A 2-category associated to a holomorphic symplectic manifold
(joint with A. Kapustin and N. Saulina)

Rough outline

A 2-category is a category such that
\( \text{Hom}(A, B) \) is a category

Holomorphic symplectic manifold \((X, \omega)\)

\( X \) - holomorphic manifold
\( \omega \in \Omega^{2,0}(X) \), \( d\omega = 0 \), \( \omega \) is nowhere degenerate

A 2-category \( \mathcal{L}(X, \omega) \)

Simplest objects: \( Y \subseteq X \) - lagrangian submanifolds
\( \uparrow \) automatically holomorphic

A category of morphisms

A clean intersection

\[ \begin{align*}
Y_1 & \\
Y_2 &
\end{align*} \]

Not clean:
If \( Y_1 \cap Y_2 \) is clean, then
\[
\text{Hom} ( Y_1, Y_2 ) = D^b ( Y_1 \cap Y_2 ; \lambda )
\]

A composition of morphisms

\[
\begin{array}{ccc}
Y_1 \cap Y_2 \cap Y_3 & \xrightarrow{\iota_{12}} & Y_1 \cap Y_2 \\
\downarrow \iota_{23} & & \downarrow \iota_{13} \\
Y_2 \cap Y_3 & \xrightarrow{\iota_{13}} & Y_1 \cap Y_3
\end{array}
\]

\[
\mathcal{E}_{23} \circ \mathcal{E}_{12} = (\iota_{13})_* \left( \iota_{12}^* \mathcal{E}_{12} \otimes \iota_{23}^* \mathcal{E}_{23} \right)
\]

More complicated objects
A holomorphic fibration

Y
\downarrow \text{a real bundle, holomorphic projection}
Y \subset X
\uparrow \text{lagrangian}

\[
\text{Hom} ( Y_1, Y_2 ) = D^b ( Y_1 \times_X Y_2 ; \lambda )
\]

TQFT motivation

A 3d B-model (L.R., E. Witten)

based on maps \( M^3 \rightarrow X \)

\( \mathcal{L} (X) \) is the 2-category of its boundary \( \partial \)s (similar to \( D^b (X) \) being the category of boundary \( \partial \)s of a 2d B-model)
3d B-model implied an existence of non-standard monoidal structure on $D^b(X,\omega)$ (J. Roberts):

$$(E_1 \otimes E_2) \otimes E_3 \xrightarrow{\text{Drinfeld associator}} E_1 \otimes (E_2 \otimes E_3)$$

Reason: $D^b(X,\omega)$ is the Drinfeld center of $\mathcal{L}(X,\omega)$.

Algebraic model for $\mathcal{L}(T^*\mathbb{C}^n)$

$X = x_1, \ldots, x_n$ - coordinates on $\mathbb{C}^n$

Simplest lagrangian submanifolds can be described by generating functions:

for $W \in \mathbb{C}[x]$ define $Y_w = \{ (x, p) \mid p = dw \} \subset T^*\mathbb{C}^n$

$$\text{Hom}(Y_1, Y_2) = MF(x; W_2 - W_1)$$

$MF(x; W)$ is a category of matrix factorizations of $W(x)$

A crash course in matrix factorizations

Two approaches to the definition of $MF(x, W)$

1. A singular part of $D^b(\mathbb{C}[x]/(W))$
    take a quotient over all finite-length resolutions

2. A deformation of $D^b(\mathbb{C}[x])$
    Hochschild cohomology $HH^*(\mathbb{C}[x]) = \mathbb{C}[x, \Theta]$ with

    $\lambda \in \mathbb{C}[x, \Theta]$ is a deformation parameter if $[\lambda, \lambda] = 0$
\[ W(x) \in C[x, \theta] \]

\[ \text{MF}(x; W) \text{ as a deformation of } D^b(C[x]) \]

**Objects**

\[ D^b(C[x]): M \cong D, \deg D = 1, D^2 = 0 \]

\[ \mathbb{Z}_2\text{-graded free } C[x]\text{-module} \]

\[ \text{MF}(x, W): M \cong D, \deg x = 1, D^2 = W \uparrow M \]

\[ \mathbb{Z}_2\text{-graded free } C[x]\text{-module} \]

\[ M \cong D = \left( \begin{array}{c}
M_0 \\ M_1
\end{array} \right), \quad F, G \in \text{Mat}_{k \times k}(C[x]) \]

\[ F \circ G = W \uparrow M_1, \quad G \circ F = W \uparrow M_0 \]

**Morphisms:**

\[ \text{Hom}(M, M') \subseteq C[x] \]

\[ \alpha^2 = [D, [D, \cdot]] = [D^2, \cdot] = 0 \]

\[ \text{Hom}(M, M') = H_d \]

**Localization to**

\[ \text{Crit}(W) = \{ \partial W = 0 \} \]

\[ D^2 = W \uparrow M \quad \Rightarrow \quad \forall \omega \cdot \uparrow M = [D, \partial D] \]

\[ C[x] \rightarrow C[x]/(\partial W) \rightarrow \text{End}_{\text{MF}}(M) \]

\[ \text{Jacobi algebra} \]

\[ \mathbb{Y}_1 = \{ p = \partial W \} \]
\[ \gamma_2 = \{ p = \partial \omega_2 \} \]

\[ \text{Crit} (W_2 - W_1) \]

**Examples**

1. \( W = x^2 \)

A single indecomposable object

\[ M = \left( \mathcal{C}_0[x] \xrightarrow{x} \mathcal{C}_1[x] \right) \]

\[ M[1] \cong M \]

\[ \text{MF} (x; x^2) \cong \text{Vect-} \mathcal{C} \]

not exactly \( D^b (\text{1-pt}) \cong \mathbb{Z}\text{-graded} \text{ Vect-} \mathcal{C} \)

2. \( W = xy \)

Two indecomposable objects

\[ M = \left( \mathcal{C}_0[x,y] \xrightarrow{x} \mathcal{C}_1[x,y] \xrightarrow{y} \right) \]

\[ M[1] = \left( \mathcal{C}_0[x,y] \xrightarrow{y} \mathcal{C}_1[x,y] \xrightarrow{x} \right) \]

\[ \text{MF} (x,y; xy) = \mathbb{Z}_2\text{-graded} \text{ Vect-} \mathcal{C} \]

Knörrer periodicity
3. \( W(x) \) such that \( \text{Crit}(W) \) is smooth

and the matrix of second derivatives \( \partial_i \partial_j W \)
determines a non-degenerate pairing on \( N \text{Crit}(W) \)

Then \( \text{MF}(x; W) \cong D^b(\text{Crit}(W)) \)

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  ↑                ↑
  "almost"         \( \mathbb{Z}/2 \)-graded
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Also the intersection \( \{ p = 0 \} \cap \{ p = \partial W \} \cong \text{Crit}(W) \)
is clean

Tensor product \( \bigotimes \mathbb{C}[x] \) as a composition
of morphism categories

Tensor product of matrix factorizations

\[
M_1 \in \text{MF}(x, a; W_1(x, a))
\]

\[
M_2 \in \text{MF}(x, b; W_2(x, b))
\]

\[
M_1 \otimes \mathbb{C}[x] \otimes M_2 \cong D = D_1 \otimes M_2 + (-1)^F \otimes D_2
\]

\[
D^2 = D_1^2 + D_2^2 + \{ D_1, D_2 \} = (W_1 + W_2) \bigotimes_{M_1 \otimes M_2} 0
\]

\[
\text{MF}(W_1) \otimes \text{MF}(W_2) \longrightarrow \text{MF}(W_1 + W_2)
\]

Composition of morphism categories

\[
\text{Hom}(W_1(x), W_2(x)) \times \text{Hom}(W_2(x), W_3(x)) \longrightarrow \text{Hom}(W_1(x), W_3(x))
\]
\[ \text{Hom} \left( W_1 (\underline{x}), W_2 (\underline{x}) \right) \times \text{Hom} \left( W_2 (\underline{x}), W_3 (\underline{x}) \right) \longrightarrow \text{Hom} \left( W_1 (\underline{x}), W_3 (\underline{x}) \right) \]

\[ \text{MF} (\underline{x} ; W_2 - W_1) \times \text{MF} (\underline{x} ; W_3 - W_2) \otimes \mathbb{C} [\underline{x}] \longrightarrow \text{MF} (\underline{x} ; W_3 - W_1) \]

because \( (W_2 - W_1) + (W_3 - W_2) = W_3 - W_1 \)

**General objects and their morphisms**

A general object of \( \text{MF} (\underline{x}) \) is a pair \( (\underline{a} ; W (\underline{x} ; \underline{a})) \)

\[ \xi \in \mathbb{C} [\underline{x}, \underline{a}] \]

\[ \uparrow \text{extra variables} \]

It describes a (generalized) lagrangian submanifold or a fibration with a lagrangian base:

\[ Y_W = \{ (\underline{x}, p, \underline{a}) \mid p = \partial \underline{x} W, \partial \underline{a} W = 0 \} \]

**Example 1** \( W = \alpha x \) describes

\[ \begin{array}{c}
\text{p} \\
\text{x=0} \\
\text{x}
\end{array} \]

**Example 2** \( W = \alpha x^2 \) describes \( \{ p = 2\alpha x, x^2 = 0 \} \)

Supported at \( x=0, p=0 \)

\[ \text{Hom} \left( (\underline{a}; W_1 (\underline{x} ; \underline{a})), (\underline{b}; W_2 (\underline{x} ; \underline{b})) \right) = \text{MF} (\underline{a} ; \underline{b} ; \underline{x}; W_2 - W_1) \]

**Composition of morphisms**

\[ \text{Hom} \left( (\underline{a}; W_1), (\underline{b}; W_2) \right) \times \text{Hom} \left( (\underline{b}; W_2), (\underline{c}; W_3) \right) \longrightarrow \text{Hom} \left( (\underline{a}; W_1), (\underline{c}; W_3) \right) \]

\[ \text{MF} (\underline{x} ; \underline{a} ; \underline{b} ; W_2 - W_1) \times \text{MF} (\underline{x} ; \underline{b} ; \underline{c} ; W_3 - W_2) \otimes \mathbb{C} [\underline{x}] \longrightarrow \text{MF} (\underline{x} ; \underline{a} ; \underline{c} ; W_3 - W_1) \]
Lagrangian correspondences

A lagrangian correspondence in an object \((a; W) \in \mathcal{MF}(x, y)\) where \(x_i, \ldots, x_n \sim y_i, \ldots y_m\)

It determines a functor

\[
\begin{align*}
\mathcal{MF}(x) \xrightarrow{(a; W)} & \mathcal{MF}(y) \\
(b; W_1(x, b)) & \mapsto (a, b, x; W_1 + W)
\end{align*}
\]

Action of a lagrangian correspondence on a category of morphisms

\[
\begin{align*}
\mathcal{MF}(x) \xrightarrow{W(x, y)} & \mathcal{MF}(y) \\
W_1(x) & \mapsto W_1(a) + W_1(a, y) \\
W_2(x) & \mapsto W_2(b) + W(b, y)
\end{align*}
\]

We need a functor

\[
\mathcal{MF}(x; W_2(x) - W_1(x)) \rightarrow \mathcal{MF}(a, b, y; W(b, y) - W(a, y) + W_2(b) - W_1(a))
\]

Describe it as a "bimodule": a matrix factorization of

\[
\mathcal{MF}(x, y, a, b; (W(b, y) - W(a, y)) + (W_2(b) - W_2(x)) - (W_1(a) - W_1(x)))
\]

\[
W^{tot}(a, b, x, y)
\]

Since \(W^{tot} \in (a-x, b-x)\), there is a canonical choice

\[
\text{a regular sequence in } \mathbb{C}[a, b, x, y]
\]
Koszul matrix factorizations

Koszul complex: for $p \in \mathbb{C}[x]$, $K(p) = (\mathbb{C}[x] \xrightarrow{p} \mathbb{C}[x])$

for $p = p_1, \ldots, p_k \in \mathbb{C}[x]$, $K(p) = \bigotimes_{i=1}^{k} K(p_i)$

Koszul matrix factorization:

for $p, q \in \mathbb{C}[x]$, $K(p, q) = (\mathbb{C}[x] \xrightarrow{p} \mathbb{C}[x]) \in MF(x; p q)$

for $p, q \in \mathbb{C}[x]$, $K(p; q) = \bigotimes_{i=1}^{k} K(p_i, q_i) \in MF(x; p q)$

If $W \in K(p)$, then $\exists q$ s.t. $W = p q$

Theorem: If $p$ is a regular sequence, then $K(p, q) \in MF(x; W)$ does not depend on the choice of $q$.

Fourier-Legendre transform

$MF(x) \xrightarrow{x : y} MF(y)$

This is an equivalence of categories, the inverse transform is $MF(y) \xrightarrow{x : y} MF(x)$

The composition is

$MF(x) \xrightarrow{a \cdot (x - y)} MF(y)$

$W(x) \mapsto a \cdot (b - x) + W(b) \approx W(x)$
\[ W(x) \mapsto a \cdot (b - x) + W(b) \equiv W(x) \]
\[ = (a + \frac{W(b) - W(x)}{b - x}) (b - x) + W(x) \]
\[ = \tilde{a} \tilde{b} + W(x) \equiv W(x) \]
\[ \text{by \ Krörrer\ periodicity} \]

**Drinfeld center of \( \tilde{\text{MF}}(x) \)**

**Endomorphism category of the identity functor**

\[ \text{MF}(x) \xrightarrow{a \cdot (y - x)} \text{MF}(y) \]
\[ \text{renamed } x \]
\[ (a; a \cdot (y - x)) \in \tilde{\text{MF}}(x, y) \]
\[ \text{End } (a \cdot (y - x)) = \text{MF}(x, y, a, b; (b - a)(x - y)) \]
\[ \text{renamed } a \]

\[ \text{Krörrer periodicity} \]
\[ \cong D^b(C[x, a]) \]
\[ \text{renamed } p \]

**Monoidal structure of the Drinfeld center coincides with tensor product**

\[ \tilde{\text{MF}}(x) \text{ and derived algebraic geometry} \]

\[ A\text{-abelian algebra (e.g. } C[x] \text{) } \mapsto 2\text{-category } \tilde{A} \]
\[ \text{an object } B \text{ - an algebra over } A \text{ (e.g. } C[x], C(x)/(p) \text{)} \]
\( \text{Hom}_{\tilde{A}} (B_1, B_2) \) - bimodules, that is \( D^b (B_1 \otimes_A B_2) \)

Composition - usual composition of bimodules, that is, tensor product over the intermediate algebra

Conjecture \( \mathcal{C}[\tilde{x}] \cong \tilde{\text{MF}}(\tilde{x}) \)

\( \mathcal{C}[\tilde{x}] / (\rho) \leftrightarrow a \cdot \rho \)

Example 1 \( (a, ax) \)

\( \text{End}_{\tilde{A}} B \) : resolution of \( B \) : \( \mathcal{C}[x, \theta] \sim x \partial_{\theta} \)

\( \text{End}_{\tilde{A}} B = D^b (\mathcal{C}[\theta]) \)

\( \text{End}_{\tilde{MF}} (ax) = \tilde{\text{MF}}(a, b, x, x \cdot (b-a)) \)

\( = \tilde{\text{MF}}(a; 0) = D^b (\mathcal{C}[\theta]) = D^b (\mathcal{C}[\theta]) \)

Example 2 \( (a, a \rho(x)) \)

\( \text{End}_{\tilde{A}} B \) : resolution of \( B \) : \( \mathcal{C}[x, \theta] \sim \rho \partial_{\theta} \)

\( \text{End}_{\tilde{MF}} (a \rho) = \tilde{\text{MF}}(a, b, x, x \cdot (a-b) \rho) \)
\[ = \mathbb{C} \left[ b \right] \otimes MF (\tilde{a}, x ; \tilde{\alpha}_p ) \]
\[ \mathbb{C} \left[ \theta \right] \quad \mathbb{C} [x] / (\rho) \]

**Part 2**

**Reminder:** 2-category \( MF (x) \)
\[ \uparrow x, \ldots, x_n \]

**An object** \((a; W(\bar{x}, \bar{a}))\)
\[ \uparrow \text{extra variables} \quad \mathbb{C} [\bar{x}, \bar{a}] \]

**A category of morphisms:**

\[ \text{Hom} \left( (a_1; W_1), (b_2, W_2) \right) = MF (a, b, \bar{x} ; W_2 - W_1) \]
\[ \uparrow \text{always distinct} \]

**Composition of morphisms:** \( \otimes \mathbb{C} [\bar{x}, b] \)
\[ \uparrow \text{intermediate extra variables} \]

**Equivalence with derived algebraic geometry**

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"Algebraization" of an additive category \( \mathbb{C} \)
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A magic object \( A \rightarrow \text{algebra} \quad A = \text{End}_\mathbb{C} (A) \)

Any object \( B \rightarrow \text{an } A\text{-module} \quad \text{Hom}_A (A, B) \)

Sometimes \( \mathbb{C} \rightarrow D^b (A\text{-mod}) \) is an isom. of categories
A 2-category version of this construction

A magic object of $\hat{MF} : (W \cong 0)$

Monoidal category of endomorphisms: $\text{End}_{\hat{MF}} (O) = MF(\tilde{a}, 0) = D^b (C[\tilde{a}])$

An object of $\hat{MF}$: $\text{A "module" category over } D^b (C[\tilde{a}])$

$(a; W(\tilde{a}, a)) \implies \text{Hom}_{MF} (O, (a, W)) = MF (a, \tilde{a}; W)$

Conj: $MF (a, \tilde{a}; a \rho (x)) \cong D^b (C[\tilde{a}] / (\rho))$

not only as categories but also as "module"-categories over $D^b (C[\tilde{a}])$

Comment: $D^b (a, \tilde{a}) \cong D^b (\theta, \tilde{a})$

$\uparrow$ Koszul duality

Apply deformation: $W = a \rho (x) \iff \theta = \rho (x) \theta$

$(X, \omega)$ - holomorphic symplectic manifold

$\tilde{L} (X, \omega)$ is hard to define, but it is local.

Locality means that all constructions involving an object $Y \subset X$

are determined by tubular neighborhood $\text{Tub} (Y) \subset X$

Generally, $\text{Tub} (Y) \neq \text{Tub} (Y) \triangleleft$ zero-section

Let there should be a deformation $(T^* Y)_{\tilde{a}}$ of

the holomorphic symplectic manifold $T^* Y$ such that
\[ T \cup (Y) \cong T \cup (X) \cap (T^* Y)_x \]

Hence if we understand \( \bar{L}(T^* Y; x) \), then we know the part of \( \bar{L}(X) \) which involves \( Y \)

**\( \bar{L}(T^* U) \) as \( D_{\mathbb{Z}_2}(U) \)**

\( U \) - complex manifold

\[ \Omega^*(U) = \Omega^\hat{\circ} + \Omega^\wedge \] - Dolbeault \((0, \cdot)\) forms on \( U \)

with \( \mathbb{Z}_2 \)-grading \( (\mathbb{Z}_2 = \{\hat{\circ}, \wedge\}) \)

Let \( W \in \Omega^\hat{\circ}(U) \), \( \bar{\partial} W = 0 \)

\( E \)

\( U \)

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**Def** A matrix factorization of \( W \) in a \( \mathbb{Z}_2 \)-graded vector bundle with the differential

\[ \nabla : \Omega^*(E) \rightarrow \Omega^*(E) \]

such that

1. \( \text{deg}_{\mathbb{Z}_2} \nabla = \wedge \)

2. \( \nabla (\alpha \circ) = (\bar{\partial} \alpha \circ) \circ + (-1)^{|\alpha|} \alpha \circ \bar{\partial} \circ \)
for any \( \alpha \in \mathcal{A}^*(U) \), \( \sigma \in \mathcal{A}^*(E) \)
that \( \eta \), locally \( \overline{\nabla} = \overline{\partial} + A \), \( A \in \mathcal{A}^*(\text{End} E) \)
\( \uparrow \text{add} \)

\[(3) \quad \overline{\nabla}^2 = W \bigotimes E\]

**Remark** Even if \( W = 0 \), then \( D^2_z(U;0) \) is a bit bigger
than \( D^b(U) \):

1. \( \mathbb{Z}_2 \)-grading instead of \( \mathbb{Z} \)-grading
2. allow \( A \) to contain Dolbeault degree more than 1

A 2-category \( \mathcal{D}^2_z(U) \quad ( = \mathcal{L}(T*U) ) \)

**Simplest object:** \( W \in \mathcal{L}^0(U) \). (adding \( \partial \) creates an isomorphic object)

**Morphisms:** \( \text{Hom}_{\mathcal{D}^2_z(U)}(W_1, W_2) = \text{MF}(U; W_2 - W_1) \)

**Deformation**

Deformation of the holomorphic symplectic structure of \((X, \omega)\)
(without changing \([\omega] \in H_{\text{DR}}(X)\))

A complex structure of \(X\) is deformed by Beltrami differential
\( \mu \in \mathcal{A}^1(TX) \), \( \overline{\partial}_Z \mu + \frac{i}{2} [\mu, \mu] = 0 \)

\[\text{Lie Bracket}\]

\[\text{Cartan-Maurer eqn}\]

so that \( \overline{\partial} \rightarrow \overline{\partial}' = \overline{\partial} + \mu \wedge \partial \)
A holomorphic symplectic structure is deformed by a Hamiltonian Berezin differential
\[ \overline{\partial} (\omega \cdot \mu) = 0 \]
or for simplicity by Hamilton differential
\[ \exists \in \mathcal{S}^1 (X), \quad \overline{\partial} \exists + \frac{i}{2} [\exists, \exists] = 0 \]
\[ \text{Poisson bracket} \]
\[ \text{Maurer-Cartan eqn} \]
so that \( \mu = \omega^{-1} (\exists \exists) \), \( \omega \mapsto \omega' = \omega + \frac{d}{\exists} \exists + \overline{\partial} \exists \).

**Deformation of** \( T^*U \)

\[ \mathcal{S}^1 (T^*U) \rightarrow \mathcal{S}^1 (S \cdot T^*U) \\exists \exists \]
\[ \text{total space} \quad \text{vector bundle} \]
\[ \overline{\partial} \exists + \frac{i}{2} [\exists, \exists] = 0 \]
\[ \text{Schouten bracket} \]

**Components of** \( \exists \exists \)

\( \exists \exists \) is a \((0,1)\)-form taking values in polynomials of fibers of \( T^*U \)
\[ \exists = \exists_0 + \exists_1 + \exists_2 + \exists_3 + \ldots, \quad \exists_i \in \mathcal{S}^i (S \cdot T^*U) \]
irrelevant deforms complex structure of \( U \)
\[ \beta \in H_{\overline{\partial}}^1 (S^2 T^*U) \rightarrow \text{Ext}_1^1 (\mathbb{N}U_0, T^*U) \]
zero section in \( T^*U \)
normal bundle to zero section
in the total space $T^*U$

$\beta \neq 0$ means that $TU_0 \to T(T^*U)\big|_{U_0} \to NU_0$ does not split

Consider the diagonal $\Delta_x \subseteq X \times X$

For $\Delta_x \subseteq X \times X$, $\beta = 0$, $\gamma = R \in H_2^i(S^3TX) \to \text{Ext}^i(S^2TX, TX)$

Atiyah class of $TX$
represented by Riemann curvature

Describe a lagrangian (with respect to $\omega' = \omega + d\alpha$)
submanifold $Y \subset T^*U$ as a graph of $\bar{\omega}$:

$Y = \{ (x, p) \mid p = \bar{\omega} \}$

This time $\bar{\omega} = \bar{\alpha}(\bar{\omega})$

$\bar{\omega}$ polynomial function on fibers of $T^*U$

Explanation: we want $\alpha = p \cdot dx + \bar{\omega}$ to be exact on $Y$:

$\alpha |_{p = \bar{\omega}} = \bar{\omega} \cdot dx + \bar{\omega}(\bar{\omega}) = \bar{\omega} + \bar{\omega} = p \cdot dx$

Category of morphisms $\text{Hom}_{MF(U)}(W_1, W_2)$

The old choice $\text{Hom}(W_1, W_2) = MF(U; W_2 - W_1)$ does not work
not holomorphic

$\bar{\omega}(W_2 - W_1) = \bar{\alpha}(\bar{\omega}_2) - \bar{\alpha}(\bar{\omega}_1) \neq 0$

$A_{\infty}$-deformations of $D_{\mathbb{Z}}(U)$
\[ \mathcal{N}^* (\wedge^* TU), \partial \bar{\partial}, \left[ -,- \right] \]

*Schouten-Nijenhuis bracket*

*Maurer-Cartan element* \( \lambda \in \mathcal{N}^* (\wedge^* TU), \deg \partial \lambda = 0 \)

\[ \partial \lambda + \frac{1}{2} \left[ \lambda, \lambda \right] = 0 \]

**Conjecture** There exists a unique “universal” MC element

\[ \lambda = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots \]

1. \( \lambda_i \in \mathcal{N}^i (\wedge^i TU) \) (relatively balanced)
2. \( \lambda_0 = W_2 - W_1 \)

Substitute \( \lambda = W_2 - W_1 + \lambda_1 + \ldots \) into MC equation:

\[ \left[ \lambda_n, W_2 - W_1 \right] = -\partial \lambda_{n-1} - \frac{1}{2} \sum_{i=1}^{n-1} \left[ \lambda_i, \lambda_{n-i} \right] \]

depends on \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \)

Find \( \lambda \) perturbatively:

*compute the r.h.s. and recognize it as* \( \left[ \lambda_n, W_2 - W_1 \right] \)

\[ \wedge^n (\wedge^n TU) \]

**First step:** \( \left[ \lambda_1, W_2 - W_1 \right] = -\partial (W_2 - W_1) \)

\[ = \partial (\partial W_1) - \partial (\partial W_2) \]
where divided difference \( p(x) - p(y) = p'(x, y)(x-y) \)

The first term which does not depend on \( \partial W_1 \) and \( \partial W_2 \)
appears in \( \lambda_3 : \)

\[
\frac{1}{3} \beta \beta \beta \mathbb{R} \text{ or explicitly } \frac{1}{3} \beta \beta \beta \mathcal{R}^M_{LM} \partial_I \wedge \partial_J \wedge \partial_K
\]

Its presence implies a deformation of even \( \text{End}_{\mathbb{D}}^{1} (0) \)

Claim If \( \beta = 0 \) and \( W_1 = W_2 = 0 \), then \( \lambda = 0 \)

that is, \( \text{End}_{\mathbb{D}}^{1} (0) \) is not deformed

( but its monoidal structure may still be deformed! )

Deformation of composition of morphisms

For simplicity, work perturbatively over \( \varepsilon \); ignore all terms of quadratic and higher order

Then MC equation is simply \( \partial \varepsilon = 0 \)

and, most importantly, the deformation parameter for

\[
\text{Hom}_{\mathbb{D}}^{1} (W_1, W_2) = \mathbb{D}_{\mathbb{Z}}^{1} (U; \lambda)
\]

has only two terms:

\[
\lambda = \lambda_0 + \lambda_1
\]

\[
\mu_{12} = - \varepsilon' (\partial W_1, \partial W_2)
\]
hence deformations are limited to “curving” and infinitesimal change of complex structure.

A morphism \( \tilde{E}_{12} \in \text{Hom} (\mathcal{W}_1, \mathcal{W}_2) \) can be described as a \( \mathbb{Z}_2 \)-graded vector bundle \( \tilde{E} \rightarrow \mathcal{U} \) with deformed \( \tilde{\nabla} \) denoted as \( \tilde{\nabla} \)

\[
\tilde{\nabla} = \nabla + \mu \cdot \nabla
\]

such that

\[
\tilde{\nabla}^e = \nabla^e + [\nabla, \mu \cdot \nabla] = (\mathcal{W}_2 - \mathcal{W}_1) \mathcal{I}_E
\]

such that \( \mu \cdot \nabla \) is Atiyah “class”

Note: \( F = [\nabla, \nabla] \), hence \( \nabla F = [\nabla^2, \nabla] = \mathcal{I}(\mathcal{W}_2-\mathcal{W}_1) \)

Composition of morphisms as a deformed tensor product

\[
\tilde{\nabla}_{12} \circ \tilde{\nabla}_{23} = (E_{12} \otimes E_{23} : \tilde{\nabla}_{12} + \tilde{\nabla}_{23} + \text{deformation})
\]

\[
\tilde{\nabla}_{12} + \tilde{\nabla}_{23} = \nabla_{12} + \nabla_{23} + \mu_{13} \cdot \left( \nabla_{12} + \nabla_{23} \right) + (\mu_{12} - \mu_{13}) \cdot \nabla_{12} + (\mu_{23} - \mu_{13}) \cdot \nabla_{23}
\]

\[
\text{deformation} = -\delta + a \cdot \eta, \; \eta \in \mathcal{I}(\text{End} \mathcal{E})
\]

Translation on \( \nabla \): \( \left[ \nabla^e, \mu \cdot \nabla \right] \)
Condition on $\alpha$:

$$\left[ \nabla_{\mu} , -\delta + \alpha \right] = 0$$

$$\left[ \nabla_{\mu} , \delta \right] = (\mu_{12} - \mu_{13}) l F_{12} + (\mu_{23} - \mu_{13}) l F_{23}$$

$$\mu_{12} - \mu_{13} = \alpha(\mathcal{E}, \mathcal{E}) - \alpha(\mathcal{W}, \mathcal{W})$$

$$= \alpha''(\mathcal{W}, \mathcal{W}, \mathcal{W}) + \mathcal{E}(\mathcal{W}, \mathcal{W}, \mathcal{W})$$

By the symmetry of the second divided difference

$$\mu_{23} - \mu_{13} = \alpha''(\mathcal{W}, \mathcal{W}, \mathcal{W}) + \mathcal{E}(\mathcal{W}, \mathcal{W}, \mathcal{W})$$

$$b = \alpha''(\mathcal{W}, \mathcal{W}, \mathcal{W}) + \mathcal{E}(\mathcal{W}, \mathcal{W}, \mathcal{W})$$

$$= \beta l F_{12} F_{23} + O(W)$$

in non-trivial at $W_1 = W_2 = W_3 = 0$.

Deformation of the monoidal structure of $\text{End}_D(0)$

$$\mathcal{E}_1 \circ \mathcal{E}_2 = (\mathcal{E}_1 \otimes \mathcal{E}_2 ; \nabla_1 + \nabla_2 + \beta l F, F)$$

non-commutativity of monoidal structure

Associators in $\text{End}_D(0)$

Suppose that $R = 0$ on $U$

\[ \text{Atiyah class of } TU \]

Then $\text{End}_D(0) = D^2(\mathcal{U})$ is undeformed even for a general $\alpha \in \mathcal{L}(S^1 TU)$
However monoidal structure is deformed

\[(E_1; \overline{\nabla}_1) \circ (E_2; \overline{\nabla}_2) = (E_1 \otimes E_2; \overline{\nabla}_1 + \overline{\nabla}_2 + \alpha_{12})\]

and associativity requires associator:

\[E_1 \circ (E_2 \circ E_3) = (E_{123}; \overline{\nabla}_{123} + \alpha_{123})\]

\[\overline{\nabla}_{123} \quad \alpha_{123} \quad b_{123} \quad \text{associator (gauge transformation)}\]

\[E_1 \circ (E_2 \circ E_3) = (E_{123}; \overline{\nabla}_{123} + \alpha'_{123})\]

\[b_{123} \quad \alpha'_{123} \quad b_{123} \quad \text{subject to} \quad \alpha_{123} = 0\]

Solve perturbatively over Dolbeault degree
starting with \( a = \beta \cdot (F_1 F_2) \)

The first term in associator is \( \frac{2}{3} \gamma^3 F_1 F_2 F_3 \)
non-zero even if \( \beta = 0 \)

**Categorified Riemann–Roch–Hirzebruch**

General idea

\[\mathcal{C} \times \mathcal{C} \xrightarrow{\text{ch}} \mathbb{Z}(\mathcal{C}) \times \mathbb{Z}(\mathcal{C})\]

\[\mathcal{C}' \xrightarrow{\text{ch}} \mathbb{Z}(\mathcal{C}')\]

\[\mathcal{D}^b(U) \times \mathcal{D}^b(U) \xrightarrow{\text{ch} \times \text{ch}} \mathbb{H}(\mathcal{C}) \times \mathbb{H}(\mathcal{C})\]
$\Ext \downarrow \quad \dim \quad \downarrow \cap$

$\mathcal{C} - \text{Vect} \quad \rightarrow \quad \mathcal{C}$

$Y_1, Y_2 \subset X \quad \rightarrow \quad \mathcal{O}_{Y_1}, \mathcal{O}_{Y_2} \in D^b(X)$

$\downarrow$

$D^b(Y_1 \cap Y_2, \mathcal{X}) \quad \xrightarrow{\HH^*, \HH^*} \quad \mathcal{C} - \text{Vect}$

$\Ext, \text{Tor}$

$\Ext_X (\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) \cong HH_* (\text{Hom}_{\mathcal{L}} (Y_1, Y_2))$

$\Tor_X (\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) \cong HH^* (\text{Hom}_{\mathcal{L}} (Y_1, Y_2))$