m\{\ldots D_0^{(1)} \ldots D_0^{(n)} \ldots\} \otimes \ldots \otimes m\{\ldots\} \otimes \ldots$

with the only requirement that the second $m\{\ldots\}$ should contain elements $D_p^{(q)}$ and $D_{p'}^{(q')}$ with $q \neq q'$.(The terms in which the second $m\{\ldots\}$ contains $D_p^{(q)}$ where all q's are the same cancel out: they enter twice, as contributions from $b\widetilde{m}^{(1)}(a^{(1)}, \ldots, a^{(q)}, \ldots, a^{(n)}$ and from $\widetilde{m}^{(1)}(a^{(1)}, \ldots, ba^{(1)}, \ldots, a^{(n)})$.

The collections of terms (1) and (2) differ from

(0) $\frac{1}{2}[m, m]\{\ldots D_0^{(1)} \ldots \ldots D_0^{(n)} \ldots\} \otimes \ldots$

by the sum of all the following terms:

($1'$) terms as in (1), but with a requirement that in the inside $m\{\ldots\}$ an element $D_p^{(q)}$ must me present such that $q \leq i$ or $q > j$;

(2′) terms as in (1), but with a requirement that the inside $m\{\dots\}$ must contain elements $D_p^{(q)}$ and $D_{p'}^{(q')}$ with $q \neq q'$.

Assume for a moment that $D_p^{(q)}$ are elements of a commutative algebra (or, more generally, of a C_∞ algebra, i.e. a homotopy commutative algebra). Then there is no δ and $\widetilde{m}^{(1)} = m^{(1)}$. But the terms (1′) and (2′) all cancel out, as well as (3). Indeed, they all involve $m\{\dots\}$ with some shuffles inside, and m is zero on all shuffles. (the last statement is obvious for a commutative algebra, and is exactly the definition of a C_∞ algebra).

Now, we are in a more complex situation where $D_p^{(q)}$ are Hochschild cochains (or, more generally, elements of a *brace algebra*). Recall that all the formulas above assume that cochains $D_p^{(q)}$ may contain their neighbors on the right inside the braces. We claim that

(A) the terms (1′), (2′) and (3), together with (0), cancel out with the terms constituting $[\delta, \widetilde{m}^{(1)}]$.

To see this, recall from [**?**] the following description of brace operations. To any rooted planar tree with marked vertices one can associate an operation on Hochschild cochains. The operation

$$D\{\dots E_1\{\dots\{Z_{1,1},\dots,Z_{1,k_1}\},\dots\}\dots E_n\{\dots\{Z_{n,1},\dots,Z_{n,k_n}\}\dots\}\dots\}$$

corresponds to a tree where D is at the root, E_i are connected to D by edges, and so on, with Z_{ij} being external vertices. The edge connecting D to E_i is to the left from the edge connecting D to E_j for $i < j$, etc. Furthermore, one is allowed to replace some of the cochains D, E_i, etc. by the cochain m defining the A_∞ structure. In this case we leave the vertex unmarked, and regard the result as an operation whose input are cochains marking the remaining vertices (at least one vertex should remain marked).

For a planar rooted tree T with marked vertices, denote the corresponding operation by $\mathbf{O}_T$. The following corollary from Proposition 6.0.2 was proven in [**?**]:

$$[\delta, \mathbf{O}_T] = \sum_{T'} \pm \mathbf{O}_{T'}$$

where T' are all the trees from which T can be obtained by contracting an edge. One of the vertices of this new edge of T' inherits the marking from the vertex to which it gets contracted; the other vertex of that edge remains unmarked. There is one restriction: the unmarked vertex of T' must have more than one outgoing edge. Using this description, it is easy to see that the claim (A) is true.

Now let us prove that

$$[\delta, \widetilde{m}^{(2)}] + \widetilde{m}^{(1)} \circ m^{(2)} + m^{(2)} \circ \widetilde{m}^{(1)} = 0$$

The summand $m^{(2)} \circ \widetilde{m}^{(1)}$ contributes terms of the form:

(1) $D_0^{(1)} \otimes \dots \otimes D_0^{(2)} \otimes \dots \otimes D_0^{(n)} \otimes \dots$
(2) $D_0^{(n)} \otimes \dots \otimes D_0^{(1)} \otimes \dots \otimes D_0^{(n-1)} \otimes \dots$
(3) $1 \otimes \dots D_0^{(1)} \otimes \dots \otimes m\{D_0^{(i+1)} \dots D_0^{(j)}\} \otimes \dots \otimes D_0^{(n)} \otimes \dots$ where $j \geq i$.

The summand $\widetilde{m}^{(1)} \circ m^{(2)}$ contributes terms of the form:

(4) Same as (3), but with the only elements allowed inside the $m\{\dots\}$ being $D_p^{(q)}$ with $i + 1 \leq q \leq j$;

(5) $1\otimes \dots D_0^{(1)}\otimes\cdots\otimes m\{\dots\}\otimes\dots\otimes D_0^{(n)}\otimes\dots$
where the only elements allowed inside the $m\{\dots\}$ are $D_p^{(q)}$ for one and only q.

The terms of type (1) and (2) cancel out - indeed, $bm^{(2)}(a^{(1)},\dots,a^{(1)})$ contributes both (1) and (2); $\widetilde{m}^{(1)}(a^{(1)},m^{(2)}(a^{(2)},\dots,a^{(n)}))$ contributes (1), and $\widetilde{m}^{(1)}(m^{(2)}(a^{(1)},\dots,a^{(n-1)}),a^{(n)})$ contributes (2).

The sum of the terms (3), (4), (5) is equal to zero by the same reasoning as in the end of the proof of $[\widetilde{m}^{(1)},\widetilde{m}^{(1)}]=0$. □

The proof of the theorem 2.1.2 is the same as above. □

Remark 2.1.4. Let A be a commutative algebra. Then $C_\bullet(A)[[u]]$ is not only a subcomplex but an A_∞ subalgebra of $C_\bullet(C^\bullet(A))[[u]]$. This A_∞ structure on $C_\bullet(A)[[u]]$ was introduced in [**?**].

The corresponding binary product was defined by Hood and Jones [**?**]. One can define it for any algebra A, commutative or not. If A is not commutative then this product is not compatible with the differential. Nevertheless, we will use it in **??**.

CHAPTER 13

Rigidity and the Gauss-Manin connection for periodic cyclic complexes

1. Introduction

2. Rigidity of periodic cyclic complexes

DEFINITION 2.0.1. *Let A be a k-module and I a k-submodule. Let 1 be an element of A. Fix an algebra structure on A/I with unit 1. A 3-pd lifting of the algebra A/I is an algebra structure on A for which*

(1) *I is a 3-pd ideal (cf. Definition 7.0.1);*
(2) *1 is a unit.*

THEOREM 2.0.2. *For any two 3-pd liftings (A, m_0) and (A, m_1) of the algebra A/I there is a natural isomorphism of complexes*

$$T_{01} : \widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(A, m_0) \xleftarrow{\sim} \widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(A, m_1)$$

of I-adic completions of periodic cyclic complexes that induces the identity map on $\mathrm{CC}^{\mathrm{per}}_{\bullet}(A/I)$. Moreover, for any $n > 2$ and any n liftings there is a morphism of degree $1 - n$ of graded k-modules

$$T_{0\ldots n} : \widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(A, m_0) \longleftarrow \widehat{\mathrm{CC}}^{\mathrm{per}}_{\bullet}(A, m_n)$$

and the following A_∞ relations are satisfied:

$$[b + uB, T_{0\ldots n}] - \sum_{j=1}^{n-1} (-1)^j T_{0\ldots \widehat{j} \ldots n} + \sum_{j=1}^{n-1} (-1)^j T_{0\ldots j} T_{j\ldots n} = 0$$

In other words, if X is the set of 3-pd liftings then assigning to a lifting its completed periodic cyclic complex extends to an A_∞ functor from I_X to the category of complexes. Here I_X is the category with the set of objects X and with exactly one morphism between any two objects.

PROOF.

LEMMA 2.0.3. *Let $(\mathfrak{g}, \delta)$ be a DG Lie algebra over $k((u))$. Assume it has a a decreasing filtration F^n such that for any n there exists m such that $n!F^n \subset F^m$ and m goes to infinity when m does. Let $\widehat{U}(\mathfrak{g}[\epsilon])$ be the completion of the universal enveloping algebra of $(\mathfrak{g}[\epsilon], \delta + u\frac{\partial}{\partial \epsilon})$ over $k((u))$ with respect to the induced filtration. Let λ be a Maurer Cartan element of $\mathfrak{g}$, i.e. an element of $\mathfrak{g}^1$ satisfying*

$$\delta\lambda + \frac{1}{2}[\lambda, \lambda] = 0. \tag{2.1}$$

Then

$$(\delta + u\frac{\partial}{\partial \epsilon} + \lambda)(e^{-\frac{\epsilon\lambda}{u}}) = 0$$

and

$$(\delta + u\frac{\partial}{\partial \epsilon})(e^{\frac{\epsilon\lambda}{u}}) = e^{\frac{\epsilon\lambda}{u}}\lambda$$

in $\widehat{U}(\mathfrak{g}[\epsilon])$.

PROOF.

$$(\delta + u\frac{\partial}{\partial \epsilon})(e^{-\frac{\epsilon\lambda}{u}}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} \sum_{k=0}^{n-1} (\epsilon\lambda)^k (\delta + u\frac{\partial}{\partial \epsilon})(\epsilon\lambda)(\epsilon\lambda)^{n-1-k} =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} \sum_{k=0}^{n-1} (\epsilon\lambda)^k (\epsilon\frac{[\lambda,\lambda]}{2} + u\lambda)(\epsilon\lambda)^{n-1-k} =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} n\epsilon\frac{[\lambda,\lambda]}{2}(\epsilon\lambda)^{n-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} \sum_{k=0}^{n-1} uk\epsilon[\lambda,\lambda](\epsilon\lambda)^{n-2} +$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} nu\lambda(\epsilon\lambda)^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{u^n (n-1)!} \epsilon\frac{[\lambda,\lambda]}{2}(\epsilon\lambda)^{n-1} +$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{u^{n-1} (n-2)!} \epsilon\frac{[\lambda,\lambda]}{2}(\epsilon\lambda)^{n-2} - \lambda e^{-\frac{\epsilon\lambda}{u}} = -\lambda e^{-\frac{\epsilon\lambda}{u}}$$

$\square$

Let m_0 and m_1 be two 3-pd liftings of the algebra A/I. Let

$$\lambda_{01} = m_0 - m_1 \tag{2.2}$$

Next we are going to refine the A_∞ action from Theorem 1.0.3. Now it is enough for us to consider the case $k = \mathbb{Z}_{(2)}$ (integers with 2 inverted), $K = [[u]]$, and $\mathfrak{g}$ being the free Lie graded Lie algebra generated by elements λ_j of degree 1, $j = 0, 1, \ldots$, satisfying $\delta\lambda_j + \frac{1}{2}[\lambda_j, \lambda_j] = 0$.

We define $S^{\mathrm{pd}}(\mathfrak{g})$ to be the symmetric algebra with divided powers, i.e. the universal graded commutative K-algebra to which $\mathfrak{g}$ maps as a K-module and where $x^{[n]} = \frac{1}{n!}x^n$ are defined and satisfy the standard properties for all elements $x \in g$ of even degree (or equivalently: for all generators of even degree in some set of homogeneous generators).

Denote by $U(\mathfrak{g}) \ltimes_0 \overline{S}^{\mathrm{pd}}(\mathfrak{g})$ the graded algebra $U(\mathfrak{g}) \ltimes \overline{S}^{\mathrm{pd}}(\mathfrak{g})$ with the new differential $\delta + \partial_{\mathrm{Cobar}} + uB_0$ where the value of B_0 on free generators of Cobar is given by

$$B_0((D_1^{[n_1]} \ldots D_m^{[n_m]})) = 0, \sum n_j > 1;$$
$$B_0((D_1)) = D_1 \in U(\mathfrak{g})$$

Here $D_j \in \mathfrak{g}$ and for $n_j > 0$ We have a commutative diagram of morphisms of DG algebras

$$\begin{tikzcd}
\mathrm{CobarBar}(U(\mathfrak{g}[\epsilon])) \arrow[rr] \arrow[dr, "\sim"'] & & U(\mathfrak{g}) \ltimes_0 \overline{S}^{\mathrm{pd}}(\mathfrak{g}) \arrow[dl, "\sim"] \\
& U(\mathfrak{g}[\epsilon]) &
\end{tikzcd} \tag{2.3}$$

Here $\mathfrak{g}[\epsilon]$ is viewed as a DG Lie algebra with the differential $\delta + u\frac{\partial}{\partial u}$. Diagonal morphisms are quasi-isomorphisms. The main difference with characteristic zero is that the horizontal morphism cannot be made ad($\mathfrak{g}$)-equivariant. We can only make it satisfy

$$\phi(ab_1|\dots|b_n) = a\phi(b_1|\dots|b_n) \tag{2.4}$$

for $a \in U(\mathfrak{g})$ and $b_j \in U(\mathfrak{g}[\epsilon])$.

Now consider the commutative diagram

$$\begin{tikzcd} U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}(\overline{S}^{\mathrm{pd}}(\mathfrak{g})) \arrow[r] \arrow[d] & U^{\mathrm{PD}}(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}_{\mathbb{Q}})) \arrow[d] \\ U(\mathfrak{g}[\epsilon]) \arrow[r] & U(\mathfrak{g}_{\mathbb{Q}}[\epsilon]) \end{tikzcd} \tag{2.5}$$

where the upper horizontal morphism is defined on free generators of Cobar as follows. For $x \in \overline{S}^{n,\mathrm{pd}}(\mathfrak{g})$, define by

$$\Delta_{n-k,k}x = \sum x^{(1)(n-k)} \otimes x^{(2)(k)} \tag{2.6}$$

the component of the coproduct Δx in $\overline{S}^{n-k,\mathrm{pd}}(\mathfrak{g}) \otimes \overline{S}^{k,\mathrm{pd}}(\mathfrak{g})$

$$(x) \mapsto \sum_{k=1}^{n} \frac{k!(n-k)!}{n!} c_k^n \mathrm{PBW}(x^{(1)(n-k)})(x^{(2)(k)}) \tag{2.7}$$

where

$$z(z-1)\dots(z-n+1) = \sum_{k=1}^{n} c_k^n z^k$$

Composing the upper horizontal maps in (2.3) and (2.5), we get an A_∞ morphism

$$U(\mathfrak{g}[\epsilon]) \to U(\mathfrak{g}_{\mathbb{Q}}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}_{\mathbb{Q}}) \tag{2.8}$$

satisfying

$$\phi_n(D, a_2, \dots, a_n) = 0,\ n > 1,\ D \in \mathfrak{g}.$$

We have.

$$(b_0 + uB)\phi_1(e^{-\frac{\epsilon\lambda_{01}}{u}}) = \phi_1(e^{-\frac{\epsilon\lambda_{01}}{u}})(b_2 + uB) \tag{2.9}$$

where b_j are the Hochschild differentials corresponding to the products m_j. Indeed, we have $\phi_1(\lambda_{01}) = L_{m_0} - L_{m_1} = b_0 - b_1$. More generally, if $\lambda_{jk} = m_j - m_k$ and $\delta = [m_n, \_]$ then

$$(\delta + u\frac{\partial}{\partial\epsilon})(e^{-\frac{\epsilon\lambda_{jk}}{u}}) = -\lambda_{jn}e^{-\frac{\epsilon\lambda_{jk}}{u}} + e^{-\frac{\epsilon\lambda_{jk}}{u}}\lambda_{kn} \tag{2.10}$$

Now define (substituting λ_{kn} for λ_{n-k})

(2.11)

$$T_{0\dots n} = \sum \phi(e^{-\frac{\epsilon\lambda_{01}}{u}}, \lambda_{1n}, \dots, \lambda_{1n}, e^{-\frac{\epsilon\lambda_{12}}{u}}, \dots, e^{-\frac{\epsilon\lambda_{n-2,n-1}}{u}}, \lambda_{n-1,n}, \dots, \lambda_{n-1,n}, e^{-\frac{\epsilon\lambda_{n-1,n}}{u}})$$

Note that the denominators of the right hand side are under control. Namely, the right hand side is a sum of terms that have N of λ_{jk} as an input, either in the form of $\lambda_{j,j+1}$ or entering $\frac{1}{m!}(\epsilon\lambda_{jn})^m$ in the exponent. The denominator of this term is at most $\frac{1}{N!^3}$. In fact, the number 3 comes from the three contributions: $m!$ in the exponentials; the factorials in $\overline{S}^{\mathrm{pd}}$; and the ones in (2.7).

The A_∞ identity for ϕ implies that

$$(b_0 + uB)T_{0\dots n} - T_{0\dots n}(b_n + uB) = \lambda_{0n}\phi_n(\dots) +$$

$$\sum \pm\phi_n(\ldots(\delta+u\frac{\partial}{\partial\epsilon})e^{-\frac{\epsilon\lambda_{j,j+1}}{u}}\ldots)+\sum\pm\phi_j(\ldots e^{-\frac{\epsilon\lambda_{j-1,j}}{u}})\phi_{n-j}(e^{-\frac{\epsilon\lambda_{j,j+1}}{u}}\ldots)+$$
$$\sum\pm\phi_{n-1}(\ldots e^{-\frac{\epsilon\lambda_{j-1,j}}{u}}e^{-\frac{\epsilon\lambda_{j,j+1}}{u}}\ldots)=\lambda_{0n}\phi_n(\ldots)$$
$$+\sum\pm\phi_n(\ldots\lambda_{jn}e^{-\frac{\epsilon\lambda_{j,j+1}}{u}}\ldots)+\sum\pm\phi_n(\ldots e^{-\frac{\epsilon\lambda_{j,j+1}}{u}}\lambda_{j+1,n}\ldots)+$$
$$\sum\pm\phi_j(\ldots e^{-\frac{\epsilon\lambda_{j-1,j}}{u}})\phi_{n-j}(e^{-\frac{\epsilon\lambda_{j,j+1}}{u}}\ldots)+\sum\pm\phi_{n-1}(\ldots e^{-\frac{\epsilon\lambda_{j-1,j}}{u}}e^{-\frac{\epsilon\lambda_{j,j+1}}{u}}\ldots)=$$
$$\sum_{j=1}^{n-1}(-1)^j T_{0\ldots\widehat{j}\ldots n}-\sum_{j=1}^{n-1}(-1)^j T_{0\ldots j}T_{j\ldots n}$$

MODIFY THIS? We used the fact that $\phi_k(\ldots,\lambda,\ldots)=0$ for $k>1$ and $\lambda\in\mathfrak{g}_A$. $\square$

REMARK 2.0.4. Lemma 2.0.3 strongly resembles BV formalism. The resemblance between the formal parameter u and the Planck constant $\hbar$, both here and in calculations with cyclic homology of deformation quantization, is rather intriguing.

COROLLARY 2.0.5. *Let A be a torsion-free $\mathbb{Z}$-module. Let $p>3$ be a prime. For any products $n>0$ and any products $m_0,\ldots,m_n$ on A that are the same modulo p, there is a morphism of degree $n-1$ of p-adically completed periodic cyclic complexes*

$$T_{0\ldots n}:\widehat{\mathrm{CC}}_\bullet^{\mathrm{per}}(A,m_0)\longleftarrow\widehat{\mathrm{CC}}_\bullet^{\mathrm{per}}(A,m_n)$$

satisfying the identities from Theorem 2.0.2.

3. The Gauss-Manin connection in characteristic zero

Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras where S is a manifold (real, complex, or algebraic). We assume that $\mathcal{A}$ carries a connection ∇ (not necessarily compatible with the product). Let $\mathrm{CC}_\bullet^{\mathrm{per}}(\mathcal{A})$ be the sheaf of periodic cyclic complexes of $\mathcal{A}$ over $\mathcal{O}_S$. (If $\mathcal{A}$ is the sheaf of local sections of a bundle of algebras then $\mathrm{CC}_\bullet^{\mathrm{per}}(\mathcal{A})$ is the sheaf of local sections of the bundle of complexes $s\mapsto\mathrm{CC}_\bullet^{\mathrm{per}}(\mathcal{A}_s)$). In this section we construct a *flat superconnection* on $\mathrm{CC}_\bullet^{\mathrm{per}}(\mathcal{A})$, i.e. an operator

$$\nabla_{\mathrm{GM}}:\Omega_S^\bullet\otimes_{\mathcal{O}_S}\mathrm{CC}_\bullet^{\mathrm{per}}(\mathcal{A})\to\Omega_S^\bullet\otimes_{\mathcal{O}_S}\mathrm{CC}_\bullet^{\mathrm{per}}(\mathcal{A})$$

of degree one such that $\nabla_{\mathrm{GM}}^2=0$ and $\nabla_{\mathrm{GM}}(fa)=f\nabla_{\mathrm{GM}}(a)+df\cdot a$ for a functtion f and a local section a.

Let $C^\bullet(\mathcal{A})$ be the sheaf of Hochschild cochain complexes of $\mathcal{A}$ over $\mathcal{O}_S$. The product on $\mathcal{A}$ defines a two-cochain m; then ∇m is a section of $\Omega^1(S,C^2(\mathcal{A})$. Note also that $\nabla^2=R\in\Omega^2(S,\mathrm{End}(\mathcal{A})=\Omega^2(S,C^0(\mathcal{A}))$. Put

$$\alpha=\nabla m+R;$$

one has

$$(\delta+\nabla)^2=\alpha;\ (\delta+\nabla)(\alpha)=0$$

(recall that $\delta=[m,?]$ is the Hochschild differential). Put

$$\nabla_{\mathrm{GM}}=b+uB+\nabla+\sum_{n\geq1}u^{-n}\phi_n(\alpha,\ldots,\alpha)$$

where $\phi_n:U(\mathfrak{g}_{\mathcal{A}}[\epsilon,u])^{\otimes n}\to\mathrm{End}(\widehat{\mathrm{CC}}_\bullet^{\mathrm{per}}(\mathcal{A}))$ are the components of the A_∞ module structure given by Theorem 1.0.3

PROPOSITION 3.0.1. ∇_{GM} *is a flat superconnection.*

Proof. □

4. The Gauss Manin connection for pd liftings

Now let $\mathcal{A}$ be a free $\mathcal{O}_S$ module. We now assume that all products m_s, $s \in S$, are liftings of the same product. In other words, let k be a commutative unital ring over $\mathbb{Z}_{(2)}$; let A_0 be an algebra over k without $\mathbb{Z}_{(2)}$-torsion; let $\mathcal{O}_S$ be a commutative unital k-algebra. Put $\mathcal{A} = A_0 \otimes_k \mathcal{O}_S$. This is an $\mathcal{O}_S$-algebra with the product that we denote by m_0. Now consider another $\mathcal{O}_S$-algebra structure on $\mathcal{A}$. Denote the corresponding product by m. We require that

$$\lambda = m_0 - m \tag{4.1}$$

take values in a pd ideal $\mathcal{I}$ of $\mathcal{A}$.

Consider the DG Lie algebra $\mathfrak{g}$ that is the free graded Lie algebra with generators λ of degree 1 and $d\lambda$ of degree 2, with the differential determined by

$$\delta\lambda + \frac{1}{2}[\lambda, \lambda] = 0;\ \delta d\lambda + [\lambda, d\lambda] = 0$$

We can construct the A_∞ morphism from (2.8) for this algebra. (To be precise, the algebra U^{PD} is constructed using the grading where the degree of λ and $d\lambda$ are both one.

$$\nabla_{\mathrm{GM}} = d + b + uB + \phi(\frac{\epsilon d\lambda}{u}) + \sum_{n_1>0}\sum_{n_2\geq 0} u^{-n_1}\phi(\mathrm{sh}((\epsilon d\lambda)^{\otimes n_1}, \lambda^{\otimes n_2}) \tag{4.2}$$

Theorem 4.0.1. *Formula* (4.2) *defines a flat superconnection on the completed periodic cyclic complex* $\widehat{\mathrm{CC}}^{\mathrm{per}}_{-\bullet}(\mathcal{A})$.

Proof. □

Corollary 4.0.2. *Let $p > 3$ is a prime. Assume that $\mathcal{A}$ is without $\mathbb{Z}_{(2)}$-torsion and that, as an algebra, $\mathcal{A}/p\mathcal{A} \xrightarrow{\sim} \overline{A}_0 \otimes_{\mathbb{F}_p} (\mathcal{O}_S/p\mathcal{O}_S)$ for an $\mathbb{F}$-algebra $\overline{A}_)$. Then the p-adically completed periodic cyclic complex of $\mathcal{A}$ carries a flat superconnection.*

5. Bibliographical notes

CHAPTER 14

DG categories

1. Introduction

The contents of this section are taken mostly from [**175**], [**?**], and [**465**].

2. Definition and basic properties

A (small) differential graded (DG) category $\mathcal{A}$ over k is a ***set*** $\mathrm{Ob}(\mathcal{A})$ of elements called objects and of complexes $\mathcal{A}(x, y)$ of k-modules for every $x, y \in \mathrm{Ob}(A)$, together with morphisms of complexes

$$\mathcal{A}(x, y) \otimes \mathcal{A}(y, z) \to \mathcal{M}(x, z),\ a \otimes b \mapsto ab, \tag{2.1}$$

and zero-cycles $\mathbf{1}_x \in \mathcal{A}(x, x)$, such that (2.1) is associative and $\mathbf{1}_x a = a \mathbf{1}_y = a$ for any $a \in \mathcal{A}(x, y)$. For a DG category, its homotopy category is the k-linear category $\mathrm{Ho}(\mathcal{A})$ such that $\mathrm{Ob}(\mathrm{Ho}(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Ho}(\mathcal{A})(x, y) = H^0(\mathcal{A}(x, y))$, with the units being the classes of $\mathbf{1}_x$ and the composition induced by (2.1).

A DG functor $\mathcal{A} \to \mathcal{B}$ is a map $\mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$, $x \mapsto Fx$, and a collection of morphisms of complexes $F_{x,y} \colon \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy), x, y \in \mathrm{Ob}(\mathcal{A})$, which commutes with the composition (2.1) and such that $F_{x,x}(\mathbf{1}_x) = \mathbf{1}_{Fx}$ for all x.

The opposite DG category of $\mathcal{A}$ is defined by $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$, $\mathcal{A}^{\mathrm{op}}(x, y) = \mathcal{A}(y, x)$, the unit elements are the same as in $\mathcal{A}$, and the composition (2.1) is the one from $\mathcal{A}$, composed with the transposition of tensor factors.

3. Semi-free DG categories

Semi-free DG categories are defined exactly as in 2. For a k-linear graded category $\mathcal{A}$ and a collection of graded k-modules $\{\mathcal{V} = V(x, y) | x, y \in \mathrm{Ob}(\mathcal{A})\}$ one defines by the usual universal property a new k-linear category freely generated by $\mathcal{A}$ and all $\mathcal{V}$. A DG category $\mathcal{R}$ is semi-free over $\mathcal{A}$ if it is freely generated over $\mathcal{A}$ by a collection$\mathcal{V}$ and there is an increasing filtration $\mathcal{V}_n | n \geq -1$, $\mathcal{V}_{-1} = 0$, $d\mathcal{V}_n$ is contained in the subcategory generated by $\mathcal{A}$ and $\mathcal{V}_{n-1}$, and $d|\mathcal{A}$ is the differential of $\mathcal{A}$.

For a set S define the category k_S as follows: $\mathrm{Ob}(k_S) = S$; $k_S(x, y) = 0$ for $x \neq y$; $k_S(x, x) = k\mathbf{1}_x$. A DG category is called semi-free if it is semi-free over the DG category $k_{\mathrm{Ob}(A)}$. Existence and uniqueness up to homotopy equivalence of a semi-free resolution of a DG category is proved as in 2 without any changes. Similarly the relative case 3.2 for a DG functor $\mathcal{A} \to \mathcal{B}$ *which is the identity map on objects.*

4. Quasi-equivalences

A quasi-equivalence [?] between DG categories $\mathcal{A}$ and $\mathcal{B}$ is a DG functor $F : \mathcal{A} \to \mathcal{B}$ such that a) F induces an equivalence of homotopy categories and b) for any $x, y \in \mathrm{Ob}(\mathcal{A})$, $F_{x,y} \colon \mathcal{A}(x,y) \to \mathcal{B}(Fx, Fy)$ is a quasi-isomorphism.

5. Drinfeld quotient

For a full DG subcategory $\mathcal{A}$ of a DG category $\mathcal{B}$, the quotient of $\mathcal{B}$ by $\mathcal{A}$ is by definition the graded category freely generated by $\mathcal{B}$ and the family $\mathcal{V}(x,y) = k\epsilon_x$, $|\epsilon_x| = -1$, in $x = y \in \mathrm{Ob}(\mathcal{A})$; $\mathcal{V}(x,y) = 0$ in all other cases. The differential on $\mathcal{B}/\mathcal{A}$ extends the one on $\mathcal{B}$ and satisfies $d\epsilon_x = \mathbf{1}_x$.

In other words, it is a DG category $\mathcal{B}/\mathcal{A}$ such that:

(1) $\mathrm{Ob}(\mathcal{B}/\mathcal{A}) = \mathrm{Ob}(\mathcal{B})$;
(2) there is a DG functor $i : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ which is the identity on objects;
(3) for every $x \in \mathrm{Ob}(\mathcal{A})$, there is an element ϵ_x of degree -1 in $\mathcal{B}/\mathcal{A}(x,x)$ satisfying $d\epsilon_x = \mathbf{1}_x$;
(4) for any other DG category $\mathcal{B}'$ together with a DG functor $i' : B \to \mathcal{B}'$ and elements ϵ'_x as above, there is unique DG functor $f : \mathcal{B}/\mathcal{A} \to \mathcal{B}'$ such that $i' = f \circ i$ and $\epsilon_x \mapsto \epsilon'_x$.

One has

$$(\mathcal{B}/\mathcal{A})(x,y) = \bigoplus_{n \geq 0} \bigoplus_{x_1, \ldots, x_n \in \mathrm{Ob}(A)} \mathcal{A}(x, x_1)\epsilon_{x_1}\mathcal{A}(x_1, x_2)\epsilon_{x_2} \ldots \epsilon_{x_n}\mathcal{A}(x_n, y);$$

it is easy to define the composition and the differential explicitly.

6. DG modules over DG categories

A DG module over a DG category $\mathcal{A}$ is a collection of complexes of k-modules $\mathcal{M}(x)$, $x \in \mathrm{Ob}(\mathcal{A})$, together with morphisms of complexes

$$\mathcal{A}(x,y) \otimes \mathcal{M}(y) \to \mathcal{M}(x),\ a \otimes m \mapsto am, \tag{6.1}$$

which is compatible with the composition (2.1) and such that $\mathbf{1}_x m = m$ for all x and all $m \in \mathcal{M}(x)$. A DG bimodule over $\mathcal{A}$ is a collection of complexes $\mathcal{M}(x,y)$ together with morphisms of complexes

$$\mathcal{A}(x,y) \otimes \mathcal{M}(y,z) \otimes \mathcal{A}(z,w) \to \mathcal{M}(x,w),\ a \otimes m \otimes b \mapsto amb, \tag{6.2}$$

that agrees with the composition in $\mathcal{A}$ and such that $\mathbf{1}_x m \mathbf{1}_y = m$ for any x, y, m. We put $am = am\mathbf{1}_z$ and $mb = \mathbf{1}_x mb$. A DG bimodule over $\mathcal{A}$ is the same as a DG module over $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

6.1. Semi-free DG modules.

6.2. The enveloping pre-triangulated DG category.

7. Hochschild and cyclic complexes of DG categories

7.1. Definitions.

DEFINITION 7.1.1. *For a DG category $\mathcal{A}$ and a DG bimodule $\mathcal{M}$, set*

$$C_\bullet(\mathcal{A},\mathcal{M}) = \bigoplus_{n\geq 0; x_0,\ldots,x_n\in\mathrm{Ob}(\mathcal{A})} \mathcal{M}(x_0,x_1)\otimes\overline{\mathcal{A}}(x_1,x_2)[1]\otimes\ldots\otimes\overline{\mathcal{A}}(x_n,x_0) \tag{7.1}$$

where

$$\overline{\mathcal{A}}(x,y)=\mathcal{A}(x,y)\,\textit{when}\ x\neq y\ \textit{and}\overline{\mathcal{A}}(x,x)=\mathcal{A}(x,x)/k\mathbf{1}_x.$$

Define also

$$C_\bullet(\mathcal{A})=C_\bullet(\mathcal{A},\mathcal{A})$$

the differentials b, d, and B are defined exactly as in 3. Similarly for the non-normalized complex $\widetilde{C}_\bullet(\mathcal{A})$.

As usual,

$$\mathrm{CC}^-(\mathcal{A})=(C_\bullet(\mathcal{A})[[u]],b+d+uB)$$

and similarly for the cyclic and periodic cyclic complexes.

8. Invariance properties of Hochschild and cyclic complexes

8.1. Passing to matrices. If $\mathcal{A}$ is a DG category then let

$$M(\mathcal{A})=\varinjlim_{n\to\infty}(\mathcal{A})=\mathcal{A}\otimes M(k)$$

be the dg category of finite matrices $m_{jk}|j,k\geq 0$ with entries in $\mathcal{A}$. We have an embedding

$$i:\mathcal{A}\to M(\mathcal{A});\ a\mapsto aE_{00} \tag{8.1}$$

Here, as usual, E_{jk} is the elementary matrix with the only nonzero entry 1 that is located in row j and column k.

PROPOSITION 8.1.1. *The embedding* (3) *induces homotopy eqivalences of Hochschild, negative cyclic, cyclic, and periodic cyclic complexes.*

PROOF. As in the case of ordinary algebras, this can be easily deduced from the fact that the Hochschild homology is the derived tensor product. Here we will give an explicit proof that is almost identical to the proofs of the invariance properties in 8.2 below and in 16.

First, observe that

$$\mathrm{tr}:C_\bullet(M(\mathcal{A}))\to C_\bullet(\mathcal{A});\ a_0\otimes\ldots\otimes a_n\mapsto\sum(a_0)_{i_0i_1}\otimes\ldots\otimes(a_n)_{i_ni_0}$$

commutes with all the differentials. We have $\mathrm{tr}\circ i=\mathrm{id}$ and $\mathrm{id}-i\circ\mathrm{tr}=[b+d,h]$ where the homotopy h is defined by

$$h(a_0\otimes\ldots\otimes a_n)=\sum_{p=0}^{n}\sum\pm a_0E_{i_00}\otimes E_{0i_0}a_1E_{i_10}\otimes E_{0i_{p-1}}a_pE_{i_p0}\otimes E_{0i_p}\otimes a_{p+1}\otimes\ldots\otimes a_n$$

The sign is $(-1)^{\sum_{j=0}^{p}|a_j|+p}$. This proves the statement for the Hochschild complex. the statement for the various versions of the cyclic complex follows because corresponding morphisms preserve the filtration by powers of u and are quasi-isomorphisms on associated graded quotients. $\square$

8.2. Adding idempotents. For a DG category $\mathcal{A}$, consider a new DG category $\mathcal{A}^{\mathrm{id}}$. Its objects are pairs (x, e) where $x \in \mathrm{Ob}(\mathcal{A})$, $e \in \mathcal{A}^0(x, x)$ such that $e^2 = e$ and $de = 0$; morphisms from (x, e) to (y, f) are elements eaf where $a \in \mathcal{A}(x, y)$.

Proposition 8.2.1. *The embedding $i : \mathcal{A} \to \mathcal{A}^{\mathrm{id}}$ sending any object x to $(x, \mathbf{1}_x)$ induces homotopy equivalences of Hochschild, negative cyclic, cyclic, and periodic cyclic complexes.*

Proof. As in Proposition 8.1.1, it is enough to construct a homotopy inverse for the map of Hochschild complexes. We define it to be

$$P : e_0 a_0 e_1 \otimes \ldots \otimes e_n a_n e_0 \mapsto e_0 a_0 e_1 \otimes \ldots \otimes e_n a_n e_0$$

where in the left hand side $e_j a_j e_{j+1}$ is viewed as an element in $\mathcal{A}^{\mathrm{id}}((x_j, e_j), (x_{j+1}, e_{j+1}))$ and the right hand side as an element in $\mathcal{A}(x, y)$. Therefore $i \circ P$ sends the left hand side to itself where $e_j a_j e_{j+1}$ is viewed as an element in $\mathcal{A}^{\mathrm{id}}((x_j, \mathbf{1}_{x_j}), (x_{j+1}, \mathbf{1}_{x_{j+1}}))$. We have $P \circ i = \mathrm{id}$, while a homotopy between id and $i \circ P$ can be chosen as

$$e_0 a_0 e_1 \otimes \ldots \otimes e_n a_n e_0 \mapsto \sum_{p=0}^{n} \pm e_0 a_0 e_1 \otimes \ldots \otimes e_p a_p e_{p+1} \otimes e_{p+1} \otimes e_{p+1} a_{p+1} e_{p+2} \otimes \ldots \otimes e_n a_n e_0.$$

Here $e_j a_j e_{j+1}$ is viewed as:

1) an element of $\mathcal{A}^{\mathrm{id}}((x_j, e_j), (x_{j+1}, e_{j+1})$ for $j \le p$;
2) an element of $\mathcal{A}^{\mathrm{id}}((x_j, \mathbf{1}_{x_j}), (x_{j+1}, \mathbf{1}_{x_{j+1}}))$ for $j > p$.

The tensor factor e_{p+1} is viewed as an element $\mathcal{A}^{\mathrm{id}}((x_{p+1}, e_{p+1}), (x_{p+1}, \mathbf{1}_{x_{p+1}}))$. Also, $e_{n+1} = e_0$ and the sign is $(-1)^{\sum_{j=0}^{p} |a_j| + p}$. □

8.3. Invariance up to quasi-equivalence.

Theorem 8.3.1. *A quasi-equivalence $\mathcal{A} \to \mathcal{B}$ of DG categories induces homotopy equivalences of complexes*

$$C_\bullet(\mathcal{A}) \xrightarrow{\sim} C_\bullet(\mathcal{B});\ \mathrm{CC}_\bullet(\mathcal{A}) \xrightarrow{\sim} \mathrm{CC}_\bullet(\mathcal{B});\ \mathrm{CC}^-_\bullet(\mathcal{A}) \xrightarrow{\sim} \mathrm{CC}^-_\bullet(\mathcal{B});\ \mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A}) \xrightarrow{\sim} \mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{B})$$

Proof. We will start with proving that an equivalence of graded k-linear categories induces a homotopy equivalence of Hochschild complexes and hence of cyclic complexes of all types. For this it is enough to show that two isomorphic functors induce homotopic maps of Hochschild complexes. If $c : F \xrightarrow{\sim} G$ is an isomorphism of functors $F, G : \mathcal{A} \to \mathcal{B}$, then an explicit homotopy is given by

$$a_0 \otimes \ldots \otimes a_n \mapsto \sum_{j=0}^{n} \pm c_{x_0}^{-1} F a_0 \otimes \ldots \otimes F a_j \otimes c_{x_{j+1}} \otimes G a_{j+1} \otimes \ldots \otimes G a_n \tag{8.2}$$

Here $a_k \in \mathcal{A}(x_k, x_{k+1})$, $x_{n+1} = x_0$, and $c_x : Fx \xrightarrow{\sim} Gx$, $x \in \mathrm{Ob}(\mathcal{A})$, define the isomorphism c. The sign is $(-1)^{\sum_{p \le j}(|a_p|+1)}$.

The statement for quasi-equivalences follows when we consider the spectral sequences whose first terms are $C_\bullet(H^0(\mathcal{A}))$, $C_\bullet(H^0(\mathcal{B}))$. Functors F and G induce the same morphisms on E_1 terms and therefore on the total compexes. □

8.4. Hochschild and cyclic complexes of Drinfeld quotients. Let $\mathcal{A}$ be a full DG subvategory of $\mathcal{B}$. Let $\mathcal{B}/\mathcal{A}$ be the Drinfeld quotient.

THEOREM 8.4.1. *(Keller excision theorem).*

$$C_\bullet(\mathcal{A}) \to C_\bullet(\mathcal{B}) \to C_\bullet(\mathcal{B}/\mathcal{A})$$

is a homotopy fibration sequence of complexes.

PROOF. We will deduce the statement from the results of 3.2. Observe first that all these results remain true not only for DG algebras but for DG categories *with a fixed set of objects.* Put $\mathcal{C} = \mathcal{B}/\mathcal{A}$. Observe that $\mathcal{C}$ is semi-free over $\mathcal{B}$. We claim that the complex (3.9)

$$\mathrm{DR}^1(\mathcal{C}/\mathcal{B}) \xrightarrow{b} (\mathcal{C}/\mathcal{B})/[\mathcal{B}, \mathcal{C}/\mathcal{B}] \tag{8.3}$$

is isomorphic to the extended Hochschild complex

$$((\mathcal{A}^{*+1}, d+b') \xrightarrow{1-\tau} (\mathcal{A}^{*+1}, b+d))[1].$$

Indeed, chains on the left, resp. on the right, in the complex above can be identified with the chains of (8.3) as follows:

$$a_0 \otimes \ldots \otimes a_n \mapsto d\epsilon_{x_0} a_0 \epsilon_{x_1} a_1 \ldots \epsilon_{x_n} a_n,$$

resp.

$$a_0 \otimes \ldots \otimes a_n \mapsto \epsilon_{x_0} a_0 \epsilon_{x_1} a_1 \ldots \epsilon_{x_n} a_n.$$

Here x_j are objects of $\mathcal{A}$, $a_j \in \mathcal{A}(x_j, x_{j+1}) = \mathcal{B}(x_j, x_{j+1})$, $x_{n+1} = x_0$. The differential on the left decomes $d + b'$.the one on the right becomes $d + b$, and b in the middle becomes $1-\tau$. The shift by one occurs because in (8.3) every ϵ_{x_j} contributes -1, while in the Hochschild complex only those with $j \geq 0$ do. ***Maybe a few more words*** $\square$

9. Hochschild cochain complexes

Define the Hochschild cochain complex of a DG category $\mathcal{A}$ in a DG bimodule $\mathcal{M}$ as

$$C^\bullet(\mathcal{A}, \mathcal{M}) = \prod_{n \geq 0; x_0, \ldots, x_n \in \mathrm{Ob}(\mathcal{A})} \underline{\mathrm{Hom}}(\overline{\mathcal{A}}(x_0, x_1)[1] \otimes \ldots \otimes \overline{\mathcal{A}}(x_{n-1}, x_n)[1], \mathcal{M}(x_0, x_n)) \tag{9.1}$$

with the differential $d + \delta$, and similarly $\widetilde{C}^\bullet(\mathcal{A})$.

10. A_∞ categories and A_∞ functors

An A_∞ category is a natural generalization of both a DG category and an A_∞ algebra. We define it as a coderivation of degree one and square zero of $\mathrm{Bar}(\mathcal{A})$ where $\mathcal{A}$ is a collection graded k-modules $\mathcal{A}(x, y)$ where x, y run through a set $\mathrm{Ob}(\mathcal{A})$. We view $\mathcal{A}$ as a DG category with zero differential and product. ***No zero part?***

In other words, start with a DG category $\mathcal{A}$ where the differential and the product are zero. An A_∞ structure is an element m of degree one in $C^\bullet(\mathcal{A}, \mathcal{A})$ such that $m\{m\} = 0$. In addition, we require that the component of m corresponding to $n = 0$ as in (9.1) be zero.

More explicitly, an A_∞ category is a set $\mathrm{Ob}(\mathcal{A})$ and a collection of complexes $\mathcal{A}(x, y)$, $x, y \in \mathrm{Ob}(\mathcal{A})$, together with k-linear maps

$$m_n : \mathcal{A}(x_0, x_1) \otimes \ldots \otimes \mathcal{A}(x_{n-1}, x_n) \to \mathcal{A}(x_0, x_n) \tag{10.1}$$

of degree $2-n$, satisfying

(10.2)
$$\sum_{j\geq 0; j+k\leq n} (-1)^{\sum_{i=1}^{j}(|a_i|+1)(k+1)} m_{n+1-k}(a_1,\ldots,m_k(a_{j+1},\ldots,m_{j+k}),\ldots,a_n)=0$$

We refer the reader, for example, to [**?**].

10.1. A_∞ **bimodules and Hochschild complexes.** An A_∞ bimodule $\mathcal{M}$ over $\mathcal{A}$ is a collection of graded k-modules $\mathcal{M}(x,y)$, $x,y\in \mathrm{Ob}(\mathcal{A})$, and an A_∞ category $\mathcal{A}+\mathcal{M}$ with the same objects as $\mathcal{A}$ such that:

a) $\mathcal{A}$ is an A_∞ subcategory;

b) $(\mathcal{A}+\mathcal{M})(x,y)=\mathcal{A}(x,y)\oplus\mathcal{M}(x,y)$ as graded k-modules for all x, y;

c) the operations m_n vanish if more than one argument is in $\mathcal{M}$, and tales values in $\mathcal{M}$ if one argument is in $\mathcal{M}$ and the rest in $\mathcal{A}$.

We define Hochschild and cyclic complexes of an A_∞ category with coefficients in an A_∞ bimodule $\mathcal{M}$ by formulas (7.1) and (9.1). The differentials are

$$b_m=L_m;\ \delta_m=[m,\_] \tag{10.3}$$

11. Bar and cobar constructions for DG categories

The bar construction of a DG category $\mathcal{A}$ is a DG cocategory $\mathrm{Bar}(\mathcal{A})$ with the same objects where

$$\mathrm{Bar}(\mathcal{A})(x,y)=\bigoplus_{n\geq 0}\bigoplus_{x_1,\ldots,x_n}\mathcal{A}(x,x_1)[1]\otimes\mathcal{A}(x_1,x_2)[1]\otimes\ldots\otimes\mathcal{A}(x_n,x)[1]$$

with the differential

$$d=d_1+d_2;$$
$$d_1(a_1|\ldots|a_{n+1})=\sum_{i=1}^{n+1}\pm(a_1|\ldots|da_i|\ldots|a_{n+1});$$
$$d_2(a_1|\ldots|a_{n+1})=\sum_{i=1}^{n}\pm(a_1|\ldots|a_ia_{i+1}|\ldots|a_{n+1})$$

The signs are $(-1)^{\sum_{j<i}(|a_i|+1)+1}$ for the first sum and $(-1)^{\sum_{j\leq i}(|a_i|+1)}$ for the second. The comultiplication is given by

$$\Delta(a_1|\ldots|a_{n+1})=\sum_{i=0}^{n+1}(a_1|\ldots|a_i)\otimes(a_{i+1}|\ldots|a_{n+1})$$

Dually, for a DG cocategory $\mathcal{B}$ one defines the DG category $\mathrm{Cobar}(\mathcal{B})$. The DG category $\mathrm{CobarBar}(\mathcal{A})$ is a semi-free resolution of $\mathcal{A}$. *****

11.1. Units and counits. It is convenient for us to work with DG (co)categories without (co)units. For example, this is the case $\mathrm{Bar}(\mathcal{A})$ and $\mathrm{Cobar}(\mathcal{B})$ (we sum, by definition, over all tensor products with at least one factor). Let $\mathcal{A}^+$ be the (co)category $\mathcal{A}$ with the (co)units added, i.e. $\mathcal{A}^+(x,y)=\mathcal{A}(x,y)$ for $x\neq y$ and $\mathcal{A}^+(x,x)=\mathcal{A}(x,x)\oplus k\,\mathrm{id}_x$. If $\mathcal{A}$ is a DG category then $\mathcal{A}^+$ is an augmented DG category with units, i.e. there is a DG functor $\epsilon: A^+\to k_{\mathrm{Ob}(\mathcal{A})}$. The latter is the DG category with the same objects as $\mathcal{A}$ and with $k_I(x,y)=0$ for $x\neq y$, $k_I(x,x)=k$. Dually, one defines the DG cocategory $k^{\mathrm{Ob}(\mathcal{B})}$ and the DG functor $\eta: k^{\mathrm{Ob}(\mathcal{B})}\to\mathcal{B}^+$ for a DG cocategory B.

11.2. Tensor products. For DG (co)categories with (co)units, define $\mathcal{A}\otimes\mathcal{B}$ as follows: $\mathrm{Ob}(\mathcal{A}\otimes\mathcal{B})=\mathrm{Ob}(\mathcal{A})\times\mathrm{Ob}(\mathcal{B})$; $(\mathcal{A}\otimes\mathcal{B})((x_1,y_1),(x_2,y_2))=A(x_1,y_1)\otimes\mathcal{B}(x_2,y_2)$; the product is defined as $(a_1\otimes b_1)(a_2\otimes b_2)=(-1)^{|a_2||b_1|}a_1a_2\otimes b_1b_2$, and the coproduct in the dual way. This tensor product, when applied to two (co)augmented DG (co)categories with (co)units, is again a (co)augmented DG (co)category with (co)units: the (co)augmentation is given by $\epsilon\otimes\epsilon$, resp. $\eta\otimes\eta$.

Definition 11.2.1. *For DG categories $\mathcal{A}$ and $\mathcal{B}$ without units, put*

$$\mathcal{A}\otimes\mathcal{B}=\mathrm{Ker}(\epsilon\otimes\epsilon:\mathcal{A}^+\otimes\mathcal{B}^+\to k_{\mathrm{Ob}(\mathcal{A})}\otimes k_{\mathrm{Ob}(\mathcal{B})}).$$

Dually, for For DG cocategories $\mathcal{A}$ and $\mathcal{B}$ without counits, put

$$\mathcal{A}\otimes\mathcal{B}=\mathrm{Coker}(\eta\otimes\eta:k^{\mathrm{Ob}(\mathcal{A})}\otimes k^{\mathrm{Ob}(\mathcal{B})}\to\mathcal{A}^+\otimes\mathcal{B}^+).$$

One defines a morphism of DG cocategories

$$\mathrm{Bar}(\mathcal{A})\otimes\mathrm{Bar}(\mathcal{B})\to\mathrm{Bar}(\mathcal{A}\otimes\mathcal{B})\tag{11.1}$$

by the standard formula for the shuffle product

$$(a_1|\dots|a_m)(b_1|\dots|b_n)=\sum\pm(\dots|a_i|\dots|b_j|\dots)\tag{11.2}$$

The sum is taken over all shuffle permutations of the symbols $(a_1,\dots,a_m,b_1,\dots,b_n)$, i.e. over all permutations that preserve the order of the a_i's and the order of the b_j's. The sign is computed as follows: a transposition of a_i and b_j introduces a factor $(-1)^{(|a_i|+1)(b_j|+1)}$. Let us explain the meaning of the factors a_i and b_j in the formula. We assume $a_i\in\mathcal{A}(x_{i-1},x_i)$ and $b_j\in\mathcal{B}(y_{j-1},y_j)$ for $x_i\in\mathrm{Ob}(\mathcal{A})$ and $y_j\in\mathrm{Ob}(\mathcal{B})$, $0\le i\le m$, $0\le j\le m$. Consider a summand $(\dots|a_i|b_j|b_{j+1}|\dots|b_k|a_{i+1}|\dots)$. In this summand, all b_p, $j\le p\le k$, are interpreted as $\mathrm{id}_{x_i}\otimes b_p\in(A\otimes B)((x_i,y_{p-1}),(x_i,y_p))$. Similarly, in the summand $(\dots|b_i|a_j|a_{j+1}|\dots|a_k|a_{i+1}|\dots)$, all a_p, $j\le p\le k$, are interpreted as $a_p\otimes\mathrm{id}_{y_i}\in(\mathcal{A}\otimes\mathcal{B})((x_{p-1},y_i),(x_p,y_i))$. Dually, one defines the morphism of DG cocategories

$$\mathrm{Cobar}(\mathcal{A}\otimes\mathcal{B})\to\mathrm{Cobar}(\mathcal{A})\otimes\mathrm{Cobar}(\mathcal{B})\tag{11.3}$$

12. DG category $\mathbf{C}^\bullet(\mathcal{A},\mathcal{B})$

For two DG categories $\mathcal{A}$ and $\mathcal{B}$, define the DG category $\mathbf{C}^\bullet(\mathcal{A},\mathcal{B})$ as follows. Its objects are A_∞ functors $f:\mathcal{A}\to\mathcal{B}$. Define the complex of morphisms as

$$\mathbf{C}^\bullet(\mathcal{A},\mathcal{B})(f,g)=C^\bullet(\mathcal{A},{}_f\mathcal{B}_g)\tag{12.1}$$

where ${}_f\mathcal{B}_g$ is the complex $\mathcal{B}$ viewed as an $\mathcal{A}_\infty$ bimodule on which $\mathcal{A}$ acts on the left via f and on the right via g. The composition is defined by the cup product as in the formula (**??**).

Remark 12.0.1. Every $\mathcal{A}_\infty$ functor $f:\mathcal{A}\to\mathcal{B}$ defines an A_∞ $(\mathcal{A},\mathcal{B})$-bimodule ${}_f\mathcal{B}$, namely the family of complexes $\mathcal{B}$ on which $\mathcal{A}$ acts on the left via f and $\mathcal{B}$ on the right in the standard way. If for example f, $g:\mathcal{A}\to\mathcal{B}$ are morphisms of algebras then $C^\bullet(\mathcal{A},{}_f\mathcal{B}_g)$ computes $\mathrm{Ext}^\bullet_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}}({}_f\mathcal{B},{}_g\mathcal{B})$. What we are going to construct below does not seem to extend literally to all (A_∞) bimodules. This applies also to related constructions of the category of internal homomorphisms, such as in [**?**] and [**?**]. One can overcome this by replacing $\mathcal{A}$ by the category of A-modules, since every $(\mathcal{A},\mathcal{B})$-bimodule defines a functor between the categories of modules.

Now let us explain how to modify the product $\bullet$ from 7.2 and get a DG functor

$$\bullet : \mathrm{Bar}(\mathbf{C}^\bullet(A,B)) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(B,C)) \to \mathrm{Bar}(\mathbf{C}^\bullet(A,C)) \tag{12.2}$$

12.1. The brace operations on $\mathbf{C}^\bullet(A,B)$. For Hochschild cochains $D \in C^\bullet(B,_{f_0} C_{f_1})$ and $E_i \in \mathbf{C}^\bullet(A,_{g_{i-1}} B_{g_i}$, $1 \le i \le n$, define the cochain

$$D\{E_1, \ldots, E_n\} \in C^\bullet(A,_{f_0 g_0} C_{f_1 g_n})$$

by

$$D\{E_1, \ldots, E_n\}(a_1, \ldots, a_N) = \sum \pm D(\ldots, E_1(\underline{\ldots}), \ldots, E_n(\underline{\ldots}), \ldots) \tag{12.3}$$

where the space denoted by $\underline{\ldots}$ within $E_k(\underline{\ldots})$ stands for $a_{i_k+1}, \ldots, a_{j_k}$, and the space denoted by $\ldots$ between $E_k(\underline{\ldots})$ and $E_{k+1}(\underline{\ldots})$ stands for

$$g_k(a_{j_k+1}, \ldots,), g_k(\ldots), \ldots, g_k(\ldots, a_{i_{k+1}}).$$

The sum is taken over all possible combinations such that $i_k \le j_k \le i_{k+1}$. The signs are as in (**??**).

12.2. The $\bullet$ product on $\mathrm{Bar}(\mathbf{C}(A,B))$. For Hochschild cochains $D_i \in C^\bullet(B,_{f_{i-1}} C_{f_i})$ and $E_j \in C^\bullet(A,_{g_{j-1}} B_{g_j})$, $1 \le i \le m$, $1 \le j \le n$, we have

$$(D_1|\ldots|D_m) \in \mathrm{Bar}(\mathbf{C}^\bullet(B,C))(f_0, f_m);$$

$$(D_1|\ldots|D_m) \in \mathrm{Bar}(\mathbf{C}^\bullet(A,B))(g_0, g_m);$$

define

$$(D_1|\ldots|D_m) \bullet (E_1|\ldots|E_n) \in \mathrm{Bar}(\mathbf{C}^\bullet(A,C))(f_0 g_0, f_m g_n)$$

by the formula in the beginning of 7.2, with the following modification. The expression $D_i\{E_{j+1}, \ldots, E_k\}$ is now in $\mathbf{C}(A,C)(f_{i-1}g_{j+1}, f_i g_j)$, as explained above. The space denoted by $\ldots$ between $D_i\{E_{j+1}, \ldots, E_k\}$ and $D_{i+1}\{E_{p+1}, \ldots, E_q\}$ contains $f_i(E_{k+1}|\ldots)|f_i(\ldots)|\ldots|f_i(\ldots, E_p)$. Here, for an A_∞ functor f and for cochains $E_1, \ldots, E_k$,

$$f(E_1, \ldots, E_k)(a_1, \ldots, a_N) = \sum f(E_1(a_1, \ldots, a_{i_2-1}), \ldots, E_k(a_{i_k+1}, \ldots, a_n)) \tag{12.4}$$

The sum is taken over all possible combinations $1 = i_1 \le i_2 \le \ldots i_k$.

Lemma 12.2.1. *1) The product $\bullet$ is associative.*
2) It is a morphism of DG cocategories. In other words, one has

$$\Delta \circ \bullet = (\bullet_{13} \otimes \bullet_{24}) \circ (\Delta \otimes \Delta)$$

as morphisms

$$\mathrm{Bar}(C^\bullet(A,B))(f_0, f_1) \otimes \mathrm{Bar}(C^\bullet(B,C))(g_0, g_1) \to$$

$$\mathrm{Bar}(C^\bullet(A,C))(f_0 g_0, fg) \otimes \mathrm{Bar}(C^\bullet(A,C))(fg, f_1 g_1)$$

12.3. Internal $\underline{\mathrm{Hom}}$ of DG cocategories. Following the exposition of [**?**], we explain the construction of Keller, Lyubashenko, Manzyuk, Kontsevich and Soibelman. For two k-modules V and W, let $\mathrm{Hom}(V, W)$ be the set of homomorphisms from V to W, and let $\underline{\mathrm{Hom}}(V, W)$ be the same set viewed as a k-module. The two satisfy the property

$$\mathrm{Hom}(U \otimes V, W) \xrightarrow{\sim} \mathrm{Hom}(U, \underline{\mathrm{Hom}}(V, W)). \tag{12.5}$$

In other words, $\underline{\mathrm{Hom}}(V, W)$ is the internal object of morphisms in the symmetric monoidal category $k-\mathrm{mod}$. The above equation automatically implies the existence of an associative morphism

$$\underline{\mathrm{Hom}}(U, V) \otimes \underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(U, W) \tag{12.6}$$

If we replace the category of modules by the category of algebras, there is not much chance of constructing anything like the internal object of morphisms. However, if we replace $k - \mathrm{mod}$ by the category of coalgebras, the prospects are much better. For our applications, it is better to consider counital coaugmented coalgebras. In this category, objects $\underline{\mathrm{Hom}}$ do not exist because the equation (12.5) does not agree with coaugmentations. However, as explained in [**?**], the following is true.

Proposition 12.3.1. *The category of coaugmented counital conilpotent cocategories admits internal* $\underline{\mathrm{Hom}}$*s. For two DG categories A and B, one has*

$$\underline{\mathrm{Hom}}(\mathrm{Bar}(A), \mathrm{Bar}(B)) = \mathrm{Bar}(\mathbf{C}(A, B)) \tag{12.7}$$

****Expand? Delete? Modify?*****Look up in Faonte?***

13. Homotopy and homotopy equivalence for DG categories

Definition 13.0.1. *For every set X let $\mathbf{I}(X)$ be the DG category defined by $\mathrm{Ob}(\mathbf{I}(X)) = X$, $\mathbf{I}(x, y) = kf_{xy}$ for any x, $y \in X$, $f_{xy}f_{yz} = f_{xz}$, and $d = 0$. Let $\mathbf{I}_2 = \mathbf{I}(\{0, 1\})$.*

Definition 13.0.2. *Denote by $\mathcal{I}_2$ the DG category with $\mathrm{Ob}(\mathcal{I}_2) = \{0, 1\}$, freely generated by morphisms $f_{xy}^{(n)}$, $x, y = 0, 1$, for all nonnegative even n when $x \neq y$ and for all nonnegative odd n when $x = y$. We set*

$$|f_{xy}^{(n)}| = -n;$$

$$df_{xy}^{(n)} = \sum_{j+k=n-1} \sum_{z=0,1} (-1)^j f_{xz}^{(j)} f_{zy}^{(k)} - \delta_n^1 \mathbf{1}_x$$

Lemma 13.0.3. *Define the DG functor $\mathcal{K} \to \mathbf{I}_2$ which is identity on objects by the following action on morphisms: $f_{xy}^{(0)} \mapsto f_{xy}$, $x \neq y$, and $f_{xy}^{(0)} \mapsto 0$ if $n > 0$. This DG functor is a quasi-isomorphism.*

In other words, $\mathcal{I}_2$ is a semi-free resolution of $\mathbf{I}_2$.

Proof. Consider the filtration by the number of factors $f^{(n)}$ in a monomial. The corresponding spectral sequence shows that it is enough to prove the statement for the associated graded, i.e. with the same differential without the last term $\delta_n^1 \mathbf{1}_x$. This algebra is the cobar construction of the DG category with two objects x and y and two morphisms of degree-1 $\xi : x \to y$ and $\eta : y \to x$ subject to $\xi\eta = 0$; $\eta\xi = 0$. ***FINISH*** ∎

13.1. Strong equivalence of DG categories. Recall the DG category $\mathcal{I}_2$ from Definition 13.0.2.

Definition 13.1.1. *Two objects x and y of a DG category $\mathcal{A}$ are strongly equivalent if there is a DG functor $\mathcal{I}_2 \to \mathcal{A}$ sending the object 0 to x and the object 1 to y.*

Lemma 13.1.2. *Being strongly equivalent is an equivalence relation.*

Proof. There is a semi-free resolution $\mathcal{I}_3$ of $\mathbf{I}(\{0,1,2\})$ together with $\mathcal{I}_2 \xrightarrow{i_{02}} \mathcal{I}_3$, etc., with ...***** $\square$

Remark 13.1.3. The above definition coincides with the of homotopic morphisms of DG algebras in 8, with the only difference that the latter assumes that the $n = 0$ component of the zero cochain defining the equivalence is $1 \in B^0$.

Definition 13.1.4. *Two A_∞ functors $f, g \colon \mathcal{A} \to \mathcal{B}$ between two DG categories are strongly equivalent if they are strongly equivalent as objects of the DG category $\mathbf{C}(\mathcal{A}, \mathcal{B})$.*

Definition 13.1.5. *Two DG categories $\mathcal{A}$ and $\mathcal{B}$ are strongly equivalent if there are A_∞ functors $f \colon \mathcal{A} \to \mathcal{B}$ and $g \colon \mathcal{B} \to \mathcal{A}$ such that gf is strongly equivalent to $\mathrm{id}_{\mathcal{A}}$ and fg is strongly equivalent to $\mathrm{id}_{\mathcal{B}}$.*

13.2. DG quotients and localization. Drinfeld quotient is sometime called localization. Here we explain why.

Let $\mathcal{A}$ be any category of complexes (or a DG category equal to its triangulated hull)***REF***, consider a morphism $X_1 \xrightarrow{f} X_2$ and assume that $\mathrm{Cone}(f)$ lies in a full DG subcategory $\mathcal{C}$.

Lemma 13.2.1. *The morphism f is a strong equivalence in $\mathcal{A}/\mathcal{C}$.*

Proof. Consider the DG category $\mathcal{A}_0$ with two objects X_1 and X_2 and with

$$\mathcal{A}_0(X_1, X_1) = k\mathbf{1}_{X_1},\ \mathcal{A}_0(X_2, X_2) = k\mathbf{1}_{X_2};\ \mathcal{A}_0(X_1, X_2) = kf,$$

where $df = 0$. Let cA be the full subcategory of the triangulated hull of $\mathcal{A}_0$ generated by objects X_1, X_2, and $C = \mathrm{Cone}(f)$. Then the quotient $\mathcal{A}/\mathcal{C}$ is generated by the following morphisms.

$$\begin{tikzcd}
X_1 \arrow[rr, "f"] \arrow[dr, bend right, "{\alpha, d\alpha}"'] & & X_2 \arrow[dl, bend left, "{b, db}"] \\
& C \arrow[ul, "\alpha^*"'] \arrow[ur, "b^*"] \arrow[loop below, "{e, de, \epsilon}"] &
\end{tikzcd}$$

Here

$$|e| = |b| = |b^*| = 0; |\alpha| = -1;\ |\alpha^*| = 1; |\epsilon| = -1.$$

Relations are as follows (recall our convention of writing the composition $x \xrightarrow{f} x \xrightarrow{g} z$ as fg):

$$db^* = 0;\ d\alpha^* = 0;\ d\epsilon = \mathbf{1}_C;$$

$$\alpha^*\alpha = e;\ b^*b = \mathbf{1}_{X_2}; bb^* = \mathbf{1}_C - e;\ eb = 0;\ edb = db;\ b^*e = 0$$

$$e^2 = e;\ ede = de = de(\mathbf{1}_C - e);\ \alpha e = \alpha;\ d\alpha e = 0; \alpha\alpha^* = \mathbf{1}_{X_1};\ e\alpha^* = \alpha^*;\ b^*\alpha^* = 0$$

$$\alpha b = 0;\ d\alpha b = f = -\alpha db;\ fb^* = d\alpha$$

Set

$$\mathcal{P}(X_1, C) = k\alpha + kd\alpha;\ \mathcal{P}(C, X_1) = k\alpha^*;\ \mathcal{P}(X_2, C) = kb^*;\ \mathcal{P}(C, X_2) = kb + kdb;$$

$$\mathcal{P}(C, C) = ke + kde\mathbf{1}_C.$$

We see that there is a short exact sequence

$$0 \to \mathcal{A}_0(X_i, X_j) \to (\mathcal{A}/\mathcal{C})(X_i, X_j) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathcal{P}(X_i, C)\epsilon(\mathcal{P}(C, C)\epsilon)^{\otimes n}\mathcal{P}(C, X_j) \to 0$$

***[that is when we assume $\epsilon^2 = 0$-check that we do]**. Direct sums of the first n factors, $n \geq 0$, form an increasing filtration of the term on the right in the sequence. For all three cases except $i = 2,\ j = 1$, the associated graded quotients of the filtration are all acyclic (because $\mathcal{P}(X_1, C)$ and $\mathcal{P}(C, X_2)$ are contractible. When $i = 2$ and $j = 1$, all graded factors with $n \geq 1$ are acyclic because $\mathcal{P}(C, C)$ is contractible. We see that $(\mathcal{A}/\mathcal{C})(X_2, X_1)$ is quasi-isomorphic to kg where

$$g = b^*\epsilon\alpha^*. \tag{13.1}$$

The full subcategory of $\mathcal{A}/\mathcal{C}$ generated by X_1 and X_2 is therefore quasi-isomorphic to I_2 and admits a quasi-isomorphism from $\mathcal{I}_2$. □

13.3. Comparison to other notions of equivalence of DG categories. . *******************

We would like to *briefly indicate* relations to:

1) homotopy between DG functors, according to the model structures on DG categories (Tabuada) and A_∞ categories (Lefevre-Hasegawa).

2) the ∞-category structure on DG categories (Toen, Faonte). Also: a bit more about relating this to $\mathbf{C}(\mathcal{A}, \mathcal{B})$ (Faonte).

13.4. Sort of appendix, not sure if needed. We call a DG functor $F : \mathcal{A} \to \mathcal{B}$ *strong quasi-equivalence* if

1) $F : \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is a quasi-isomorphism;

2) every object of $\mathcal{B}$ is strongly equivalent to Fx for some object x of $\mathcal{B}$.

Proposition 13.4.1. *Assume that k is a field. Then a strong quasi-equivalence is an A_∞ homotopy equivalence.*

Proof. It is enough to prove the following two statements.

1) Let $F : \mathcal{A} \to \mathcal{B}$ be a DG functor which is a bijection on objects. Assume that $\mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is a quasi-isomorphism for any x, y in $\mathrm{Ob}(\mathcal{A})$. Then $F_* : C_\bullet(\mathcal{A}) \to C_\bullet(\mathcal{B})$ is a quasi-isomorphism.

2) Let $F : \mathcal{A} \to \mathcal{B}$ be an embedding of a full DG subcategory $\mathcal{A}$ into $\mathcal{B}$. Assume that every object of $\mathcal{A}$ is strongly equivalent to some object of $\mathcal{B}$. Then F is an A_∞ homotopy equivalence.

Statement 1) is proved exactly as for DG algebras. Let us prove 2). For every object x of $\mathcal{B}$ choose an object x' of $\mathcal{A}$ together with a functor $F_x : \mathcal{I}_2 \to \mathcal{B}$ sending 0 to x' and 1 to x. If x is an object of $\mathcal{A}$ then we choose $x' = x$ and F_x as the trivial functor sending both 0 and 1 to x, all morphisms of degree zero to $\mathbf{1}_x$, and the rest to zero We will construct the A_∞ functor G such that $\mathrm{id}_\mathcal{A} = GF$, as well as a homotopy between $\mathrm{id}_\mathcal{B}$ and FG, on any objects $x_0, \ldots, x_n$ of $\mathcal{B}$ by induction in

n. We will denote the objects in the source category $\mathcal{I}_2$ of F_{x_j} by j' and j instead of 0 and 1.

$$\begin{tikzcd}
x'_0 \arrow[loop above, "f^{(j)}_{0'0'}"] \arrow[d, bend right, "f^{(j)}_{0'0}"'] & x'_1. \arrow[loop above, "f^{(j)}_{1'1'}"] \arrow[d, bend right, "f^{(j)}_{1'1}"'] & & x'_n \arrow[loop above, "f^{(j)}_{n'n'}"] \arrow[d, bend right, "f^{(j)}_{n'n}"'] \\
x_0 \arrow[loop below, "f^{(j)}_{00}"] \arrow[u, bend right, "f^{(j)}_{00}"'] \arrow[r, "a_1"] & x_1 \arrow[loop below, "f^{(j)}_{11}"] \arrow[u, bend right, "f^{(j)}_{11'}"'] \arrow[r, "a_2"] & \dots \arrow[r, "a_n"] & x_n \arrow[loop below, "f^{(j)}_{nn}"] \arrow[u, bend right, "f^{(j)}_{nn'}"']
\end{tikzcd}$$

More precisely, we define $Gx = x'$; on morphisms, we put

$$G(a_1, \dots, a_n) = f^{(0)}_{0'0} a_0 f^{(1)}_{11} a_2 \dots f^{(1)}_{(n-1)(n-1)} a_n f^{(0)}_{nn'} \in \mathcal{B}^{1-n}(x'_0, x'_n) \tag{13.2}$$

We look for all the components in the homotopies in the form

$$\varphi(a_1, \dots, a_n) = \gamma_0 a_0 \gamma_1 a_1 \dots \gamma_{n-1} a_n \gamma_n \tag{13.3}$$

where γ_j are images of morphisms from $\mathcal{I}_2$ under F_j. On each step we get an element in the tensor power of the k-module of morphisms in $\mathcal{I}_2$ whose differential is equal to zero, and we have to find its primitive φ. Since $\mathcal{I}_2$ has no cohomology in nonzero degrees, this is always possible. $\square$

CHAPTER 15

Frobenius algebras, CY DG categories

1. Introduction

2. Bibliographical notes

CHAPTER 16

Perfect complexes

1. Introduction

Definition 1.0.1. *Let A be an associative algebra. A complex $F^\bullet$ of A-modules is* strictly perfect *if each F^k is finitely generated projective and $F^k = 0$ for all but finitely many k. A complex $M^\bullet$ of A-modules is* perfect *if it is quasi-isomorphic to a strictly perfect complex. By* $\mathrm{Perf}(A)$, *resp.* $\mathrm{sPerf}(A)$, *we denote the DG category of perfect, resp. strictly perfect, complexes.*

There is the obvious inclusion functor

$$i\colon \mathrm{sPerf}(A) \to \mathrm{Perf}(A) \tag{1.1}$$

We will construct the canonical homotopy inverse A_∞ functor

$$P\colon \mathrm{Perf}(A) \to \mathrm{sPerf}(A) \tag{1.2}$$

We will start with the construction of P and then explain in what sense it is canonical.

1.1. Construction of P.

Lemma 1.1.1. *For a perfect complex $M^\bullet$ there exists a strictly perfect complex $F^\bullet$ and a quasi-isomorphism $g : F^\bullet \to M^\bullet$.*

Proof. It is enough to prove that, if $F^\bullet$ is strictly perfect and $f\colon M^\bullet \to F^\bullet$ is a quasi-isomorphism, that there is a quasi-isomorphism $g\colon F^\bullet \to M^\bullet$ such that fg is homotopic to $\mathrm{id}_{F^\bullet}$. Since $F^\bullet$ is non-zero only in finitely many degrees, we may assume that the morphism g and the homotopy g are already constructed on F^k for $k > n$ for some n. We can assume that we have $g : F^k \to M^k$ and $h : F^k \to F^{k-1}$ for $k > n$ such that

$$\mathrm{id}_F - fg = dh + hd \tag{1.3}$$

on all F^k with $k > n$.

$$\begin{tikzcd}
\ldots \arrow[r] & F^n \arrow[r] & F^{n+1} \arrow[r] \arrow[l, bend right, "h"'] \arrow[d, shift left, "g"] & \ldots \arrow[l, bend right, "h"'] \\
\ldots \arrow[r] & M^n \arrow[r] \arrow[u, bend left, "f"] & M^{n+1} \arrow[r] \arrow[u, bend left, "f"] & \ldots
\end{tikzcd}$$

The morphism $gd\colon F^n \to M^{n+1}$ has its image inside $\mathrm{Ker}(d)$; $fgd : F^n \to F^{n+1}$ has its image inside $\mathrm{Im}(d)$ because of (1.3); since f is a quasi-isomorphism, gd has image in dM^n. By projectivity of F^n, we can find $g_0 : F^n \to M^n$ such that $dg_0 = gd$. Now consider the map $\mathrm{id}_{F^n} - fg_0 - hd : F^n \to F^n$. We have $d(\mathrm{id}_{\mathrm{F}^n} - fg_0 - hd) = 0$ because of (1.3). Since f is a quasi-isomorphism, the image of $\mathrm{id}_{F^n} - fg_0 - hd$ is in $dF^{n-1} + f(\mathrm{Ker}(d|M^n))$, and therefore the map can be lifted to $(h, g_1)\colon F^n \to$

$F^{n-1} \oplus \operatorname{Ker}(d|M^n)$. This gives us our h. Now define $g = g_0 + g_1$. These h and g satisfy (1.3). □

LEMMA 1.1.2. *Let $g_1\colon F_1^\bullet \to M_1^\bullet$ and $g_2\colon F_2^\bullet \to M_2^\bullet$ be two quasi-isomorphisms where $F_1^\bullet$ is strictly perfect. Let $f\colon M_1^\bullet \to M_2^\bullet$ be a morphism. Then there is a quasi-isomorphism $\varphi\colon F_1^\bullet \to F_2^\bullet$ such that $g_2\varphi$ and fg_1 are homotopic.*

PROOF. As above, we may assume that the morphism φ and the homotopy h are already constructed on F^k for $k > n$ for some n. One has

$$g_2\varphi - fg_1 = dh + hd \tag{1.4}$$

on F^k, $k > n$. Since $d\varphi d = 0$, φdF_1^n is inside dF_2^n. Since $g_2\varphi d = fg_1d + dhd$ has its image inside dM_2^n and g_2 is a quasi-isomorphism, φg_2 has its image inside dF_2^n. By projectivity of F_1^n we get $\varphi_0 : F_1^n \to F_2^n$ such that $d\varphi_0 = \varphi d$. We have

$$d(g_2\varphi_0 - fg_1 - hd) = g_2\varphi d - fg_1 d - dhd = (dh + hd)d - dhd = 0$$

and therefore $(g_2\varphi_0 - fg_1 - hd)(F_1^n)$ is inside $dM_2^{n-1} + g_2(\operatorname{Ker}(d|F_2^n))$. By projectivity of F_1^n, there is a map

$$(\varphi_1, h) : F_1^n \to \operatorname{Ker}(d|F_2^n) \oplus M_2^{n-1}$$

such that, for $\varphi = \varphi_0 + \varphi_1$, (1.4) holds. □

LEMMA 1.1.3. *Let $F^\bullet \xrightarrow{\varphi} F_1^\bullet \xrightarrow{g} M^\bullet$ where $F^\bullet$ is strictly perfect, g is a quasi-isomorphism, $g\varphi = [d, h]$ for some homotopy $h\colon F_1^\bullet \to M^{\bullet-1}$. Then there exists $H\colon F^\bullet \to F_1^{\bullet-1}$ such that $\varphi = [d, H]$ and gH is homotopic to h.*

PROOF. Assume that we have, in addition to $h : F^k \to M^{k-1}$ for all k, also$H : F^k \to F_1^{k-1}$ and $s : F^k \to M^{k+2}$ for $k > n$ satisfying

$$dh + hd = g\varphi;\ dH + Hd = \varphi;\ gH - h = ds - sd \tag{1.5}$$

Observe that $d(\varphi - Hd) = 0$ and $g(\varphi - Hd) = dh - dsd$ and therefore the image of $\varphi - Hd$ is inside dF_1^n since g is a quasi-isomorphism. Therefore $\varphi - Hd = dH_0$ for some $H_0 : F^n \to F_1^{n-1}$. Now,

$$d(gH - h + sd) = gdH - dh + dsd = 0$$

and therefore the image of $gH - h + sd : F^n \to M^{n-1}$ is inside $g\ker(d|F_1^{n-1}) + dM^{n-2}$. By projectivity of F^n we have a morphism $(H_1), s) : F^n \to \ker(d|F_1^{n-1}) \oplus M^{n-2}$ such for s and for $H = H_0 + H_1$ (1.5) is true. □

We will construct inductively an A_∞ functor

$$P\colon \operatorname{Perf}(A) \to \operatorname{sPerf}(A) \tag{1.6}$$

together with a natural transformation $S\colon iP \to \mathrm{id}$. The latter is, by definition, a Hochschild cocycle of degree zero $C^\bullet(\operatorname{Perf}(A), {}_{iP}\operatorname{Perf}(A)_{\mathrm{id}})$, cf. (12.1).

For every perfect $M^\bullet$, make a choice of a strictly perfect $P^\bullet$ and a quasi-isomorphism $g(M^\bullet)\colon F^\bullet \to M^\bullet$ as in Lemma 1.1.1. When $M^\bullet$ is strictly perfect, we choose $g(M^\bullet) = M^\bullet$. Put $P(M^\bullet) = F^\bullet$. Now let $f_i\colon M_i^\bullet \leftarrow M_{i+1}^\bullet$ for $i = 1, \ldots, n$. Denote $g_i = g(M_i^\bullet)$. Let $P(M_i^\bullet) = F_i^\bullet$. We denote the components of the A_∞ functor P by

$$P_n(f_1, \ldots, f_n) \in \underline{\operatorname{Hom}}^{1-n}(F_1^\bullet, F_{n+1}^\bullet)$$

and the components of the natural transformation S by

$$s_n(f_1, \ldots, f_n) \in \underline{\operatorname{Hom}}^{-n}(F_1^\bullet, M_{n+1}^\bullet).$$

REMARK 1.1.4. We have two conflicting notations, namely, we denote the composition in any abstract DG category by

$$\mathcal{A}(x,y)\otimes\mathcal{A}(y,z)\to\mathcal{A}(x,z); f\otimes g\mapsto fg \tag{1.7}$$

but denote the composition of morphisms of complexes, as usual, by gf.

For $n=0$, we put $s=g(M^\bullet)$ for every $M^\bullet$. For $n=1$, let $P(f_1)$ be the morphism φ from Lemma 1.1.2. Let $s(f_1)$ be the homotopy from the same Lemma. By inductive hypothesis, assume that all P_m and s_m are already defined for m smaller than n. The maps P and s have to satisfy

$$[d,P(f_1,\ldots,f_n)]=\sum_{k=1}^{n-1}\pm P(f_1,\ldots,f_k)P(f_{k+1},\ldots,f_n)+ \tag{1.8}$$

$$\sum_{k=1}^{n-1}\pm P(f_1,\ldots,f_kf_{k+1},\ldots,f_n)$$

and

$$[d,s(f_1,\ldots,f_n)]\pm g_1P(f_1,\ldots,f_n)=f_1s(f_2,\ldots,f_n)+ \tag{1.9}$$

$$\sum_{k=1}^{n-1}\pm s(f_1,\ldots,f_k)P(f_{k+1},\ldots,f_n)+\sum_{k=1}^{n-1}\pm s(f_1,\ldots,f_kf_{k+1},\ldots,f_n)$$

Denote by R_1 the right hand side of (1.8) and by R_2 the right hand side of (1.9). If we apply (1.9) for all $k<n$, we will get

$$\phi_1R_1=[d,R_2] \tag{1.10}$$

By Lemma 1.1.3, there exist the homotopies $P(f_1,\ldots,f_n)$ and $s(f_1,\ldots,s_n)$ satisfying (1.8) and (1.9). This completes the construction of the A_∞ functor P.

2. Perfect complexes modulo acyclic complexes

Recall the definitions of the Drinfeld localization 5. Let $\mathrm{Perf}^{\mathrm{acyclic}}(A)$ be the full DG subcategory of acyclic complexes. As usual, by $\mathcal{A}/\mathcal{B}$ we denote the Drinfeld quotient of $\mathcal{A}$ by $\mathcal{B}$.

THEOREM 2.0.1. *The A_∞ functor P extends to* $\mathrm{Perf}(A)/\mathrm{Perf}^{\mathrm{acyclic}}(A)$.

PROOF. □

LEMMA 2.0.2. *The inclusion of* $\mathrm{sPerf}(A)$ *to* $\mathrm{Perf}(A)$ *induces a strong quasi-equivalence*

$$\mathrm{sPerf}(A)\to\mathrm{Perf}(A)/\mathrm{Perf}^{\mathrm{acyclic}}(A).$$

In particular, it induces an isomorphism of Hochschild homologies and cyclic homologies of all types.

PROOF.

$$\mathrm{sPerf}(A)\to\mathrm{Perf}(A)/\mathrm{Perf}^{\mathrm{acyclic}}(A).$$

□

3. The trace map $\mathrm{HH}_\bullet(\mathrm{sPerf}(A)) \to \mathrm{HH}_\bullet(A)$

Define the trace map as follows. Let (F_j, d_j) be strictly perfect complexes such that $(F_0, d_0) = (F_{n+1}, d_{n+1})$. For a Hochschild chain $a = a_0 \otimes \ldots \otimes a_n$, $a_j \in \mathrm{Hom}_A(F_j, F_{j+1})$ put

$$\mathrm{Tr}(a) = \mathrm{tr} \sum \pm a_0 \otimes \ldots \otimes a_{j_1} \otimes d_{j_1+1} \otimes a_{j_1+1} \otimes \ldots \otimes a_n$$

where tr was defined in 8.1***[need graded version]**(see also 8.2). The sum is infinite but the trace map is zero on all but finitely many terms. In fact, the trace map has the form

$$(b_0 m_0 \otimes \ldots \otimes b_N m_N) \mapsto \pm \mathrm{tr}(m_0 \ldots m_N)(b_0 \otimes \ldots \otimes b_N)$$

where $b_j \in A$ and m_j are matrices over k. On every F_j there is a grading (just by the degree in the complex). This induces a grading on tensor products of $\mathrm{Hom}_A(F_j, F_{j+1}$. In the infinite sum, there are always $n+1$ factors a_j and a growing number of factors d_j. Therefore the degree of a term in the sum goes to infinity. But the trace is zero on all terms of nonzero degree.

LEMMA 3.0.1. *The map* Tr *is a quasi-isomorphism of complexes that commutes with B.*

PROOF. Define the completions $\widehat{C}_\bullet$ of the Hochschild complexes of sPerf and $\mathrm{sPerf}_{d=0}$ consisting of infinite sums of chains where the degree is allowed to go to infinity. The trace map Tr defines an isomorphism

$$\widehat{C}_\bullet(\mathrm{sPerf}_{d=0}(A)) \xrightarrow{\sim} \widehat{C}_\bullet(\mathrm{sPerf}(A)) \tag{3.1}$$

□

We have

$$\begin{tikzcd} C_\bullet(A) \arrow[r, shift right, "i"'] & C_\bullet(\mathrm{sPerf}_{d=0}(A)) \arrow[l, shift right, "p"'] \arrow[loop above, "s"] \arrow[r, "\iota"] & \widehat{C}_\bullet(\mathrm{sPerf}_{d=0}(A)) \arrow[r, "g"] & \widehat{C}_\bullet(\mathrm{sPerf}(A) \end{tikzcd}$$

Here i, p, and s are the inclusion, the projection and the homotopy from 8.1 while g is the isomorphism defined in 3. More precisely, g is the map such that $\mathrm{Tr} = \mathrm{tr} \circ g$. One has

$$p \circ i = \mathrm{id}; \ \mathrm{id} - ip = [d + b, s]$$

The easiest thing now is to observe that a) s extends to $\widehat{C}_\bullet(\mathrm{sPerf}_{d=0}(A))$ and b) gsg^{-1} descends to $C_\bullet(\mathrm{sPerf}(A))$. Therefore $g\iota i$ is a homotopy equivalence of complexes.

We have proved the following theorem of Keller.

THEOREM 3.0.2. *There is a natural quasi-isomorphism*

$$C_\bullet(\mathrm{Perf}(A)/\mathrm{Perf}^{\mathrm{acyclic}}(A)) \to C_\bullet(A)).$$

Same for the cyclic complexes of all types.

4. Perfect complexes over a field

(***Maybe not needed)***. Let A be a field k. Then $\mathrm{Perf}(A) = \mathrm{Perf}(k)$ is the DG category of complexes of vector spaces over k with finite dimensional total cohomology, and $\mathrm{sPerf}(A) = \mathrm{sPerf}(k)$ is the DG category of finite dimensional complexes over k. For a perfect complex $C^\bullet$ we can choose $F^\bullet = H^\bullet$, the cohomology of $C^\bullet$. For every
Cb choose an embedding $\iota\colon H^\bullet \to C^\bullet$ and a projection $\Pi\colon C^\bullet \to H^\bullet$ such that $\Pi\iota = \mathrm{id}_{H^\bullet}$. Furthermore, choose a homotopy $h\colon C^\bullet \to C^{\bullet-1}$ such that $\mathrm{id}_{C^\bullet} - \iota\Pi = [d,h]$. Put $P(C^\bullet) = H^\bullet$.

Now let $C_1^\bullet \xleftarrow{f_1} C_2^\bullet \xleftarrow{f_2} \dots \xleftarrow{f_n} C_{n+1}^\bullet$. Denote our choices of ι, Π, and h for each complex $C_k^\bullet$ by ι_k, Π_k, and h_k respectively.

PROPOSITION 4.0.1. *The formula*

$$P(f_1, \dots, f_n) = \Pi_1 f_1 h_2 f_2 \dots h_n f_n \iota_{n+1}$$

defines an A_∞ *functor* $P\colon \mathrm{Perf}(k) \to \mathrm{sPerf}(k)$. *The formula*

$$s(f_1, \dots, f_n) = h_1 f_1 h_2 f_2 \dots h_n f_n \iota_{n+1}$$

defines a natural transformation $\mathrm{id}_{\mathrm{Perf}(k)} \to iP$ *where* $i\colon \mathrm{sPerf}(k) \to \mathrm{Perf}(k)$ *is the inclusion.*

PROOF. The equations (1.8) and (1.9) can be easily checked directly. Note that this is a partial case of the general construction above (put $\phi = \iota$). □

PROPOSITION 4.0.2. *The* A_∞ *functor* P *is defined canonically up to strong equivalence.*

PROPOSITION 4.0.3. *The* A_∞ *functors* P *and* i *extend to a strong equivalence*

$$\mathrm{Perf}(k)/\,\mathrm{Perf}(k)^{\mathrm{acyclic}} \to \mathrm{sPerf}(k).$$

PROOF. □

5. Other versions of Theorem 2.0.1

In the above theorem, one can replace $\mathrm{Perf}(A)$ by the DG category of complexes of A-modules whose cohomology is bounded from above, and $\mathrm{sPerf}(A)$ by the category of complexes of projective modules bounded from above. Or we can start with a DG category $\mathcal{A}$, replace $\mathrm{Perf}(A)$ by the DG category $\mathcal{A} - \mathrm{mod}$ of DG modules over $\mathcal{A}$, and replace $\mathrm{sPerf}(A)$ by the DG category $\mathcal{A} - \mathrm{mod}^{\mathrm{cof}}$ of cofibrant DG modules (REF in TEXT). A proof identical to the above yields a canonical (up to strong equivalence of A_∞ functors) strong equivalence of DG categories

$$\mathcal{A} - \mathrm{mod}^{\mathrm{cof}} \xrightarrow{\sim} \mathcal{A} - \mathrm{mod}/\mathcal{A} - \mathrm{mod}^{\mathrm{acyclic}} \tag{5.1}$$

6. Bibliographical notes

Let $\mathcal{A}$ be any category of complexes (or a DG category equal to its triangulated hull)***REF***, consider a morphism $X_1 \xrightarrow{f} X_2$ and assume that $\mathrm{Cone}(f)$ lies in a full DG subcategory $\mathcal{C}$.

LEMMA 6.0.1. *The morphism* f *is a strong equivalence in* $\mathcal{A}/\mathcal{C}$.

PROOF. Consider the DG category $\mathcal{A}_0$ with two objects X_1 and X_2 and with

$$\mathcal{A}_0(X_1, X_1) = k\mathbf{1}_{X_1},\ \mathcal{A}_0(X_2, X_2) = k\mathbf{1}_{X_2};\ \mathcal{A}_0(X_1, X_2) = kf,$$

where $df = 0$. Let cA be the full subcategory of the triangulated hull of $\mathcal{A}_0$ generated by objects X_1, X_2, and $C = \mathrm{Cone}(f)$. Then the quotient $\mathcal{A}/\mathcal{C}$ is generated by the following morphisms.

$$\begin{tikzcd}
X_1 \arrow[rr, "f"] \arrow[dr, bend right, "{\alpha, d\alpha}"'] & & X_2 \arrow[dl, "b^*"'] \\
& C \arrow[ul, "\alpha^*"'] \arrow[ur, bend right, "{b, db}"'] \arrow[loop below, "{e, de, \epsilon}"] &
\end{tikzcd}$$

Here

$$|e| = |b| = |b^*| = 0; |\alpha| = -1;\ |\alpha^*| = 1; |\epsilon| = -1.$$

Relations are as follows (recall our convention of writing the composition $x \xrightarrow{f} x \xrightarrow{g} z$ as fg):

$$db^* = 0;\ d\alpha^* = 0;\ d\epsilon = \mathbf{1}_C;$$

$$\alpha^*\alpha = e;\ b^*b = \mathbf{1}_{X_2}; bb^* = \mathbf{1}_C - e;\ eb = 0;\ edb = db;\ b^*e = 0$$

$$e^2 = e;\ ede = de = de(\mathbf{1}_C - e);\ \alpha e = \alpha;\ d\alpha e = 0; \alpha\alpha^* = \mathbf{1}_{X_1};\ e\alpha^* = \alpha^*;\ b^*\alpha^* = 0$$

$$\alpha b = 0;\ d\alpha b = f = -\alpha db;\ fb^* = d\alpha$$

Set

$$\mathcal{P}(X_1, C) = k\alpha + kd\alpha;\ \mathcal{P}(C, X_1) = k\alpha^*;\ \mathcal{P}(X_2, C) = kb^*;\ \mathcal{P}(C, X_2) = kb + kdb;$$
$$\mathcal{P}(C, C) = ke + kde\mathbf{1}_C.$$

We see that there is a short exact sequence

$$0 \to \mathcal{A}_0(X_i, X_j) \to (\mathcal{A}/\mathcal{C})(X_i, X_j) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathcal{P}(X_i, C)\epsilon(\mathcal{P}(C, C)\epsilon)^{\otimes n}\mathcal{P}(C, X_j) \to 0$$

***[that is when we assume $\epsilon^2 = 0$-check that we do]**. Direct sums of the first n factors, $n \geq 0$, form an increasing filtration of the term on the right in the sequence. For all three cases except $i = 2,\ j = 1$, the associated graded quotients of the filtration are all acyclic (because $\mathcal{P}(X_1, C)$ and $\mathcal{P}(C, X_2)$ are contractible. When $i = 2$ and $j = 1$, all graded factors with $n \geq 1$ are acyclic because $\mathcal{P}(C, C)$ is contractible. We see that $(\mathcal{A}/\mathcal{C})(X_2, X_1)$ is quasi-isomorphic to kg where

$$g = b^*\epsilon\alpha^*. \tag{6.1}$$

The full subcategory of $\mathcal{A}/\mathcal{C}$ generated by X_1 and X_2 is therefore quasi-isomorphic to I_2 and admits a quasi-isomorphism from $\mathcal{I}_2$. $\square$

CHAPTER 17

What do DG categories form?

1. Introduction

The question in the title of this chapter was asked by Drinfeld in ***REF***. We will give a version of an answer that is related to other versions, such as ***REFS***. Namely, we will show that Hochschild cochains and chains form a homotopy category with a trace functor (a variant of the definition that was introduced by Kaledin).

Let us start by noting that categories form a two-category, morphisms being functors and 2-morphisms being natural transformations. A related fact is that rings form a two-category. In fact, for two rings A and B, let $\mathcal{C}(A,B)$ be the category of (A,B)-bimodules. The composition $\mathcal{C}(A,B)\times\mathcal{C}(B,C)\to\mathcal{C}(A,C)$ is given by the tensor product $\otimes_B$. If we restrict ourselves to bimodules which are graphs of morphisms, i.e. for every $f:A\to B$ define fB to be B with the bimodule action $a\cdot b\cdot b_1=f(a)bb_1$, then the resulting sub-2-category becomes a subcategory of the 2-category of categories above (rings are additive categories with one object).

The 2-category of rings has an extra structure. Namely, there is a functor $\mathrm{TR}_A:\mathcal{C}(A,A)\to\mathbb{Z}\text{-mod}$ for every ring A, defined by

$$\mathrm{TR}_A : M\mapsto M/[A,M]\xrightarrow{\sim} M\otimes_{A\otimes A^{\mathrm{op}}}A\xrightarrow{\sim} HH_0(A,M). \tag{1.1}$$

The functors TR_A have the trace property, namely, for any A and B and for any objects M in $\mathcal{C}(A,B)$ and B in $\mathcal{C}(B,A)$ there is a functorial isomorphism

$$\tau_{AB}:\mathrm{TR}_A(M\otimes_B N)\xrightarrow{\sim}\mathrm{TR}_B(N\otimes_A M).$$

Those isomorphisms satisfy a compatibility condition for every three rings A,B,C and for three objects M in $\mathcal{C}(A,B)$, B in $\mathcal{C}(B,C)$, and $P\in\mathcal{C}(C,A)$:

$$\tau_{AC}\tau_{CB}\tau_{BA}=\mathrm{id}:\mathrm{TR}_A(M\otimes_B N\otimes_C P)\to\mathrm{TR}_A(M\otimes_B N\otimes_C P)$$

The goal of this chapter is to describe a derived analog of the above. More precisely, replace morphisms of bimidules by the standard complexes computing $\mathbb{R}\mathrm{Hom}$ and replace the trace of a bimodule by the standard complex computing the derived tensor product, i.e. the Hochschild chain complex. We will actually restrict ourselves to bimodules that are graphs of morphisms (or, more generally, of A_∞ morphisms of DG categories). We will use brace operations on Hochschild cochains, and their analogs on Hochschild cochains and chains, to construct a homotopy version of the structure described above. We will see that much of the structure is actually strict, not up to homotopy, when the morphisms are DG cocategories. The single place where this is not so is precisely where the cyclic differential B appears. We find this significant, together with the fact that bar construction of the algebra of Hochschild cochains of an individual algebra form a Hopf algebra (strict, not up to homotopy).

Let us show how the structure of a two-category up to homotopy (in any reasonable sense) gives rise to a differential B on $\mathrm{TR}_A(\mathrm{id}_A)$ for any A. Start with two morphisms of rings $f : A \to B$ and $g : B \to A$. Then we should have quasi-isomorphisms of complexes

$$\mathrm{TR}_A(gf) \xrightarrow{\tau_{AB}} \mathrm{TR}_B(fg) \xrightarrow{\tau_{BA}} \mathrm{TR}_A(gf)$$

and a homotopy between id and $\tau_{BA}\tau_{AB}$. We denote this homotopy by B_{gf}. Let us also denote the τ_{AB} above by f_* and τ_{BA} by g_*.

Now let $A = B$ and $f = g = \mathrm{id}$. Then $\tau_{BA} = \tau_{AB} = \mathrm{id}$ and the homotopy now commutes with the differential. We get an endomorphism B, or B_A, of (homological) degree one of the complex $\mathrm{TR}_A(\mathrm{id}_A)$.

Why does B define a differential? Note that, in any reasonable definition of a 2-category with a trace functor up to homotopy, any new morphism of our complexes should be homotopic to zero. For example, $f_* B_{gf} - B_{fg} f_*$ is the difference of two homotopies between f_* and $f_* g_* f_*$, it has to be homotopic to zero. Similarly, if we denote by b_A and b_B the differentials in $\mathrm{TR}_A(gf)$ and $\mathrm{TR}_B(fg)$ respectively, then***CONT.; could be a bit subtle.***

Similar considerations show that the cohomologies of $\mathcal{C}(A, A)(\mathrm{id}_A, \mathrm{id}_A)$ and $\mathrm{TR}_A(\mathrm{id}_A)$ carry a structure that is called *a calculus* in ***REF***.

The explicit formulas are as follows. For a morphism $f : A \to A$,

$$\mathrm{TR}_A(f) = C_\bullet(A,_f A);\quad f_*(a_0 \otimes \ldots \otimes a_n) = f(a_0) \otimes \ldots \otimes f(a_n);$$

$$B_f(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes f(a_j) \otimes f(a_n) \otimes a_0 \otimes \ldots \otimes a_{j-1}$$

We finish this introduction by describing a structure carried by the pair $C^\bullet(\mathcal{A}, \mathcal{A})$ and $C_\bullet(\mathcal{A}, \mathcal{A})$. This structure is less known or studied than the one involving trace functors.

For a Hopf algebra U over k, denote by U^+ the k-module U equipped with the cobimodule structure

$$U^+ \to U \otimes U^+ \otimes U;\ u \mapsto \sum S((u^{(3)}) \otimes u^{(2)} \otimes S((u^{(1)})$$

where S is the antipode. Note that

$$m^{\mathrm{op}} = m \circ \sigma : U^+ \otimes U^+ \to U^+ \tag{1.2}$$

is a morphism of cobimodules. Here m is the product in U and σ is the transposition.

A di(tetra)module over a Hopf algebra$U$ is a cobimodule M over U together with two cobimodule morphisms

$$\mu_l : U^+ \otimes M \to M;\ \mu^r : M \otimes U \to M \tag{1.3}$$

subject to the following compatibility conditions:

1) (1.3) turn M into a right module over (U, m) and over (U^+, m^{op});

2) the right actions of U and of U^+ commute with each other.

FINISH

It would be interesting to see examples, as well as a quasi-classical analog for Poisson-Lie groups, etc.

2. A category in DG cocategories

Lemma 12.2.1 defines

(1) For every two DG categories $\mathcal{A}$ and $\mathcal{B}$, a DG cocategory

$$\mathbf{B}(\mathcal{A},\mathcal{B}) = \mathrm{Bar}\mathbf{C}(\mathcal{A},\mathcal{B}) \tag{2.1}$$

(2) For any three DG categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a DG functor

$$m_{\mathcal{ABC}} : \mathbf{B}(\mathcal{A},\mathcal{B}) \otimes \mathbf{B}(\mathcal{B},\mathcal{C}) \to \mathbf{B}(\mathcal{A},\mathcal{C}) \tag{2.2}$$

that is associative, namely,

$$m_{\mathcal{ABD}} \circ m_{\mathcal{BCD}} = m_{\mathcal{ACD}} \circ m_{\mathcal{ABC}} = m_{\mathcal{ABCD}} : \mathbf{B}(\mathcal{A},\mathcal{B}) \otimes \mathbf{B}(\mathcal{B},\mathcal{C}) \otimes \mathbf{B}(\mathcal{C},\mathcal{D}) \to \mathbf{B}(\mathcal{A},\mathcal{D})$$

for any four DG categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

3. A category in DG categories

For any DG categories $\mathcal{A}_0$, ... , $\mathcal{A}_n$, define

$$\mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_n) = \mathrm{Cobar}(\mathrm{Bar}(\mathbf{C}(\mathcal{A}_0,\mathcal{A}_1)) \otimes \ldots \otimes \mathrm{Bar}(\mathbf{C}(\mathcal{A}_0,\mathcal{A}_1)))$$

These DG categories carry a structure that we call a homotopy category in DG categories, i.e.

I. DG functors

$$\mu_{j_0 \ldots j_n} : \mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_n) \to \mathcal{C}(\mathcal{A}_{j_0} \to \ldots \to \mathcal{A}_{j_m})$$

for all $m > 0$ and all $0 = j_0 \leq j_1 \leq \ldots \leq j_m = n$;

II. DG functors

$$\Delta_k : \mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_n) \to \mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_k) \otimes \mathcal{C}(\mathcal{A}_k \to \ldots \to \mathcal{A}_n)$$

for $k = 1, \ldots, n-1$

such that DG functors II are:

(1) coassociative;
(2) compatible with DG functors I, namely:

$$(\mu_{j_0 \ldots j_l} \otimes \mu_{j_l \ldots j_n}) \circ \Delta_k = \Delta_{j_l} \circ \mu_{j_0 \ldots j_m};$$

(3) weak equivalences.

Note that in our case DG functors II are bijections on objects, so being weak equivalences just means being quasi-isomorphisms on morphisms.

DG functors I and II are constructed as follows. I are induced on Cobar by the $\bullet$ product ***REF***, whereas II are obtained from the dual EZ product

$$\mathrm{Cobar}(\mathbf{B}_1) \otimes \mathrm{Cobar}(\mathbf{B}_2) \longrightarrow \mathrm{Cobar}(\mathbf{B}_1 \otimes \mathbf{B}_2) \tag{3.1}$$

3.1. The Grothendieck construction. Recal the category Δ from 2. Its objects are $[n]$, $n \geq 0$, and $\Delta'([n],[m])$ consists of transformations

$$(x_0, \ldots . x_n) \mapsto (x_{J_0}, \ldots, x_{J_m}), \tag{3.2}$$

(cf.(2.1)). We write

$$x_{J_k} = x_{j_k+1} \ldots x_{j_{k+1}} \tag{3.3}$$

where $-1 = j_0 \leq \ldots \leq j_k \leq j_{k+1} \ldots \leq j_{m+1} = n$ (and the product of an empty number of x_j is equal to 1).

The category Δ' acts on the ***Set?*** of all symbols

$$\mathcal{A}_0 \to \ldots \to \mathcal{A}_{n+1} \tag{3.4}$$

($n \geq 0$) where $\mathcal{A}_j$ are DG categories. Namely, a morphism (3.3) sends such (3.4) to $\mathcal{A}_{j_0+1} \to \ldots \to \mathcal{A}_{j_{m+1}+1}$.

Definition 3.1.1. *Define the category Δ'_{Alg} as follows. Its objects are $(n, \mathbf{A})$ where $n \geq 0$ and $\mathbf{A}$ is as in (3.4); morphisms from $(n, \mathbf{A})$ to $(m, \mathbf{A}')$ are morphisms in $\Delta'([n],[m])$ such that $\delta\mathbf{A} = \mathbf{A}'$.*

For $\mathbf{A}$ as in (3.4), define $s\mathbf{A} = \mathcal{A}_0$ and $t\mathbf{A} = \mathcal{A}_{n+1}$. Define $\Delta^{(N)}_{\mathrm{Alg}}$ to be the full subcategory of $\prod_{i=0}^{N} \Delta'_{\mathrm{Alg}}$ with objects $(\mathbf{A}_1, \ldots, \mathbf{A}_N)$ such that $t\mathbf{A}_i = s\mathbf{A}_{i+1}$ for $i = 0, \ldots, N-1$. We have the obvious functors

$$D_j : \Delta^{(N+1)} \to \Delta^{(N)}$$

($0 \leq j \leq N$) such that

$$(\mathbf{A}_0, \ldots, \mathbf{A}_N) \mapsto (\mathbf{A}_0, \ldots, \mathbf{A}_j \circ \mathbf{A}_{j+1}, \ldots, \mathbf{A}_N)$$

where

$$(\mathcal{A}_0 \to \ldots \to \mathcal{A}_{n+1}) \circ (\mathcal{A}_{n+1} \to \ldots \to \mathcal{A}_{n+m+1}) = \mathcal{A}_0 \to \ldots \to \mathcal{A}_{n+1} \to \ldots \to \mathcal{A}_{n+m+1},$$

and

$$S_j : \Delta^{(N)} \to \Delta^{(N+1)},$$

$0 \leq j \leq N+1$, such that

$$(\mathbf{A}_0, \ldots, \mathbf{A}_N) \mapsto (\mathbf{A}_0, \ldots, \mathbf{A}_{j-1}, (\mathcal{A} \to \mathcal{A}), \mathbf{A}_j, \ldots, \mathbf{A}_N)$$

where

$$\mathcal{A} = t\mathbf{A}_{j-1} = s\mathbf{A}_j.$$

Remark 3.1.2. We get a cosimplicial category $\Delta^{(*)}_{\mathrm{Alg}}$ (in other words, a functor from Δ' to categories). The structure of a homotopy category in DG categories that we constructed above can be interpreted as:

(1) a functor $\mathcal{C}^{(N)}$ from $\Delta^{(N)}_{\mathrm{Alg}}$ to DG categories for any $N \geq 0$;;
(2) a natural transformation $\delta^\dagger : \delta^*\mathcal{C}^{(M)} \to \mathcal{C}^{(N)}$ for any $\delta \in \Delta'([N],[M])$ such that
(3) $\delta^\dagger$ is a weak equivalence on every object of $\Delta^{(N)}_{\mathrm{Alg}}$, and
(4)
$$(\delta_1\delta_2)^\dagger = \delta_2^*(\delta_1^\dagger)\delta_2^\dagger$$
for any composable δ_1 and δ_2 in Δ'.

4. A category in DG cocategories with a trace functor

4.1. DG comodules. Our assumptions: DG cocategories and DG comodules are conilpotent and locally finite, i.e.

$$\Delta_{\mathbf{B}} : \mathbf{B}(x, y) \to \mathbf{B}(x, z) \otimes \mathbf{B}(z, y)$$

is equal to zero for all but finitely many z, and

$$\Delta_{\mathbf{M}} : \mathbf{M}(x) \to \mathbf{B}(x, y) \otimes \mathbf{M}(y)$$

is equal to zero for all but finitely many y.

For a DG functor $f : \mathbf{B}_2 \to \mathbf{B}_1$ between two DG cocategories and for a DG comodule $\mathbf{M}$ over $\mathbf{B}_1$, define a DG comodule $f^*\mathbf{M}$ over $\mathbf{B}_2$ as follows. For an object x_2 of $\mathbf{B}_2$, define two maps

$$\bigoplus_{y_1 \in \mathrm{Ob}\mathbf{B}_1} \mathbf{B}_2(x_2, fy_1) \otimes \mathbf{M}(y_1) \rightrightarrows \bigoplus_{y_2 \in \mathrm{Ob}\mathbf{B}_2; z_1 \in \mathrm{Ob}\mathbf{B}_1} \mathbf{B}_2(x_2, y_2) \otimes \mathbf{B}_1(fy_2, z_1) \otimes \mathbf{M}(z_1)$$

One is $\mathrm{id}_{\mathbf{B}_2} \circ \Delta_{\mathbf{M}}$, the other $(\mathrm{id}_{\mathbf{B}_2} \otimes f \otimes \mathrm{id}_{\mathbf{M}}) \circ (\Delta_{\mathbf{B}_1} \circ \mathrm{id}_{\mathbf{M}})$. Define $f^*\mathbf{M}(x_2)$ to be the equalizer of these two maps.

This construction is dual to the construction of $F_!\mathcal{M}$ for a DG functor of DG categories $\mathcal{A}_1 \to \mathcal{A}_2$ and a DG module $\mathcal{M}$ over $\mathcal{A}_1$.

LEMMA 4.1.1. *Let $\mathbf{M}$ be cofreely cogenerated over $\mathbf{B}_1$ by the system of k-modules $M(x_1)$, $x_1 \in \mathrm{Ob}\mathbf{B}_1$. Then $f^*\mathbf{M}$ is cofreely cogenerated over $\mathbf{B}_2$ by the system of k-modules $M(fx_2)$, $x_2 \in \mathrm{Ob}\mathbf{B}_2$.*

PROOF. Define a morphism

$$\bigoplus_{y_2 \in \mathrm{Ob}\mathbf{B}_2} \mathbf{B}_2(x_2, y_2) \otimes M(fy_2) \to f^*\mathbf{M}(x_2) \subset \bigoplus_{z_2, y_1} \mathbf{B}_2(x_2, z_2) \otimes \mathbf{B}_1(fz_2, y_1) \otimes M(y_1)$$

as $(\mathrm{id}_{\mathbf{B}_2} \otimes f \otimes \mathrm{id}_{\mathbf{M}}) \circ (\Delta_{\mathbf{B}} \otimes \mathrm{id}_M)$. ***MORE*** $\square$

4.2. The trace functor. We will show that DG categories form a category in DG categories with the following additional structure.

(1) For every DG category $\mathcal{A}$, a DG comodule $\mathrm{TR}_{\mathcal{A}}$ over $\mathbf{B}(\mathcal{A}, \mathcal{A})$.
(2) For any two DG categories $\mathcal{A}$ and $\mathcal{B}$, a morphism of DG comodules

$$\tau_{\mathcal{A},\mathcal{B}} : (m_{\mathcal{ABA}})^*\mathrm{TR}_{\mathcal{A}} \to (m_{\mathcal{BAB}} \circ \tau)^*\mathrm{TR}_{\mathcal{B}}$$

$$\begin{tikzcd}
\mathbf{B}(\mathcal{A},\mathcal{B}) \otimes \mathbf{B}(\mathcal{B},\mathcal{A}) \arrow[r, "\tau"] \arrow[d, "m_{\mathcal{ABA}}"] & \mathbf{B}(\mathcal{B},\mathcal{A}) \otimes \mathbf{B}(\mathcal{A},\mathcal{B}) \arrow[d, "m_{\mathcal{BAB}}"'] \\
\mathbf{B}(\mathcal{A},\mathcal{A}) & \mathbf{B}(\mathcal{B},\mathcal{B}) \\
\mathrm{TR}_{\mathcal{A}} & \mathrm{TR}_{\mathcal{B}}
\end{tikzcd}$$

where τ is the transposition;
(3) a homotopy $\sigma_{\mathcal{ABC}}$ between two morphisms of DG comodules

$$\mathrm{id} : (m_{\mathcal{ABCA}})^*\mathrm{TR}_{\mathcal{A}} \to (m_{\mathcal{ABCA}})^*\mathrm{TR}_{\mathcal{B}}$$

and

$$(\tau_{\mathcal{BCA}} \circ \tau^2) \circ (\tau_{\mathcal{CAB}} \circ \tau) \circ \tau_{\mathcal{ABC}} : (m_{\mathcal{ABCA}})^*\mathrm{TR}_{\mathcal{A}} \to (m_{\mathcal{ABCA}})^*\mathrm{TR}_{\mathcal{B}}$$

$$\begin{tikzcd}
\mathbf{B}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{A}) \arrow[r, "\tau"] \arrow[d, "m_{\mathcal{ABCA}}"] & \mathbf{B}(\mathcal{C},\mathcal{A},\mathcal{B},\mathcal{C}) \arrow[r, "\tau"] \arrow[d, "m_{\mathcal{CABC}}"] & \mathbf{B}(\mathcal{B},\mathcal{C},\mathcal{A},\mathcal{B}) \arrow[d, "m_{\mathcal{BCAB}}"] \\
\mathbf{B}(\mathcal{A},\mathcal{A}) & \mathbf{B}(\mathcal{C},\mathcal{C}) & \mathbf{B}(\mathcal{B},\mathcal{B}) \\
\mathrm{TR}_{\mathcal{A}} & \mathrm{TR}_{\mathcal{C}} & \mathrm{TR}_{\mathcal{B}}
\end{tikzcd}$$

satisfying

$$\tau_{\mathcal{ABC}} \circ \sigma_{\mathcal{ABC}} = (\sigma_{\mathcal{ABC}} \circ \tau) \circ \tau_{\mathcal{ABC}}$$

as two homotopies between $\tau_{\mathcal{ABC}}$ and $\tau_{\mathcal{ABC}} \circ (\tau_{\mathcal{BCA}} \circ \tau^2) \circ (\tau_{\mathcal{CAB}} \circ \tau) \circ \tau_{\mathcal{ABC}}$.

Here

$$\mathbf{B}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{A}) = \mathbf{B}(\mathcal{A}, \mathcal{B}) \otimes \mathbf{B}(\mathcal{B}, \mathcal{C}) \otimes \mathbf{B}(\mathcal{C}, \mathcal{A})$$

etc.; τ are cyclic permutations of length 3;

$$\tau_{\mathcal{ABC}} : (m_{\mathcal{ABCA}})^* \mathrm{TR}_{\mathcal{A}} \to (m_{\mathcal{CABC}} \circ \tau)^* \mathrm{TR}_{\mathcal{C}})$$

is defined as the composition

$$(m_{\mathcal{ACA}} \circ m_{\mathcal{ABC}})^* \mathrm{TR}_{\mathcal{A}} \overset{\tau_{\mathcal{AC}} \circ m_{\mathcal{ABC}}}{\longrightarrow} (m_{\mathcal{CAC}} \circ \tau \circ m_{\mathcal{ABC}})^* \mathrm{TR}_{\mathcal{C}} = (m_{\mathcal{CABC}} \circ \tau)^* \mathrm{TR}_{\mathcal{C}}.$$

5. A category in DG categories with a trace functor

5.1. Trace functors in terms of the Grothendieck construction. Recall the definition of a homotopy category in DG categories as in Remark 3.1.2. We will extend it as follows. Let Λ_{Alg} be the category whose objects are $(n, \mathbf{A})$ where $n \geq 0$ and $\mathbf{A}$ is a *cyclic word of length* n, i.e. a symbol

$$\mathcal{A}_0 \to \ldots \to \mathcal{A}_n \to \mathcal{A}_0 \tag{5.1}$$

where $\mathcal{A}_j$ are DG categories. A morphism $\lambda \in \Lambda([n], [m])$ transforms cyclic words of length n into cyclic words of length m. A morphism $(n, \mathbf{A}) \to (m, \mathbf{A}')$ in Λ_{Alg} is a morphism $\lambda \in \Lambda([n], [m])$ that transforms $\mathbf{A}$ to $\mathbf{A}'$. The composition is defined by the composition in Λ. We define $\Lambda_{\infty,\mathrm{Alg}}$ in exactly the same way with Λ replaced by Λ_∞.

We will usually write $\mathbf{A}$ instead of $(n, \mathbf{A})$ (of course the length of $\mathbf{A}$ is n.

A homotopy trace functor on a homotopy category $\mathcal{C}$ in DG categories (cf. Remark 3.1.2) is:

(1) A functor from Λ_{Alg} to DG categories that extends the restriction of $\mathcal{C}$ to the full subcategory of Δ_{Alg} whose objects are $(n, \mathbf{A})$ where $\mathbf{A}$ are cyclic words (we denote this functor also by $\mathcal{C}$);
(2) a DG module $\mathrm{TR}_{\mathbf{A}}$ over $\mathcal{C}(\mathbf{A})$ for every cyclic word $\mathbf{A}$;
(3) a weak equivalence $\lambda^\dagger(\mathbf{A}) : \lambda^* \mathrm{TR}_{\lambda \mathbf{A}} \to \mathrm{TR}_{\mathbf{A}}$ for every morphism λ in $\Lambda_{\infty,\mathrm{Alg}}$ such that

$$(\lambda\mu)^\dagger(\mathbf{A}) = \mu^\dagger(\mathbf{A})\mu^*\lambda^\dagger(\mu\mathbf{A})$$

in the diagram

$$\begin{tikzcd}
\mu^*\lambda^*\mathrm{TR}_{\lambda\mu\mathbf{A}} \arrow[r, "\sim"] \arrow[d, "\mu^*(\lambda^\dagger(\mu\mathbf{A}))"] & (\lambda\mu)^*\mathrm{TR}_{\lambda\mu\mathbf{A}} \arrow[d, "(\lambda\mu)^*"'] \\
\mu^*\mathrm{TR}_{\mathbf{A}} \arrow[r, "\mu^\dagger"] & \mathrm{TR}_{\mathbf{A}}
\end{tikzcd}$$

(4) a homotopy $\sigma(\mathbf{A})$ between two DG functors id and $(\tau^{n+1})^\dagger(\mathbf{A})$ for any cyclic word $\mathbf{A}$ of length n such that for any $\lambda \in \Lambda_\infty([n], [m])$ one has the equality

$$\lambda^\dagger\lambda^*(\sigma(\lambda\mathbf{A})) = \sigma(\mathbf{A})\lambda^\dagger$$

of the two homotopies between the two DG functors $\lambda^\dagger$ and

$$(\lambda\tau^{m+1})^\dagger = (\tau^{n+1}\lambda)^\dagger$$

as in

$$\begin{tikzcd}
\lambda^*\mathrm{TR}_{\lambda\mathbf{A}} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\lambda^*(\tau^{m+1})^\dagger"'] \arrow[d, "\lambda^\dagger"'] & \lambda^*\mathrm{TR}_{\lambda\mathbf{A}} \arrow[d, "\lambda^\dagger"] \\
\mathrm{TR}_{\mathbf{A}} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "(\tau^{n+1})^\dagger"'] & \mathrm{TR}_{\mathbf{A}}
\end{tikzcd}$$

6. A category in DG cocategories with a di(tetra)module

In the remaining part of this chapter we will outline the structure on cochains and chains that are both in coefficients in bimodules ${}_f\mathcal{B}_g$ where $f, g : \mathcal{A} \to \mathcal{B}$ are DG (or more generally A_∞) functors.

6.1. Cotrace of a bicomodule. If $\mathcal{M}$ is a DG module over a DG caterory $\mathcal{A}$ then

$$\mathrm{tr}_{\mathcal{A}}(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}} \mathcal{A};$$

explicitly,

$$\mathrm{tr}_{\mathcal{A}}(\mathcal{M}) = \mathrm{coker}(\bigoplus_{x,y\in \mathrm{Ob}(\mathcal{A})} \mathcal{A}(x,y) \otimes \mathcal{M}(y,x) \to \bigoplus_{x\in\mathrm{Ob}(\mathcal{A})} \mathcal{M}(x,x))$$

where the morphism in the right hand side is given by

$$a \otimes m \mapsto am - (-1)^{|a||m|} ma$$

Dually, for a DG bicomodule $\mathbf{M}$ over a DG cocategory $\mathbf{B}$,

$$\mathrm{cotr}_{\mathbf{B}}(\mathbf{M}) = \ker(\bigoplus_{x\in\mathrm{Ob}(\mathbf{B})} \mathbf{M}(x,x) \to \bigoplus_{x,y\in\mathrm{Ob}(\mathbf{B})} \mathbf{B}(x,y) \otimes \mathbf{M}(y,x))$$

where the map in the right hand side is given by

$$m \mapsto \sum b^{(1)} \otimes m^{(2)} - (-1)^{|b^2||m^{(1)}|} b^{(2)} \otimes m^{(1)}$$

Here $\Delta^l : \mathbf{M}(x,y) \to \mathbf{B}(x,z) \otimes \mathbf{M}(z,y)$ is given by $m \mapsto \sum b^{(1)} \otimes m^{(2)}$ and $\Delta^r : \mathbf{M}(x,y) \to \mathbf{M}(x,z) \otimes \mathbf{B}(z,y)$ is given by $m \mapsto \sum m^{(1)} \otimes b^{(2)}$

For a DG functor $F : \mathbf{B}_1 \to \mathbf{B}_2$ and for a DG comodule $\mathbf{M}_2$ pver $\mathbf{B}_2$, there is a natural map

$$F_\sharp : \mathrm{cotr}_{\mathbf{B}_1}(F^*\mathbf{M}_2) \to \mathrm{cotr}_{\mathbf{B}_2}\mathbf{M}_2 \tag{6.1}$$

6.2. The di(tetra)module structure. A di(tetra)module over a category $\mathbf{B}$ in DG categories is the following.

(1) A DG cobimodule $\mathbf{M}(\mathcal{A}, \mathcal{B})$ over $\mathbf{M}(\mathcal{A}, \mathcal{B})$ for every $\mathcal{A}$ and $\mathcal{B}$;
(2) a morphism of DG cobimodules over

$$\mu^r_{\mathcal{ABC}} : \mathbf{M}(\mathcal{A}, \mathcal{B}) \otimes \mathbf{B}(\mathcal{B}, \mathcal{C}) \to m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A}, \mathcal{C})$$

and
(3) a morphism of DG cobimodules over $\mathbf{B}(\mathcal{B}, \mathcal{C})$

$$\mu^l_{\mathcal{ABC}} : \mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})} m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A}, \mathcal{C}) \to \mathbf{M}(\mathcal{B}, \mathcal{C})$$

for every $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$

such that the following compatibility conditions hold.

1). The composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})\to m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})\xrightarrow{\sim}$$
$$m^*_{\mathcal{ABC}}(\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D}))\to m^*_{\mathcal{ABC}}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D})=m^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

is the same as the composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})\to \mathbf{M}(\mathcal{A},\mathcal{B})\otimes m^*_{\mathcal{BCD}}\mathbf{B}(\mathcal{B},\mathcal{D})\xrightarrow{\sim}$$
$$m^*_{\mathcal{BCD}}(\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{D}))\to m^*_{\mathcal{BCD}}m^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D})=m^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

2). The composition

$$\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})}\mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})}m^*_{\mathcal{ABC}}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D})\xrightarrow{\sim}\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})}m^*_{\mathcal{ABC}}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$
$$\xrightarrow{(m_{\mathcal{ABC}})_\sharp}\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{C})}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D})\to\mathbf{M}(\mathcal{C},\mathcal{D})$$

is the same as the composition

$$\mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})}\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})}m^*_{\mathcal{BCD}}m^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D})\xrightarrow{\sim}\mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})}m^*_{\mathcal{BCD}}\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})}m^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$
$$\to\mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})}m^*_{\mathcal{BCD}}\mathbf{M}(\mathcal{B},\mathcal{D})\to\mathbf{M}(\mathcal{C},\mathcal{D})$$

Here $(m_{\mathcal{ABC}})_\sharp$ is as in (6.1).

3). The composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})\to \mathbf{M}(\mathcal{A},\mathcal{B})\otimes m^*_{\mathcal{BCD}}\mathbf{B}(\mathcal{B},\mathcal{D})$$
$$\xrightarrow{\sim} m^*_{\mathcal{BCD}}(\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{D}))\to m^*_{\mathcal{BCD}}m^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D})\xrightarrow{\sim} m^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

is the same as the composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})\to m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})$$
$$m^*_{\mathcal{ABC}}(\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D}))\to m^*_{\mathcal{ABC}}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D})\xrightarrow{\sim} m^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

6.3. Di(tetra)modules over Hopf algebras.

7. Constructions and proofs

In this section we define the above structures on Hochschild chains and cochains and provide the needed proofs.

7.1. Braces on chains and cochains.

CHAPTER 18

Hochschild and cyclic homology *via* noncommutative forms

1. Noncommutative forms

Let $\Omega^\bullet(A)$ be the graded algebra generated by A and by symbols da, $a \in A$, linear in a and subject to relations

a) $d(ab) = da\, b + adb$;

b) the unit of A is the unit of $\Omega^\bullet(A)$.

The grading $|a| = 0$, $|da| = 1$ makes $\Omega^\bullet(A)$ a graded algebra. We define the differential $d : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ as the unique graded derivation sending a to da and da to zero for all a in A. Define also

$$\mathrm{DR}^\bullet(A) = \Omega^\bullet(A)/[\Omega^\bullet(A), \Omega^\bullet(A)] \tag{1.1}$$

$$\overline{\Omega}^\bullet(A) = \Omega^\bullet(A)/k \cdot 1 \tag{1.2}$$

$$\overline{\mathrm{DR}}^\bullet(A) = \mathrm{DR}^\bullet(A)/k \cdot 1 \tag{1.3}$$

2. The differential ι

Definition 2.0.1.

$$\iota(a_0 da_1 \dots da_n) = \sum_{j=1}^{n} (-1)^{n(j-1)} [a_j, da_{j+1} \dots da_n a_0 da_1 \dots da_{j-1}]$$

3. Operators on noncommutative forms

Define

$$\kappa(a_0 da_1 \dots da_n) = (-1)^{n-1} da_n \cdot a_0 da_1 \dots da_{n-1} \tag{3.1}$$

Note that

$$\Omega^n(A) \xrightarrow{\sim} C_n(A, A) \tag{3.2}$$

therefore one can consider the operator b on $\Omega^\bullet(A)$. One has

$$db + bd = \mathrm{Id} - \kappa \tag{3.3}$$

$$\kappa^{n+1} - \mathrm{Id} = -db \tag{3.4}$$

$$\kappa^{n+1} - \kappa = bd \tag{3.5}$$

as well as the following identities.

$$(\kappa^{n+1} - \mathrm{Id})(\kappa^{n+1} - \kappa) = 0 \tag{3.6}$$

Define the operator N on $\Omega^\bullet$ by

$$N\omega = n\omega \tag{3.7}$$

for ω in Ω^n. Define also

$$L = N d\, b + b\, N d \tag{3.8}$$

Lemma 3.0.1. *On* Ω^n,

$$L = (\kappa - 1)^2 f(\kappa)$$

where

$$f(x) = (x-1)^2 \sum_{j=1}^{n} \frac{x^j - 1}{x - 1}$$

Proof. One has

$$L = Ndb + (N+1)bd = -N(\kappa^{N+1} - 1) + (N+1)(\kappa^{N+1} - \kappa) = \kappa^{N+1} - 1 + (N+1)(1 - \kappa)$$

so

$$L = (\kappa - 1) \sum_{j=0}^{N} (\kappa^j - 1)$$

□

Definition 3.0.2. *Let P be the projection onto* $\operatorname{Ker}(\operatorname{Id} - \kappa)^2$ *and let $P^\perp$ be the projection onto* $\operatorname{Im}(\operatorname{Id} - \kappa)^2$.

Lemma 3.0.3.

$$\Omega = P\Omega \oplus P^\perp \Omega \tag{3.9}$$

Proof. This follows from the fact that that $f(x)$ and $(x^{n+1} - x)\sum_{j=0}^{n} x^j$ are relatively prime. □

Corollary 3.0.4. *Operators κ and P commute with b and d, as well as with each other.*

Example 3.0.5. For $n = 0 : P(a_0) = a_0;\ P^\perp(a_0) = 0$. For $n = 1 : P^\perp(a_0 da_1) = \frac{1}{2}(a_0 da_1 + a_1 da_0 - \frac{1}{2} d(a_0 a_1 + a_1 a_0))$; $P(a_0 da_1) = \frac{1}{2}(a_0 da_1 - a_1 da_0 + \frac{1}{2} d(a_0 a_1 + a_1 a_0))$.

The operator κ satisfies $\kappa^n = \operatorname{Id}$ on $(d\Omega)^n$ (because of (3.1)) and also on $(\Omega/d\Omega)^{n-1}$ (because $d : (\Omega/d\Omega)^{n-1} \to (d\Omega)^n$ is a surjection, actually an isomorphism up to constants). In particular, κ is invertible. Also, $\kappa^{n+1} = \operatorname{Id}$ on $(b\Omega)^n$ (because of (3.4)) and $\kappa^n = \operatorname{Id}$ on $(\Omega/b\Omega)^n$ (because of (3.5)).

Next, observe that

$$P[\Omega, \Omega] \subset [\Omega, \Omega] \tag{3.10}$$

Indeed,

$$[\Omega, \Omega] = [A, \Omega] + [dA, \Omega] = b\Omega + (\operatorname{Id} - \kappa)\Omega \tag{3.11}$$

and P, $P^\perp$ commute with b and $\operatorname{Id} - \kappa$.

We get the new interpretation of the differential B, as well as of the Ginzburg-Schedler differential ι:

$$B = NdP;\ \iota = bNP. \tag{3.12}$$

EXAMPLE 3.0.6. On Ω^0, $\kappa = \mathrm{Id}$ and $P = \mathrm{Id}$. On Ω^1,

$$\kappa(a_0 da_1) = -a_1 da_0 + d(a_1 a_0); \tag{3.13}$$

$$(\mathrm{Id} - \kappa)(a_0 da_1) = a_0 da_1 + a_1 da_0 - d(a_1 a_0) = b(da_1 da_0); \tag{3.14}$$

$$(\mathrm{Id} - \kappa)^2(a_0 da_1) = 2a_0 da_1 + 2a_1 da_0 + d[a_0, a_1]; \tag{3.15}$$

$$P(a_0 da_1) = \frac{1}{2}(a_0 da_1 - a_1 da_0 + \frac{1}{2} d(a_0 a_1 + a_1 a_0)); \tag{3.16}$$

$$P^\perp(a_0 da_1) = \frac{1}{2}(a_0 da_1 + a_1 da_0 - \frac{1}{2} d(a_0 a_1 + a_1 a_0)). \tag{3.17}$$

(Note that the above are well defined, i.e. they are zero if $a_1 = 1$ Observe also that modulo $[\Omega, \Omega]^1$, $P = \mathrm{Id}$).

4. Periodic and negative cyclic homology in terms of d and ι

THEOREM 4.0.1.

$$\mathrm{HC}^{\mathrm{per}}_\bullet(A) \xrightarrow{\sim} H_\bullet(\Omega^\bullet(A)((u)), \iota + ud)$$

$$\mathrm{HC}^-_\bullet(A) \xrightarrow{\sim} H_\bullet(\Omega^\bullet(A)[[u]]/[d\Omega, d\Omega], \iota + ud)$$

PROOF. Periodic and negative cyclic homology is computed by the complex

$$\mathrm{HC}^{\mathrm{per}}_\bullet(A) \xrightarrow{\sim} H_\bullet(\Omega/P^\perp\Omega((u)), b + uB); \tag{4.1}$$

$$\mathrm{HC}^-_\bullet(A) \xrightarrow{\sim} H_\bullet(\Omega/P^\perp\Omega[[u]], b + uB). \tag{4.2}$$

Now, we can replace $b+uB$ by $d+u\iota$. Note that d is contractible (up to constants) on Ω and therefore contractible on $P^\perp\Omega$. Also $\iota = 0$ on $P^\perp\Omega$. So the complex $P^\perp\Omega((u))$ is contractible while $P^\perp\Omega[[u]]$ is quasi-isomorphic to

$$\mathrm{Ker}(d|P^\perp\Omega) \xrightarrow{\sim} dP^\perp\Omega = P^\perp d\Omega = [d\Omega, d\Omega].$$

□

5. HKR maps

The Hochschild-Kostant-Rosenberg map from the Hochschild homology of the algebra of functions to diferential forms is a major motivation and a major tool in noncommutative geometry. It was recently discovered that an HKR map exists with values in noncommutative forms for any algebra, commutative or not. The classical HKR map isn the commutative case is obtained by projection from noncommutative to ordinary forms.

There are two noncommutative HKR maps: one, μ from $C_*(A), b+B$ to $\Omega^*(A), \iota + d$ given by:

$$\mu(a_0 \otimes \ldots \otimes a_n) = \frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{i(n-1)} da_{i+1} \ldots da_n a_0 da_1 \ldots da_i;$$

if we identify $C_*(A, A)$ with $\Omega^*(A)$, we can write the other, in reverse direction,

$$\nu(a_0 d \ldots da_n) = (n-1)! \sum_{i=0}^{n-1} (-1)^{(i+1)(n-1)} da_{i+1} \ldots da_n a_0 da_1 \ldots da_i.$$

Recall that

$$b(a_0da_1\dots da_n) = (-1)^{(n-1)}[a_0da_1\dots da_{n-1}, a_n] \tag{5.1}$$

and

$$\iota(a_0da_1\dots da_n) = \sum_{i=0}^{n-1}(-1)^{i(n-1)}[da_{i+1}\dots da_n a_0 da_1\dots da_{i-1}, a_i] \tag{5.2}$$

We will sometimes drop the index Δ.

In other words, if κ is the Karoubi operator as in (3.1), then $\mu = \frac{1}{(n+2)!}(1+\kappa+\ldots+\kappa^n)$ and $\nu = (n-1)!(1+\kappa+\ldots\kappa^{n-1})$. Both μ and ν are morphisms of complexes; this follows from basic identities (3.3), (3.4), (3.5), and (3.6).

LEMMA 5.0.1. *The isomorphism in Theorem 4.0.1 is induced by the HKR map* μ.

6. More on operators on noncommutative forms

LEMMA 6.0.1. *One has* $P = \mathrm{Id}$ *on* $\Omega/[\Omega,\Omega]$.

PROOF. Since $Id-\kappa$ is invertible on $P^\perp\Omega$, one has

$$P^\perp\Omega = (\mathrm{Id}-\kappa)P^\perp\Omega \subset [\Omega,\Omega]\cap P^\perp\Omega = P^\perp[\Omega,\Omega] \tag{6.1}$$

□

LEMMA 6.0.2. *One has* $P[\Omega,\Omega] = \iota\Omega$.

PROOF.

$$P[\Omega,\Omega] = P[A,\Omega] + P[dA,\Omega] = bP\Omega + (\mathrm{Id}-\kappa)P\Omega.$$

But $\mathrm{Id}-\kappa$ is zero on $P(\Omega/b\Omega)$ because κ is of finite order on each component and therefore P is the projection to the invariant part. Therefore

$$P[\Omega,\Omega] = bP\Omega = bNP\Omega = \iota\Omega.$$

□

Since $\iota P^\perp = 0$ and $\iota^2 = 0$, the above two lemmas show that $\iota|[\Omega,\Omega] = 0$.

7. Hochschild and cyclic homology in terms of d and ι

THEOREM 7.0.1.

$$\mathrm{HH}_\bullet(A) = \mathrm{Ker}(\iota : (\overline{\mathrm{DR}}^\bullet(A) \to \Omega^{\bullet-1}(A))$$

PROOF. Note that b is acyclic on $P^\perp\Omega$ because $\mathrm{Id}-\kappa = bd+db$ is invertible there. Therefore $HH_\bullet(A)$ is the homology of $P\Omega = \Omega/P^\perp\Omega$ with the differential bP that we can replace by $bNP = \iota$. One sees that

$$\mathrm{HH}_\bullet(A) = \mathrm{Ker}(\iota)/(P^\perp\Omega + \iota\Omega) = \mathrm{Ker}(\iota)/[\Omega,\Omega].$$

□

THEOREM 7.0.2.

$$\overline{\mathrm{HC}}_\bullet(A) = \mathrm{Ker}(\iota : \overline{\mathrm{DR}}^\bullet(A)/d\overline{\mathrm{DR}}^{\bullet-1}(A) \to \overline{\Omega}^{\bullet-1}(A)/d\overline{\Omega}^{\bullet-2}(A))$$

PROOF. Since b is contractible on $P^{\perp}\Omega$, the reduced cyclic homology is computed by the complex

$$(P\overline{\Omega}((u))/uP\overline{\Omega}[[u]], b + uB) \tag{7.1}$$

and we can replace the differential by $\iota + ud$. Since d is contractible on $\overline{\Omega}$, we can replace this complex by

$$(P\overline{\Omega}/d\overline{\Omega}, \iota) \tag{7.2}$$

Therefore (recall that the image of ι is contained in the image of P)

$$\overline{\mathrm{HC}}_{\bullet}(A) \xrightarrow{\sim} \mathrm{Ker}(\iota : \overline{\Omega}/(d\overline{\Omega} + P^{\perp}\overline{\Omega} + \iota\overline{\Omega}) \to \overline{\Omega}/d\overline{\Omega})$$

which is equal to

$$\mathrm{Ker}(\iota : \overline{\Omega}/([\overline{\Omega}, \overline{\Omega}] + d\overline{\Omega}) \to \overline{\Omega}/d\overline{\Omega}) = \mathrm{Ker}(\iota : \overline{\mathrm{DR}}/d\overline{\mathrm{DR}} \to \overline{\Omega}/d\overline{\Omega}).$$

□

8. The extended noncommutative De Rham complex

The above theorems 7.0.1 and 7.0.2 express the homology as the kernel of some differential. That suggests that perhaps there is a complex starting with this differential. This is indeed the case. We are going to describe this complex here.

Let t be a formal variable of degree zero. Define

$$\Omega_t^{\bullet}(A) = \Omega^{\bullet}(A) * k[t] \tag{8.1}$$

$$\mathtt{DR}_t^{\bullet}(A) = \Omega_t^{\bullet}(A)/[\Omega_t^{\bullet}(A), \Omega_t^{\bullet}(A)] \tag{8.2}$$

and also

$$\overline{\Omega}_t^{\bullet}(A) = \Omega^{\bullet}(A) * k[t]/k[t] \tag{8.3}$$

$$\overline{\mathrm{DR}}_t^{\bullet}(A) = \overline{\Omega}_t^{\bullet}(A)/[\overline{\Omega}_t^{\bullet}(A), \overline{\Omega}_t^{\bullet}(A)] \tag{8.4}$$

If we put $|t| = 1$, then $\Omega_t^{\bullet}$ acquires a second grading, as do all the spaces above. Therefore $\Omega_t^{\bullet}$ is a bi-graded algebra, and all the above spaces are bi-graded. For the first grading, $|d| = 1$ and $|t| = 0$. For the second, $|d| = 0$ and $|t| = 1$. We denote by $\Omega_t^{p,q}$ the component whose degree is p with respect to thefirst grading and q with respect to the second grading. We get similar decompositions for all the spaces above.

LEMMA 8.0.1.

$$\mathtt{DR}_t^{n,0} = (\Omega/[\Omega, \Omega])^n;\ \mathtt{DR}_t^{n-1,1} \xrightarrow{\sim} \Omega^{n-1}$$

8.1. The derivation ι_t. Let $|\omega|$ be the first grading of ω, *i. e.* $|a| = |t| = 0$ and $|da| = 1$. Define the graded derivation of degree -1 with respect to this grading by

$$\iota_t(a) = \iota_t(t) = 0;\ \iota_t(da) = [t, a]. \tag{8.5}$$

This is a bi-homogeneous map of degree $(-1, 1)$ satisfying

$$\iota_t^2 = 0.$$

LEMMA 8.1.1. *Under the identifications from Lemma 8.0.1, the map* $\mathtt{DR}_t^{n,0} \xrightarrow{\iota_t} \mathtt{DR}_t^{n-1,1}$ *becomes the operator* ι *from Definition 2.0.1 in 2.*

We get complexes

$$\mathtt{DR}_t^{n,0} \xrightarrow{\iota_t} \mathtt{DR}_t^{n-1,1} \xrightarrow{\iota_t} \mathtt{DR}_t^{n-2,2} \xrightarrow{\iota_t} \ldots \xrightarrow{\iota_t} \mathtt{DR}_t^{0,n} \tag{8.6}$$

PROPOSITION 8.1.2. *The homology of the complex* (8.6) *at* $\mathtt{DR}_t^{n-j,j}$ *is isomorphic to* $H_j(A, A)$.

PROOF. The case $j = n$ follows from Theorem 7.0.1. The general case follows from the following interpretation of $\mathtt{DR}_t^\bullet(A)$. Let $\mathtt{DR}_{t,+}^\bullet(A)$ be the subcomplex of $\mathtt{DR}_t^\bullet(A)$ spanned by elements whose degree with respect to t is positive. This subcomplex can be expressed in the form that we are going to discuss next.

Let us start with any associative unital differential algebra $(\mathcal{A}, \partial)$. View $\mathcal{A}$ as a graded algebra. Introduce a new generator ϵ of degree one and square zero. Consider the cross product algebra

$$\widetilde{\mathcal{A}} = k[\epsilon] \ltimes \mathcal{A} \tag{8.7}$$

generated by ϵ and $\mathcal{A}$ subject to a relation $[\epsilon, a] = \partial a$ for all a in $\mathcal{A}$.

In other words, $\widetilde{\mathcal{A}}$ is generated by the algebra $\mathcal{A}$ and by elements $\underline{a} = \epsilon a$, $a \in \mathcal{A}$, of degree $|a| + 1$, linear in a and subject to relations

$$\underline{a} \cdot b = \underline{ab};\ a \cdot \underline{b} = (-1)^{|a|}(\underline{ab} - \partial a \cdot b);\ \underline{a} \cdot \underline{b} = (-1)^{|a|-1}\underline{\partial a \cdot b} \tag{8.8}$$

Now one can consider the reduced cyclic homology $\overline{\mathrm{HC}}_\bullet(\widetilde{\mathcal{A}})$ of the graded algebra $\widetilde{\mathcal{A}}$. More precisely, we will compute it using the following specific complex defined for any A:

$$\mathrm{CC}'(A) = (\mathrm{Ker}(1-t), b'); \overline{\mathrm{CC}}'(A) = \mathrm{CC}'(A)/\mathrm{CC}'(k) \tag{8.9}$$

where $1 - t$ and N are as in the standard $(b, b', 1 - t, N)$ double complex. (Recall that

$$(\mathrm{Ker}(1-t), b') = (\mathrm{Im}(N), b') \xrightarrow{\sim} (C(A)/\mathrm{Ker}(N), b) = (C(A)/\mathrm{Im}(1-t), b)$$

and therefore $\mathrm{CC}'(A)$ does compute the cyclic homology).

Consider now a dual picture. Let $\mathcal{C}$ be a differential graded counital coalgebra $(\mathcal{C}, \partial)$. For $c \in \mathcal{C}$, let $\underline{c}$ be a formal element of degree $|c| + 1$, linear in c. These elements generate the space $\underline{\mathcal{C}}$ which is same as $\mathcal{C}$ but with the grading shifted by one. Let $\widetilde{\mathcal{C}}$ be the graded coalgebra which is a linear direct sum of $\mathcal{C}$ and $\underline{\mathcal{C}}$. The comultiplication is as follows:

$$\Delta c = \sum c^{(1)} \otimes c^{(2)} + \sum (-1)^{|c^{(1)}|} \partial c^{(1)} \otimes \underline{c}^{(2)} \tag{8.10}$$

$$\Delta \underline{c} = \sum \underline{c}^{(1)} \otimes c^{(2)} + (-1)^{|c^{(1)}|} c^{(1)} \otimes \underline{c}^{(2)} + \sum (-1)^{|c^{(1)}|} \underline{\partial c^{(1)}} \otimes \underline{c}^{(2)} \tag{8.11}$$

For any counital DG coalgebra C put

$$\mathrm{CC}'(C) = (\mathrm{Coker}(1-t), b'); \overline{\mathrm{CC}}'(C) = \mathrm{Ker}(\mathrm{CC}'(C) \to \mathrm{CC}'(k)) \tag{8.12}$$

LEMMA 8.1.3. *Let* $A = k + \overline{A}$ *be the algebra obtained from an algebra* $\overline{A}$ *by attaching a unit. Then the complex* $\mathtt{DR}_{t,+}^\bullet(A)$ *is isomorphic to* $\overline{\mathrm{CC}}'(\widetilde{\mathrm{Bar}(\overline{A})})$ *where* Bar *stands for the usual bar construction (which is a DG coalgebra).*

PROOF. Take a monomial $\omega_1 t \omega_2 t \ldots \omega_n t$ in $\mathtt{DR}_{t,+}$. Identify it with $\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n$ in $\overline{\mathrm{CC}}'(\widetilde{\mathrm{Bar}(\overline{A})})$ where $\alpha_k = (a_0|a_1|\ldots|a_m)$ if $\omega_k = a_0 da_1 \ldots da_m$ and $\alpha_k = \underline{(|a_1|\ldots|a_m)}$ if $\omega_k = da_1 \ldots da_m$. One checks that this gives an isomorphism of complexes. $\square$

To finish the proof of Proposition 8.1.2, we can consider the filtration on $\mathtt{DR}_t^\bullet$ whose associated graded quotient is the complex $\mathtt{DR}_t^\bullet(A_0)$ where $A_0 = k + \overline{A}$ is the algebra obtain by adjoining the unit to the algebra $\overline{A}$ with zero multiplication. This implies that the HKR map described in 9 is a quasi-isomorphism at the level of the associated graded quotient with respect to this filtration. Therefore it is a quasi-isomorphism. ELABORATE □

REMARK 8.1.4. The above computation suggests a comparison to the work of Berest , Felder, Patotsky, Ramadoss and Willwacher where the algebra of functions on the *derived* representation scheme is identified with the standard cochain complex of the Lie coalgebra $\mathfrak{gl}_n(\mathrm{Bar}(A))$, *cf.* 20.

9. HKR map from Hochschild homology to extended De Rham cohomology

We apply the map χ from **??** to the following situation. Let $\mathcal{A} = \Omega_t^\bullet(A)$ and

$$K = \Omega^\bullet(A)/[\Omega^\bullet(A), \Omega^\bullet(A)] = \mathtt{DR}_t^\bullet(A).$$

Let $\tau : \mathcal{A} \to K$ be the projection.

PROPOSITION 9.0.1. *The map*

$$CC_\bullet^-(A) \to CC_\bullet^-(\mathcal{A}) \overset{\chi(\exp(d+t))}{\longrightarrow} \mathbb{K}$$

defines a morphism of complexes

$$\mathrm{HKR}_t : (C_{-\bullet}(A)((u)), b + uB) \to (\mathtt{DR}_t^\bullet(A), ud + \iota_t)$$

PROOF. By Proposition 5.0.4, taking into account that $[d, d] = 0$ and $d(t) = 0$,

$$\mathrm{HKR}_t((b + uB)(c)) = ud\mathrm{HKR}_t(c) + \sum_{K=0}^{\infty} \frac{1}{K!}\chi(\delta(t)(d + t)^K)(c).$$

If we put $c = \alpha_0 \otimes \ldots \otimes \alpha_n,\ \alpha_i \in \mathcal{A}$, then the second summand is equal to

$$\sum_{N=0}^{\infty} \sum_{\sum_0^n N_k = N} \sum_{i=1}^{N} \alpha_0 t^{N_0} d\alpha_1 t^{N_1} \ldots [t, \alpha_i] t^{N_i} \ldots d\alpha_n t^{N_n}$$

$$\iota_t \sum_{N=0}^{\infty} \sum_{\sum_0^n N_k = N} \alpha_0 t^{N_0} d\alpha_1 t^{N_1} \ldots d\alpha_n t^{N_n}) = \iota_t \mathrm{HKR}_t(c).$$

□

10. On the duality between chains and cochains

11. Chains and cochains

There are maps $\Omega^*(A) \to A * k[\tau] \to C^*(A, A)$ where the first one sends a to a and da to $[\tau, a]$ like in [**264**], and the second sends a to a and τ to $id : A \to A$, the identity one-cochain. The composite map is the universal map of DGAs that sends a to a. Of course it intertwines d with the Hochschild differential δ. In case when A has a trace, there is a map from $C^*(A, A)$ to the dual space of $C_*(A, A)$ induced by the bimodule map $A \to A^*$, $a \mapsto tr(a?)$. Now, the differential dual to B on the right hand side gets intertwined with the differential dual to b in $A * k[\tau] = A^{*+1}$ under the Connes isomorphism between Λ and Λ^{op} (this is a good explanation for the latter). This same differential gets intertwined with ι on Ω^* (which can be viewed as another way to discover ι). So we have morphisms

$$(\Omega^*(A), b+B) \to (\Omega^*(A), \iota + d) \to A * k[\tau], B^{\mathrm{dual}} + b^{\mathrm{dual}}) \to (C^*(A,A), B+\delta) \to (C_*(A,A)^{\mathrm{dual}}) \tag{11.1}$$

The composition of all the maps above can be interpreted in terms of our map χ as follows.

WHAT ABOUT χ? EXPLICIT FORMULA FOR THE PAIRING? PASCHKE?

Consider the map

$$C_*(A,A) \to C_*^{\mathrm{Lie}}(\mathrm{Der}(\mathfrak{gl}(A)) \ltimes \mathfrak{gl}(A)[-1])_{\mathfrak{gl}(k)}$$

given by

$$a_0 \otimes \ldots \otimes a_n \mapsto E_{01}(a_0) \wedge \mathrm{ad}(E_{12}(a_1)) \wedge \ldots \wedge \mathrm{ad}(E_{n0}(a_n))$$

This map intertwines b with the Koszul differential. Now embed A into $M_\infty(A)$ diagonally where $M_\infty(A)$ is the Lie algebra of infinite matrices, say, with finitely many nonzero diagonals. We get an embedding

$$\beta : C_*(A,A) \to C_*(M_\infty(A), M_\infty(A)).$$

Under the composition (11.1), a Hochschild chain c maps to the linear functional $x \mapsto \chi(\alpha(c))(\beta(x))$. Observe that the pairing χ clearly extends to a pairing

$$C_*^{\mathrm{Lie}}(\mathfrak{gl}(A) \ltimes \mathfrak{gl}(A)[-1])_{\mathfrak{gl}(k)} \otimes C_*(M_\infty(A), M_\infty(A))^{\mathfrak{gl}(k)} \to k$$

Note that $C^*(A,A)$ is a brace algebra, and the brace operations are defined on $A * k[\tau]$ so that the map of complexes preserves them (an insertion of $b_0\tau \ldots \tau b_m$ into $a_0\tau \ldots \tau a_n$ acts by taking an appropriate factor τ in the latter and replacing it by the former). Therefore $A * k[\tau]$ is a homotopy BV algebra, with the BV operator being B^{dual}. The brace structure can be also defined on $(\Omega^*(A), d)$. The easiest way to see this is to observe that $\Omega^*(A)$ embeds into $A * k[\tau]$ as the subspace of elements that are annihilated by substituting the unit for any given factor τ.

The above brace structure on $\Omega^*(A)$ is a quantum analog of the Gerstenhaber bracket on $\Omega^*(M)$ where M is a Poisson manifold. The bracket is defined by $[da, b] = \{a, b\}$ and $[da, db] = d\{a, b\}$ for functions a and b. The map $\Omega^*(M) \to \wedge^*(T(M))$ defined by the Poisson structure is a morphism of Gerstenhaber algebras.

When A has a trace such that $\mathrm{tr}(ab)$ is a nondegenerate form, then $C^*(A,A)$ is a homotopy BV algebra, and the morphism $\Omega^*(A) \to C^*(A,A)$ is a morphism of homotopy BV algebras.

QUESTION 11.0.1. The formality theorem for chains says that, for a smooth commutative algebra, $(C_{-*}(A,A)[[u]], b+uB)$ is quasi-isomorphic to the complex $(\Omega^{-*}_{A/k}[[u]], ud)$ as L_∞ modules over the DG Lie algebra $C^{*+1}(A,A)$; the latter acts on the former via the Kontsevich formality morphism to $\wedge^{*+1}T$. If A is a deformation of a smooth commutative algebra corresponding to a formal Poisson structure π, then $(C_{-*}(A,A)[[u]], b+uB)$ is quasi-isomorphic to $(\Omega^{-*}_{A/k}[[u,h]], hL_\pi + ud)$ (as L_∞ modules over $C^{*+1}(A,A)$). What is the correct formality statement that takes into account the brace/BV structure on Hochschild chains?

Note that the homotopy BV algebra $(\Omega^{-*}_{A/k}[[u,h]], hL_\pi + ud)$ has one more property, namely there is an operator ι_π/u whose exponential gives an isomorphism of complexes $(\Omega^*(M)[[h,u]], hL_\pi + ud)$ and $(\Omega^*(M)[[h,u]], ud)$. So it kills b in $b+uB$.

Now we see that there are some dualities intertwining b and B. Can it be that, for a smooth compact CY A, there is a similar structure that kills B in $b + uB$?

For example, if A is a deformation quantization of a compact symplectic manifold (localized in $\hbar$), the composition (11.1) is an isomorphism on homology. It coincides with the Poincaré duality $H^{2n-*}(M) \xrightarrow{\sim} H^k(M)^{\mathrm{dual}}$. The Hochschild to cyclic spectral sequence starts with the De Rham cohomology; by rigidity of the periodic cyclic homology, $HP_*(A)$ is the De Rham cohomology; therefore, since its dimension is finite, we know that the spectral sequence degenerates. Maybe there is a similar mechanism for smooth compact CY? Note that this is far from straightforward because (11.1) involves a lot of commutators and seems not to have much chance to be an isomorphism for, say, commutative algebras. Still, maybe there is some more sophisticated version.

12. Hamiltonian actions

Another thing suggested by the above constructions is a definition of a Hamiltonian action of a Hopf algebra H on an algebra A, so that one can define Hamiltonian reduction. Let H be a Hopf algebra acting on an associative algebra A. Put $\overline{H} = \mathrm{Ker}(\epsilon : H \to k)$. The action can be interpreted as an associative algebra morphism $\rho : \overline{H} \to C^1(A, A)$ such that

$$\delta\rho(h) + \sum \rho(h^{(1)}) \smile \rho(h^{(2)}) = 0$$

where $\smile$ is the cup product on Hochschild cochains and

$$1 \otimes h + \sum h^{(1)} \otimes h^{(2)} + h \otimes 1 = \Delta h.$$

The action is recovered from ρ by as $h(a) = \epsilon(h)a + \rho(h - \epsilon(h))(a)$.

Define a Hamiltonian action of H on A as a morphism of associative algebras

$$\rho : \overline{H} \to \Omega^1(A^+)$$

such that

$$d\rho(h) + \sum \rho(h^{(1)})\rho(h^{(2)}) = 0; \tag{12.1}$$

here A^+ is A with the unit adjoined, and the associative product on Ω^1 is given by the brace operation:

$$a_0 da_1 \circ b_0 db_1 = a_0 a_1 b_0 db_1 - a_0 b_0 db_1 a_1.$$

Such ρ defines an action of H on A via the map $\Omega^1(A^+) \to C^*(A^+, A^+) \to C^*(A, A)$. Given ρ as in (12.1), define a reduced algebra by

$$A_{\mathrm{red}} = (A/I)^H$$

Here I is the left ideal of A generated by elements $\sum a_{0,i}(h) x a_{1,i}(h)$ where $\rho(h) = \sum a_{0,i}(h) da_{1,i}(h)$. One observes that the action of H on A descends to an action on A/I. Indeed,

$$h(ya_0(h')xa_1(h')) = a_0(h)[a_1(h), ya_0(h')xa_1(h')] =$$

$$= a_0(h)a_1(h)y(a_0(h')xa_1(h')) - a_0(h)(ya_0(h')xa_1(h'))a_1(h) \in I;$$

as for the product,

$$\sum a_{0,i}(h)xa_{1,i}(h)y \equiv a_{0,i}(h)x[a_{1,i}(h), y] \equiv \sum [a_{0,i}(h), x][a_{1,i}(h), y] \mod I;$$

by (12.1), the latter is equal to

$$\sum a_{0,i}(h^{(1)})[a_{1,i}(h^{(1)}), x]a_{0,i}(h^{(2)})[a_{1,i}(h^{(2)}), x];$$

if y is invariant modulo I, this expression lies in I.

REMARK 12.0.1. For any Hopf algebra H, the tensor algebra $T(H[1])$ is a brace algebra: if $G_i = (g_{i,1}|\dots|g_{i,n_i}) \in H^{n_i}$, then

$$(h_1|\dots|h_m)\{G_1,\dots,G_p\} = \sum_{1\le k_1<\dots<k_p\le m} \pm(h_1|\dots|\Delta^{n_1-1}h_{k_1}\cdot G_1|\dots|\Delta^{n_p-1}h_{k_p}\cdot G_p|\dots|h_m).$$

In particular, $(h)\{(g)\} = (gh)$.

An action of H on an algebra A is a morphism of brace algebras $T(H[1]) \to C^*(A, A)$; a Hamiltonian action is a morphism of brace algebras $T(H[1]) \to \Omega^*(A^+)$. By a result of Halbout, if H is an Etingof-Kazhdan quantization of a Lie bialgebra $\mathfrak{g}$, then there is a G_∞ quasi-isomorphism $T(H[1]) \to C_*(\mathfrak{g})$, the right hand side being the Lie algebra chain complex on which the differential is the cochain differential of the Lie algebar $\mathfrak{g}^*$ and the Gerstenhaber bracket is induced by the bracket of $\mathfrak{g}$. This, together with the formality theorem of Kontsevich, should give a classification of Hamiltonian actions of a quantum group on a smooth manifold (though Pavol Ševera seemed to think that some refinement of Halbout's result is needed). Similarly, a correct formality theorem from the previous question should give a classification of Hamiltonian actions.

CHAPTER 19

Representation schemes

1.

Let A be an associative algebra. The nth representation scheme $Rep_n(A)$ is the scheme whose points are morphisms of algebras $A \to M_n(k)$. More precisely, $\mathcal{O}(Rep_n(A))$ is the commutative k-algebra with generators $\rho_{jk}(a)$, $a \in A$, $1 \leq j, k \leq n$, that are k-linear in a and satisfy

$$\rho_{j\ell}(ab) = \sum_{k=1}^{n} \rho_{jk}(a)\rho_{j\ell}(b) \tag{1.1}$$

We will usually fix n and write $Rep(A)$ instead of $Rep_n(A)$. We will also denote k^n by V.

There is a morphism of algebras

$$A \to M_n(\mathcal{O}(Rep(A))); \; a \mapsto (\rho_{jk}(a))_{1 \leq j,k \leq n} \tag{1.2}$$

In fact $\mathcal{O}(Rep(A))$ is the universal commutative algebra B with a morphism $A \to M_n(B)$. There is also an action of $\mathrm{GL}_n(k)$ by automorphisms: for a matrix $T = (T_{ij})$ in GL_n, the corresponding automorphism acts by

$$\rho_{pq} \mapsto \sum_{j,k} T_{pj}\rho_{jk}(T^{-1})_{kq} \tag{1.3}$$

The morphism (1.2) is actually a morphism

$$A \to M_n(\mathcal{O}(Rep(A)))^{\mathrm{GL}_n} \tag{1.4}$$

1.1. From the extended De Rham complex to equivariant forms on $Rep(A)$**.** Let

$$\Omega^\bullet(Rep(A)) = \Omega^\bullet_{\mathcal{O}(Rep(A))/k} \tag{1.5}$$

be the algebra of Kähler differentials. For $X \in \mathfrak{gl}_n$ let v_X be the derivation of $\mathcal{O}(Rep(A))$ defined by

$$\mathbf{v}(X)(\rho_{pq}) = \sum_{k=1}^{n} (\rho_{pj}X_{jq} - X_{pj}\rho_{jq}) \tag{1.6}$$

This is the infinitesimal form of the action (1.3) of GL_n . Denote

$$G = \mathrm{GL}_n(k); \; \mathfrak{g} = \mathfrak{gl}_n(k).$$

Consider a differential

$$\iota_{\mathbf{v}} : \mathrm{Hom}(S^j(\mathfrak{g}), \Omega^p(Rep(A)))^G \to \mathrm{Hom}(S^{j+1}(\mathfrak{g}, \Omega^{p-1}(Rep(A)))^G \tag{1.7}$$

defined by

$$(\iota_{\mathbf{v}} f)(X) = \iota_{\mathbf{v}(X)} f(X) \tag{1.8}$$

where we view $\mathrm{Hom}(S^j\mathfrak{g}, \Omega)$ as the space of homogeneous maps $\mathfrak{g} \to \Omega$ of degree j.

Define the map of differential graded algebras

$$\Omega_t^\bullet(A) \to \mathrm{Hom}(S^\bullet\mathfrak{g}, M_n(\Omega^\bullet(Rep(A))))^G \tag{1.9}$$

as follows:

$$a \mapsto (\rho_{jk}(a));\ da \mapsto d(\rho_{jk}(a))$$

(we view $M_n(\Omega)$ as $\mathrm{Hom}(S^0\mathfrak{g}, M_n(\Omega))$;

$$t \mapsto \mathrm{id} : S^1\mathfrak{g} \to M_n(k) \subset M_n(\Omega^\bullet(Rep(A))).$$

Composing (1.9) with the ordinary matrix trace tr, we observe that the result is equal to zero on all commutators. It is immediate taht the above map intertwines ι_t with $\iota_{\mathbf{v}}$. Tensoring with $k((u))$, we get a morphism

$$(\mathtt{DR}_t^\bullet(A)((u)), \iota_t + ud) \to (\mathrm{Hom}(S^\bullet\mathfrak{g}, \Omega^\bullet(Rep(A)))^G, \iota_{\mathbf{v}} + ud) \tag{1.10}$$

We have obtained

Proposition 1.1.1. *The map* (1.10) *is a morphism of complexes.*

CHAPTER 20

Derived representation schemes

Let $(R_\bullet, d_R)$ be a semi-free differential graded resolution of A. We define *the algebra of functions on the derived representation scheme of* A as the differential graded algebra $\mathcal{O}(Rep_n(R_\bullet))$. More precisely, it is the graded algebra generated by $\rho_{jk}(r)$, $r \in R_p$, of degree p, with relations (1.2); the differential is defined as

$$d\rho(r) = \rho(d_R r) \tag{0.1}$$

We denote the differential graded algebra $\mathcal{O}(Rep(R_\bullet))$ also by $\mathbb{L}\mathcal{O}(Rep(A))$.

1. Derived representation schemes and the bar construction

Just as for any differential graded algebra $\mathcal{A}$ we can form the differential graded Lie algebra $\mathfrak{gl}_n(\mathcal{A})$, for any differential graded coalgebra $\mathcal{C}$ we can form a we can form the differential graded Lie coalgebra $\mathfrak{gl}_n(\mathcal{C})$. And just as we construct the Chevalley-Eilenberg chain complex $C_\bullet^{\mathrm{CE}}(\mathcal{L}, k)$ for a DG Lie algebra, we can construct Chevalley-Eilenberg cochain complex $C^\bullet_{\mathrm{CE}}(\mathcal{L}, k)$ for a DG Lie coalgebra.

Theorem 1.0.1. *(Berest , Felder, Patotsky, Ramadoss, Willwacher). There is a quasi-isomorphism of differential graded algebras*

$$\mathbb{L}\mathcal{O}(Rep(A)) \xrightarrow{\sim} \mathrm{Sym}(C^\bullet_{\mathrm{CE}}(\mathfrak{gl}_n(\mathrm{Bar}(A), k))$$

Proof. This follows directly from applying the definition of $\mathbb{L}\mathcal{O}(Rep)$ when $R_\bullet$ is the standard resolution CobarBar(A). ELABORATE $\square$

CHAPTER 21

Cyclic homology and representation varieties

1. Introduction

We compare the Hochschild (resp. periodic cyclic) homology of an algebra A to forms (resp. equivariang De Rham cohomology) of the variety of representations of A. We provide two constructions. First, we prove a theorem of Ginzburg and Schedler [**264**] that interpretes the Hochschild and periodic cyclic homologies via the noncommutative De Rham complex, and use it, as they did, to construct the requisite maps. Second, we construct these maps directly using the standard chain complexes. Hopefully, these methods can be extended to compare cyclic homology of DG categories to the De Rham cohomology of their moduli spaces as studied by Toën and Vezzosi.

2. Periodic cyclic homology via noncommutative De Rham complex

3. Cyclic homology and representation varieties: a map using the noncommutative De Rham complex

4. Cyclic homology and representation varieties: a map using the standard complexes

Let $\mathcal{A}$ be a graded associative algebra. Let $\mathcal{A}^\bullet[1] \oplus \mathrm{Der}(\mathcal{A}^\bullet)$ be the differential graded Lie algebra defined as follows: $\mathcal{A}^\bullet[1]$ is Abelian, $\mathrm{Der}(\mathcal{A}^\bullet)$ is a subalgebra whose adjoint action on $\mathcal{A}^\bullet[1]$ is the natural one, the differential δ sends $a \in \mathcal{A}^\bullet[1]$ to $\mathrm{ad}(a)$ and is zero on $\mathrm{Der}(\mathcal{A}^\bullet)$. Let $\mathcal{L}$ be a DG Lie subalgebra of $\mathcal{A}^\bullet[1] \oplus \mathrm{Der}(\mathcal{A}^\bullet)$. Let K be a graded space on which $\mathcal{L}$ acts so that elements of $\mathcal{A}^\bullet[1]$ act by zero. Let $\tau : \mathcal{A}^\bullet \to K$ is an $\mathcal{L}$-equivariant trace. We extend it by zero to the entire Hochschild complex $C_{-\bullet}(\mathcal{A}^\bullet)$.

Define for $a \in \mathcal{A}^\bullet[1]$

$$\iota_a(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^{|a||a_0|} a_0 a \otimes a_1 \otimes \ldots \otimes a_n$$

and for $D \in \mathrm{Der}(\mathcal{A}^\bullet)$

$$\iota_D(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^{|D||a_0|} a_0 D(a_1) \otimes a_2 \otimes \ldots \otimes a_n$$

Now let $\mathcal{A}^\bullet = \Omega(Rep(A)) \otimes \mathrm{End}(V)$, $Rep(A)$ being the scheme of representations of an algebra A in a given finite dimensional space V. Consider the subalgebra $\mathrm{End}(V)$ consisting of constant functions. Let $\mathcal{L}$ be the linear span of $x \in \mathrm{End}(V)$, $\mathrm{ad}(x)$ for $x \in \mathrm{End}(V)$, and the De Rham differential d. Let $K = \Omega(Rep(A))$ and $\tau : \mathcal{A}^\bullet \to K$ be the matrix trace. Let $G_{\mathrm{geom}} = \mathrm{End}(V)$ viewed as a Lie algebra.

Claim: the map

$$c, x \to \tau(\exp(\iota_{d+x}))(c),$$

$x \in G_{\text{geom}}$ and $c \in C_{-\bullet}(\mathcal{A}^\bullet)$, defines a morphism of complexes

$$C_{-\bullet}(\mathcal{A}^\bullet)((u)), b + uB \to \Omega(Rep(A))[[G_{\text{geom}}]]^G((u)), ud + \iota_x$$

(Note that the value of this map at $x = 0$ is the HKR morphism). Combining the above with the morphism induced by ev : $A \to \mathcal{A}^\bullet$, we get a morphism from the periodic cyclic homology of A to the equivariant cohomology of $Rep(A)$.

For the proof, define, in the general situation of the first paragraph, for $X_1, \ldots, X_n \in \mathcal{L}$, and for a Hochschild chain c,

$$\chi(X_1, \ldots, X_n)(c) = \sum_{\sigma \in S_n} \pm\tau(\iota_{X_{\sigma(1)}} \ldots \iota_{X_{\sigma(n)}} c);$$

the sign is computed as follows: a permutation of X_i and X_j introduces a sign $(-1)^{(|X_i|+1)(|X_j|+1)}$. One checks directly that χ defines a cocycle of the complex

$$C^\bullet(\mathcal{L}, \operatorname{Hom}(C_{-\bullet}(A), K)[[u]]$$

with the differential $b + uB + \delta + u\partial_{\text{Lie}}$; the action of L on $\operatorname{Hom}(C_{-\bullet}, K)$ is induced by the action on K. In other words,

$$\chi(X_1, \ldots, X_n)((b + uB)(c)) = (\sum \pm\chi(X_1, \ldots, \delta X_i, \ldots, X_n) +$$
$$u \sum \pm\chi([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n) +$$
$$u \sum \pm X_i \chi(X_1 \ldots, \widehat{X_i}, \ldots, X_n))(c)$$

(cf. Nest, Tsygan, Algebraic index theorem, CMP 1995; Algebraic index theorem for families, AIM 1995).

The claim follows from this and from the fact that

$$\sum \chi(x, \ldots, \delta x, \ldots, x) = \iota_x \chi(x, \ldots, x)$$

where $x \in G_{\text{geom}}$ and ι_x in the right hand side refers to the contraction of a form by a vector field corresponding to x. It is the derivation that acts on the image of A in $\mathcal{A}^\bullet$ by sending $\operatorname{ev}(a)$ to zero and $d(\operatorname{ev}(a))$ to $[x, \operatorname{ev}(a)]$.

CHAPTER 22

Something about Katzarkov-Pantev-Kontsevich

1. Introduction

2. Bibliographical notes

CHAPTER 23

Coperiodic cyclic homology

CHAPTER 24

Hochschild and cyclic complexes of the second kind

Here we define the Hochschild and cyclic chain complexes of the second kind. The Hochschild complex of the second kind is defined as above but using direct products instead of direct sums in the total complex. Since direct sums map to direct products, Hochschild homology of the first kind maps to the Hochschild homology of the second kind. We prove a theorem giving a sufficient condition for this map to be an isomorphism. Our main reference is the article [**?**] by Positselski and Polishchuk.

CHAPTER 25

Matrix factorizations

We define the DG category of matrix factorizations and compute its Hochschild and cyclic homology following Efimov.

CHAPTER 26

End of part I

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