$K_2^M(A)$ Abelian grp

(A commutative)

Generators: $\{u,v\}$ $u, v \in A^\times$

Relations: $\{u_1 u_2, v\} = \{u_1, v\}\{u_2, v\}$

$\{u, v\}\{v, u\} = e$

$\{u, 1-u\} = 0$ when $u, 1-u \in A^\times$

Morphism $K_2^M(A) \to K_2(A)$

Recall: $St(A) =$ univ central ext of A

Gen.: $x_{ij}(a)$ $i \neq j$ $a \in A$

Rels: $x_{ij}(a) x_{ij}(b) = x_{ij}(a+b)$

$[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab)$

$(i \neq k)$

$[x_{ij}(a), x_{kl}(b)] = e$ $j \neq k$ & $i \neq l$

$$e \to \overset{\text{central}}{K_2(A)} \to St(A) \to E(A) \to e$$

$$x_{ij}^a \longmapsto E_{ij}^a$$

For $u \in A^\times$:
$$w_{ij}(u) = x_{ij}(u)\, x_{ji}(-u^{-1})\, x_{ij}(u)$$
$$h_{ij}(u) = w_{ij}(u)\, w_{ij}(-1)$$

$$w_{ij}(u) \mapsto \begin{bmatrix} 0 & u \\ -u^{-1} & 0 \end{bmatrix} \qquad h_{ij} \mapsto \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$$

One checks that: Conjugation by $w_{ij}(u)$ acts same as in $E(A)$ on x_{kl}^a, $w_{kl}(v)$

e.g. $$w_{12}(u)\, w_{12}(v)\, w_{12}(u)^{-1} = w_{12}(+u^2 v^{-1})$$
$$w_{12}(u)\, x_{13}^a\, w_{12}(u)^{-1} = x_{23}(-u^{-1}a)$$

etc. From this:
$$w_{12}(u)\, h_{12}(v)\, w_{12}(u)^{-1} = h_{12}(u^2 v^{-1})\, h_{12}(u^2)^{-1}$$

From that: Conjugation by $h_{ij}(u)$ acts as expected on x_{kl}^a, $w_{kl}(v)$. AND THEN:

$$h_{ij}(u)\, h_{ik}(v)\, h_{ij}(u)^{-1} = h_{ik}(uv)\, h_{ik}(u)^{-1}$$

$$\|$$

$$\mathrm{Ad}_{h_{ij}(u)} \cdot \left(w_{ik}(v)\, w_{ik}(-1) \right)$$

$$\|$$

$$w_{ik}(uv)\, w_{ik}(-u) = h_{ik}(uv)\, h_{ik}(u)^{-1}$$

$$\{u,v\} = [h_{ij}(u), h_{ik}(v)] = h_{ik}(uv)\, h_{ik}(u)^{-1}\, h_{ik}(v)^{-1}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$\{u,v\} = \{v,u\}^{-1}$ $\qquad\qquad$ $\{u, 1-u\} = 1$

$\{u_1 u_2, v\} = \{u_1, v\}\{u_2, v\}$

Get $K_2^M(A) \longrightarrow K_2(A)$

Also can get relations:

$$\begin{cases} h_{ij}(u)^{-1}\, h_{jk}(u)^{-1}\, h_{ki}(u)^{-1} = 1 \\ h_{ij}(u)\, h_{ji}(u) = 1 \end{cases}$$

Matsumoto Theorem For a field F,

$$K_2^M(F) \xrightarrow{\sim} K_2(F)$$

LINEAR ALGEBRA I, REVISITED: 1

Subgroups of $St(F)$:

$T = \{\prod_{i<j} x_{ij}(a_{ij})\}$ (upper triang)

W generated by $w_{ij}(u_{ij})$ (monomial)

$\cup$

H generated by $h_{ij}(u_{ij})$ (diagonal)

$$St(F) = TWT$$

Enough to prove: $TWTw_{i,i+1}(-1) \subset TWT$

$twt'w_{i,i+1}(-1) \in TWT$; $t' = x_{i,i+1}^{a} \cdot \underbrace{\prod_{j,k \neq i,i+1} x_{jk}^{a_{jk}}}_{= t''}$

$t''w_{12}(-1)$

$\shortparallel$

$w_{12}(-1)t'''$ $t''' \in T$

Two cases, depending on whether w preserves the order of $i, i+1$.

Reduce to: $x_{21}^{a} \in TWT$.

<u>But</u>: $x_{21}^{a} = x_{12}^{a^{-1}} w_{12}(-a^{-1}) x_{12}^{a^{-1}}$

Assume twt' $\phi \downarrow$ $e \in E(F)$ under $St(F)$

$\phi(t)\phi(w)\phi(t') = e$

$\phi(w) = e$

$\phi(t) = \phi(t')^{-1}$

$\Downarrow$

$t = t'^{-1}$ in $St(F)$

(no kernel of $\phi|T$)

<u>Next</u>: $\phi(w) = e \implies w$ generated by symbols $\{a, b\}$. But w = product of $w_{ij}(1)$'s and $h_{ij}(u)$'s. Modulo H, reduce it

to a product of $h_{ij}(u$'s). Using relations

$$\begin{cases} h_{ij}(u)\,h_{ji}(u) = 1 \\ h_{ij}(u)^{-1}\,h_{jk}(u)^{-1}\,h_{ki}(u)^{-1} = 1 \end{cases}$$

represent this as a product of $h_{1i}(u_i)$.

(Modulo $K_2^M(F)$, those commute).

Finally: $\prod_{i=2}^{n} h_{1i}(u_i) \overset{\Phi}{\longmapsto} \begin{bmatrix} u_2 \cdots u_n & & & 0 \\ & u_2^{-1} & & \\ & & \ddots & \\ 0 & & & u_n^{-1} \end{bmatrix}$

So all $u_i = 1$. This shows:

$$K_2^M(F) \twoheadrightarrow K_2(F)$$

<u>Rmk</u> We used: the monoidal part of a matrix. In $SL_n(F)$: if $g = twt'$, $t, t' \in T$, $w \in W$: then w is defined uniquely

<u>Next</u>: given $c: K_2^M(F) \to A$, construct a central extension of $SL_n(F)$ by A.

LINEAR ALGEBRA I, REVISITED: 2

$SL_2(F)$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & u \\ -u^{-1} & 0 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

(case $a_{21} \neq 0$)

$$= \begin{bmatrix} -au^{-1} & u \\ -u^{-1} & 0 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

(case $a_{21} = 0$)

$$= \begin{bmatrix} -au^{-1} & -au^{-1}b + u \\ -u^{-1} & -u^{-1}b \end{bmatrix}$$

$a_{21} = -u^{-1}$;

...

Both cases: unique

Matrix multiplication in these terms?

$A \cdot B$ if A or B are of type 2 ($a_{21} = 0$): clear.

If both are of type 1: below.

First, Notation:

$$x_{12}^a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad x_{21}^a = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \qquad h_{12}(u) = \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$$

$$w_{12}(u) = \begin{bmatrix} 0 & u \\ -u^{-1} & 0 \end{bmatrix}$$

Note:

$$h_{12}(u) = w_{12}(u)\, w_{12}(-1)$$

$$w_{12}(u) = x_{12}(u)\, x_{21}(-u^{-1})\, x_{12}(u)$$

Relations:

$$x_{12}^a x_{12}^b = x_{12}^{a+b} \qquad w_{12}(u)\, x_{12}^a\, w_{12}(u)^{-1} = x_{21}(-u^{-2}a)$$

$$w_{12}(u)\, w_{12}(v)\, w_{12}(u) = w_{12}(-u^2 v^{-1}) \qquad w_{12}(u)\, w_{12}(-u) = e$$

$$w_{12}(u)\, h_{12}(v)\, w_{12}(u)^{-1} = h_{12}(-u^2 v^{-1})\, h_{12}(-u^2)^{-1} \qquad w_{12}(u)\, w_{12}(v) = h_{12}(u)\, h_{12}(-v)^{-1}$$

$$w_{12}(1)^2 = h_{12}(-1)^{-1}$$

$$w_{12}(1)\, h_{12}(u)\, w_{12}(1)^{-1} = h_{12}(-u^{-1})\, h_{12}(-1)^{-1} = c(u,-1)\, h_{12}(u^{-1})$$

And now:

$$h_{12}(uv) = c(u,v)\, h_{12}(u)\, h_{12}(v)$$

↳ central; multiplicative both in u,v; skew-sym.

How consistent are those?

$$\mathrm{Ad}_{w_{12}(1)} : h_{12}(uv) \longmapsto c(uv,-1)\, h_{12}(u^{-1}v^{-1}) = c(uv,-1)\, c(u^{-1},v^{-1})\, h_{12}(u^{-1})\, h_{12}(v^{-1})$$

$$\| \checkmark$$

$$c(u,v)\, h_{12}(u)\, h_{12}(v) \longmapsto c(u,v) \cdot c(u,-1)\, h_{12}(u^{-1})\, c(v^{-1},-1)\, h_{12}(v^{-1})$$

$$\mathrm{Ad}^2_{w_{12}(1)} : h_{12}(u) \mapsto c(u,-1)\, h_{12}(u^{-1}) \longmapsto c(u,-1)\, c(u^{-1},-1)\, h_{12}(u) = h_{12}(u)$$

$$\mathrm{Ad}_{h_{12}(-1)} : h_{12}(u) \mapsto h_{12}(-1)\, h_{12}(u)\, h_{12}(-1)^{-1} = c(-1,-1)\, h_{12}(-1)\, h_{12}(u)\, h_{12}(-1)$$

$$= c(-1,-1)\, c(-1,u)\, c(-u,-1)\, h_{12}(u) = c(-1,1)^2\, h_{12}(u) = h_{12}(u) \checkmark$$

Using these rules: do multiplication.

$$x_{12}^{a}\, \cancel{x_{21}^{-a^{-1}}}\, \cancel{x_{12}^{a}} = w_{12}(a)\, x_{12}^{-a}\, x_{21}^{a^{-1}}$$

The non-obvious step:

$$w_{12}(1)\, x_{12}^{a}\, \underbrace{w_{12}(1)\, h_{12}(u)} \cdot x_{12}^{b}$$

$$\|$$

$$\underbrace{w_{12}(1)\, w_{12}(a)}_{= h_{12}(-a)^{-1}}\, x_{12}^{-a}\, x_{21}^{a^{-1}}\, \underbrace{w_{12}(1)\, h_{12}(u)}\, x_{12}^{b} = x_{12}^{\cdots} \boxed{h_{12}(-a)^{-1}\, w_{12}(1)\, h_{12}(u)}\, x_{12}^{\cdots}$$

And similarly from the right.

1) $\left(w_{12}^{(1)}\cdot\left(\qquad \right)\right)\cdot w_{12}(-1)$

$$w_{12}(1)x_{12}\,\underbrace{h_{12}(u)\,w_{12}(1)}\,x_{12}^{b}\cdot w_{12}(-1)$$

$$\|$$

$$w_{12}(1)\,w_{12}(a)\cdot x_{12}^{-a}x_{21}^{a^{-1}}\cdot h_{12}(u)\,w_{12}(1)\,x_{12}^{b}\cdot w_{12}(-1)$$

$$\|$$

$$h_{12}(-a)^{-1}\,x_{12}^{-a}\,x_{21}^{a^{-1}}\,h_{12}(u)\,w_{12}(1)\,x_{12}^{b}\,w_{12}(-1)$$

$$x_{12}^{-a^{-1}}\cdot\underbrace{h_{12}(-a)^{-1}h_{12}(u)\,w_{12}(1)}\cdot x_{12}^{b-a^{-1}u^{2}}\cdot W_{12}(-1)$$

$$\|$$

$$x_{12}^{-a^{-1}}\cdot h_{12}(-a)^{-1}h_{12}(u)\,w_{12}(1)\,x_{21}^{(b-a^{-1}u^{2})^{-1}}\,x_{12}^{-b+a^{-1}u^{2}}\,\underbrace{w_{12}(b-a^{-1}u^{2})\,w_{12}(-1)}_{=\,h_{12}(b-a^{-1}u^{2})}$$

$$\|$$

$$\ldots\ h_{12}(-a)^{-1}\cdot h_{12}(u)\cdot w_{12}(1)\cdot h_{12}\left(b\left(1-\frac{u^{2}}{ab}\right)\right)\ldots$$

$$\|$$

$$c(-1,a)\,c(-1,b)\,c\left(-1,1-\frac{u^{2}}{ab}\right)\cdot$$

$$\ldots\ h_{12}(-a^{-1})\,h_{12}(u)\,h_{12}\left(b^{-1}\left(1-\frac{u^{2}}{ab}\right)^{-1}\right)w_{12}(1)\ \ldots$$

2) $w_{12}(u)\left(\left[\_\_\_\_\_\_\right]\cdot w_{12}(-1)\right)$

$w_{12}(u)\cdot\ x_{12}(a)\underbrace{h_{12}(u)w_{12}(1)}\cdot x^{b}_{12}\,w_{12}(-1)$

$\parallel$

$w_{12}(u)\cdot\ x_{12}(a)\underbrace{h_{12}(u)w_{12}(1)}\cdot x^{b^{-1}}_{21}\,x^{-b}_{12}\,\underbrace{w_{12}(b)w_{12}(-1)}_{\parallel}$

$w_{12}(1)\ \ x_{12}(a)\ \underbrace{h_{12}(u)\,w_{12}(1)}\ x^{b^{-1}}_{21}\,x^{-b}_{12}\quad h_{12}(b)$

$w_{12}(1)\cdot x_{12}^{a-u^2b^{-1}}\cdot\underbrace{h_{12}(u)w_{12}(1)h_{12}(b)}\cdot\ x_{12}^{-b^{-1}}$

$\parallel$

$w_{12}(1)\,w_{12}(a-u^2b^{-1})\cdot x_{12}^{-(a-u^2b^{-1})}\ x_{21}^{(a-u^2b^{-1})^{-1}}\cdot$

$h_{12}(a-u^2b^{-1})^{-1}\cdot x_{12}^{-(a-u^2b^{-1})}\cdot x_{21}^{(a-u^2b^{-1})^{-1}}\cdot\underbrace{h_{12}(u)w_{12}(1)h_{12}(b)}\cdot x_{12}^{\dots}$

$x_{12}^{\dots}$

$\parallel$

$\dots\ \underbrace{h_{12}(-(a-u^2b^{-1}))^{-1}h_{12}(u)\,w_{12}(1)\,h_{12}(b)}\ \dots$

$\parallel$

$c(-a,-1)\,c\left(1-\frac{u^2}{ab},-1\right)\quad c(-1,b)$

$\dots\ h_{12}\left(-a^{-1}\left(1-\frac{u^2}{ab}\right)^{-1}\right)h_{12}(u)\,h_{12}(b^{-1})\,w_{12}\ \dots$

Have to compare:

$$c\left(1-\frac{u^2}{ab}, b\right)\cdot$$

$$\mathrm{I} = h_{12}(-a)^{-1}\, w_{12}(u)\, h_{12}\left(b\left(1-\frac{u^2}{ab}\right)\right) = \cancel{h_{12}(-a)^{-1}}\, w_{12}(u) h_{12}\left(1-\frac{u^2}{ab}\right) \cancel{h_{12}(b)}$$

$$\mathrm{II} = h_{12}\left(-a\cdot\left(1-\frac{u^2}{ab}\right)\right)^{-1}\cdot w_{12}(u)\cdot h_{12}(b) = \cancel{h_{12}(-a)^{-1}}\, h_{12}\left(1-\frac{u^2}{ab}\right)^{-1} w_{12}(u) \cancel{h_{12}(b)}$$

$$c\left(1-\frac{u^2}{ab}, -a\right)^{-1}!$$

$$w_{12}(u) h_{12}\left(1-\frac{u^2}{ab}\right) w_{12}(u)^{-1} = h_{12}\left(\frac{u^2}{1-\frac{u^2}{ab}}\right) h_{12}(u^{-2})$$

$$= c\left(1-\frac{u^2}{ab}, +u^{-2}\right)\cdot h_{12}\left(\frac{1}{1-\frac{u^2}{ab}}\right)$$

$$\mathrm{I} = c\left(1-\frac{u^2}{ab}, b\right) c\left(1-\frac{u^2}{ab}, u^{-2}\right) \quad h_{12}\left(\left(1-\frac{u^2}{ab}\right)^{-1}\right) w_{12}(u)$$

$$\mathrm{II} = c\left(1-\frac{u^2}{ab}, -a\right)^{-1} \quad h_{12}\left(1-\frac{u^2}{ab}\right)^{-1}\cdot w_{12}(u)$$

$$\|$$

$$c\left(1-\frac{u^2}{ab}, -1\right)\cdot \quad h_{12}\left(\left(1-\frac{u^2}{ab}\right)^{-1}\right)\cdot w_{12}(u)$$

$$\mathrm{I} = \mathrm{II} \Leftarrow c\left(1-\frac{u^2}{ab}, \frac{u^2}{ab}\right) = 1.$$

General $SL(F)$: Basically the same.

$$s = \prod x_{ij}(a_{ij})\cdot w\cdot \prod x_{ij}(b_{ij}) \qquad w = \text{product of } w_{ij}(u_{ij})$$

(with some cancellations);

the important case:

$$\prod x_{ij}(a_{ij})\cdot w\cdot \prod x_{ij}(b_{ij})\cdot w_{k,k+1}(-1)$$

$$\prod x_{ij}(a_{ij}) \cdot w \cdot \prod x_{ij}(b_{ij}) \cdot w_{k,k+1}(-1)$$

$$\| $$

$$x_{k,k+1}(b) \cdot \prod_{(i,j) \neq (k,k+1)} x_{ij}(\dots) \cdot w_{k,k+1}(-1)$$

still $\prod_{i<j} x_{ij}(\dots)$

nontrivial case:

◦ the permutation corresponding to w changes the order of $k, k+1$; essentially reduce to same calculation as for SL_2.

This produces a central extension

$$K_2^M(F) \longrightarrow \widetilde{SL}(F) \longrightarrow SL(F)$$

Now we have:

$$\begin{tikzcd}
K_2^M(F) \arrow[r] \arrow[d, shift right, "f"'] & \widetilde{SL}(F) \arrow[r] & SL(F) \arrow[d, equal] \\
K_2(F) \arrow[r] \arrow[u, shift right, "\text{by universality}"'] & St(F) \arrow[r] & SL(F)
\end{tikzcd}$$

easy to see the two are inverse