Waldhausen S-construction

$\mathcal{C}$ - category with cofibrations:

$*$ - zero object of $\mathcal{C}$ (i.e. initial and final)

Subcategory $co\mathcal{C}$ of $\mathcal{C}$ ($A \rightarrowtail B$ morphisms in $\mathcal{C}$)

$* \rightarrowtail A$, $\forall A$;

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ A' \arrow[r, tail] & B' = A' +_A B \end{tikzcd}$$

: pushout exists and is a cofibration.

For $(\mathcal{C}, co\mathcal{C})$: category $F_1\mathcal{C}$

Objects of $F_1\mathcal{C}$: $A \rightarrowtail B$

Morphisms in $F_1\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d] \\ A' \arrow[r, tail] & B' \end{tikzcd}$$

Cofibrations in $F_1\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dd, tail, "①"'] & B \arrow[d] \arrow[ddr, bend left] & \\ & A' +_A B \arrow[dr, tail, "②"] & \\ A' \arrow[ur] \arrow[rr] & & B' \end{tikzcd}$$

① and ② - cofibrations

(Rmk: for Ab groups, etc.: these are morphisms strongly compatible with filtrations:

(a', b): $f(b) \in A' \Rightarrow b \in A$); $\Rightarrow B/A \rightarrowtail B'/A'$

Rules out:

$$\begin{tikzcd} A = \mathbb{Z}e_1 \arrow[d, "="'] & + \ \mathbb{Z}e_2 = B \arrow[d, "="] \\ \underbrace{\mathbb{Z}e_1 + \mathbb{Z}e_2}_{= A'} & = B' \end{tikzcd}$$

We will use:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
A' \arrow[r] \arrow[ur, phantom, "\lrcorner"] & B' \arrow[r] \arrow[ur, phantom, "\lrcorner"] & C'
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d] \\
A' \arrow[r] \arrow[ur, phantom, "\lrcorner"] & C''
\end{tikzcd}$$

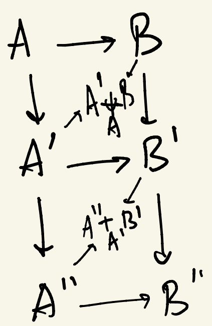
Also: $\lrcorner$ and $\lrcorner$

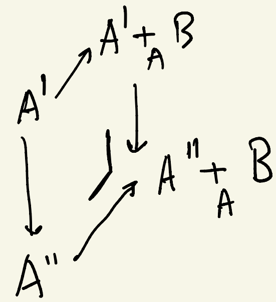
$\Downarrow$

$\lrcorner$

$$A' +_A C = (A' +_A B) +_B C$$

Prop. The above makes $F, \mathcal{C}$ a category w/ Cof

Pf 1) 

Note: 

i.e.

$$A'' +_A B = A'' +_{A'} (A' +_A B)$$

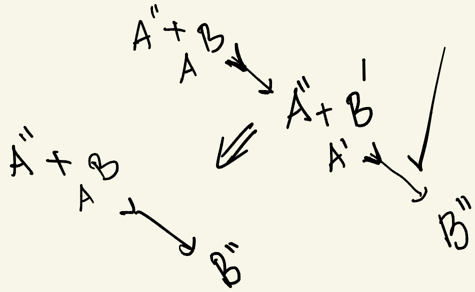
Composition:

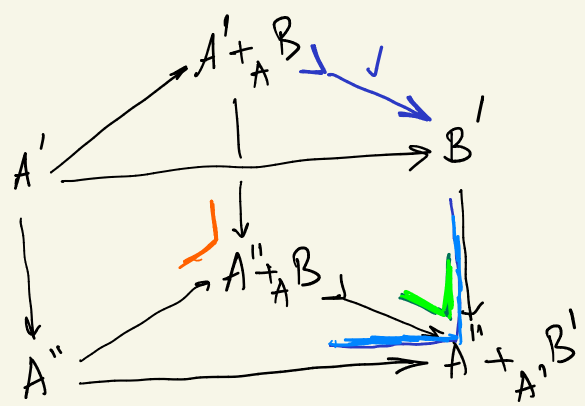
$\lrcorner$ & $\lrcorner$

$\Downarrow$

$\lrcorner$

$\Downarrow$





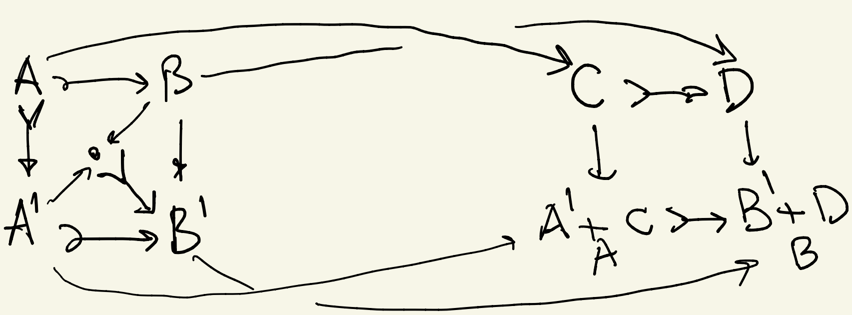
$\lrcorner$ $\lrcorner$

$\Downarrow$

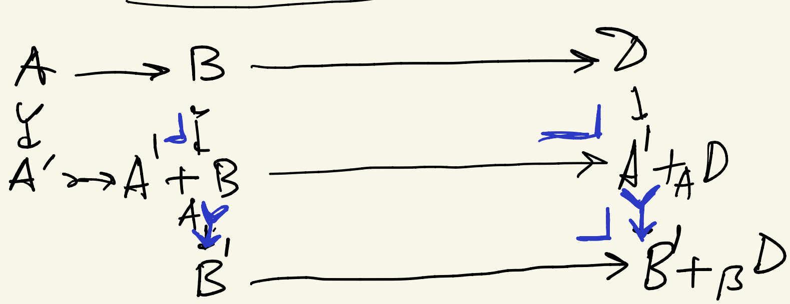
$\lrcorner$

$\Downarrow$

2) Pushout of cofibrations:



$$(A' +_A C) +_C D = A' +_A D$$



Quotients

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & * \arrow[d, tail] \\ B \arrow[r] & B/A \end{tikzcd}$$ unique up to unique isom

Equivalent category F_1^+C:

Objects: $A \rightarrowtail B \to B/A$ (choice of)

Morphisms:
$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[r] \arrow[d] & B/A \arrow[d] \\ A' \arrow[r, tail] & B' \arrow[r] & B'/A' \end{tikzcd}$$

Cofibrations:
$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r] \arrow[d] \arrow[dd, bend left] & B/A \arrow[dd] \\ A' \arrow[r] & \bullet \arrow[d, tail] & \\ & B' \arrow[r] & B'/A' \end{tikzcd}$$

$$F_1^+C \underset{\text{choose } B/A}{\overset{\text{forget}}{\underset{\sim}{\rightleftarrows}}} F_1$$

Exact functor: $\mathcal{C} \to \mathcal{D}$ preserving $*$; $co\mathcal{C} \to co\mathcal{D}$; preserving pushout of cof.

$$F_1^+C \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{C} \qquad \text{and } F_1^+C \xrightarrow{q} \mathcal{C}$$

$$\left(A \rightarrowtail B \twoheadrightarrow B/A\right) \begin{array}{l} \mapsto A \\ \mapsto B \\ \mapsto B/A \end{array}$$

Prop. All three are exact.

Pf Preserving cof:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r] \arrow[d, tail] \arrow[dl] & B/A \arrow[d] \\ A' \arrow[r, tail] \arrow[ur, phantom, "A' +_A B"] & B' \arrow[r] & B'/A' \end{tikzcd}$$

$$B/A = B \underset{A}{+} * = \left(B \underset{A}{+} A'\right) \underset{A'}{+} *$$

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail] & \\
A' \arrow[r, tail] \arrow[d] \arrow[ur, phantom, "\lrcorner"] & B +_A A' \arrow[r] \arrow[d] & B' \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\lrcorner"] & B/A \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & B'/A'
\end{tikzcd}$$

Preserving pushout of cofibrations:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] \arrow[rrr, bend left] & B \arrow[r] \arrow[d, tail] \arrow[rrr, bend left] & B/A \arrow[d] \arrow[rrr, bend left] & C \arrow[r, tail] \arrow[d] & D \arrow[r] \arrow[d] & D/C \arrow[d] \\
A' \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & B' \arrow[r] & B'/A' & A' +_A C \arrow[r] & B' +_B D \arrow[r] & (B' +_B D)/(A' +_A C)
\end{tikzcd}$$

$$B'/A' \underset{B/A}{+} D/C \overset{?}{\simeq} (B' \underset{B}{+} D)/(A' \underset{A}{+} C)$$

Morphisms from each to Z:

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & B' \arrow[r] & Z \\
* \arrow[urr, bend right] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, tail] \arrow[d] & D \arrow[r] & Z \\
* \arrow[urr, bend right] & &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & D \arrow[d] \\
B' \arrow[r] & Z
\end{tikzcd}$$

In other words: both are colimit of

$$\begin{tikzcd}
* & C \arrow[l] \arrow[r, tail] & D \\
* \arrow[u, equal] \arrow[d, equal] & A \arrow[l] \arrow[r, tail] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d] \\
* & A' \arrow[l] \arrow[r, tail] & B'
\end{tikzcd}$$

$\blacksquare$

Category of filtered objects $F_n\mathcal{C}$:

Objects: $A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n$

Morphisms:

$$\begin{tikzcd}
A_0 \arrow[r, tail] \arrow[d] & A_1 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & A_n \arrow[d] \\
A_0' \arrow[r, tail] & A_1' \arrow[r, tail] & \cdots \arrow[r, tail] & A_n'
\end{tikzcd}$$

Cofibrations: each

$$\begin{tikzcd}
A_i \arrow[r, tail] \arrow[d, tail] & A_{i+1} \arrow[d, tail] \\
A_i' \arrow[r, tail] & A_{i+1}'
\end{tikzcd}$$

is in $\mathrm{co}F_1\mathcal{C}$

Note: $\mathcal{F}_n \mathcal{F}_m \mathcal{C} \simeq \mathcal{F}_m \mathcal{F}_n \mathcal{C}$ being a cofibration is a <u>symmetric</u> condition:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail] \\ A' \arrow[r, tail] & B' \end{tikzcd}$$

is in $co F_1(A \rightarrowtail B, A' \rightarrowtail B')$

$\Updownarrow$

it is in $co F_1\left(\begin{matrix} A \\ \downarrow \\ A' \end{matrix}, \begin{matrix} B \\ \downarrow \\ B' \end{matrix}\right)$

Categories w/ cofibrations and weak equivalences

$w\mathcal{C} \subset \mathcal{C}$ another subcategory; $Iso\, \mathcal{C} \subset w\mathcal{C}$;

$A \xrightarrow[f]{\sim} B$ for f in $w\mathcal{C}$.

$$\begin{tikzcd} B \arrow[d, "\sim"'] & A \arrow[l, tail] \arrow[r, tail] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l, tail] \arrow[r, tail] & C' \end{tikzcd} \Longrightarrow \begin{tikzcd} B \underset{A}{+} C \arrow[d, "\sim"] \\ B' \underset{A'}{+} C' \end{tikzcd}$$

<u>The simplicial category $S.\mathcal{C}$</u>

Given $(\mathcal{C}, co\mathcal{C})$: $[n] = (0 < 1 < 2 < \dots < n)$

$Ar[n]$: objects (ij) $i < j$

morphisms $(ij) \to (i'j')$ $i \leq i'$ $j \leq j'$

$S_n \mathcal{C} = \{$ Functors $\mathrm{Ar}[n] \to \mathcal{C}$ $\quad (ij) \mapsto A_{ij}$

such that:

$A_{ii} = *$;

$A_{ij} \rightarrowtail A_{ik}$;

$$\begin{tikzcd} A_{ij} \arrow[r] \arrow[d] & A_{ik} \arrow[d, two heads] \\ A_{jj} \arrow[r] & A_{jk} \end{tikzcd}$$

$* \cong A_{jj}$

is a pushout

Example:

$S_0 \mathcal{C} = *$ $\qquad S_1 \mathcal{C} = \mathcal{C}$ $\qquad S_2 \mathcal{C} = \mathcal{F}_1^+ \mathcal{C}$

A_{01}

$$\begin{tikzcd} A_{01} \arrow[r, tail] & A_{02} \arrow[d, two heads] \\ & A_{12} \end{tikzcd}$$

$A_{12} \simeq A_{02}/A_{01}$

$S_3 \mathcal{C} \sim \mathcal{F}_2 \mathcal{C}$

$$\begin{tikzcd} A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d, two heads] & A_{03} \arrow[d, two heads] \\ & A_{12} \arrow[r, tail] & A_{13} \arrow[d, two heads] \\ & & A_{23} \end{tikzcd}$$

By definition, $S_\bullet \mathcal{C}$ is a <u>simplicial</u> category.

(b/c $\mathrm{Ar}[n]$ is <u>co</u>simplicial).

$$\begin{tikzcd} S_2 \mathcal{C} \arrow[d, shift right=4, "d_0"'] \arrow[d, "d_1"] \arrow[d, shift left=4, "d_2"] \\ S_1 \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} A_{01} \arrow[r, tail] & A_{02} \arrow[d, two heads] \\ & A_{12} \end{tikzcd}$$

$$\begin{tikzcd} & A_{12} \arrow[dl, "d_0"'] \arrow[d, "d_1"] \arrow[dr, "d_2"] & \\ A_{12} & A_{02} & A_{01} \end{tikzcd}$$

As we seen above: all $S_n\mathcal{C}$ are categories w/ cofibrations; all d_i, s_j are exact functors.

If we have $(\mathcal{C}, co\mathcal{C}, w\mathcal{C})$: so are $S_n\mathcal{C}$ (morphisms in $S_n\mathcal{C}$ are morphisms of functors $[n] \to \mathcal{C}$; w.e. is a morphism such that all $A_{ij} \to A'_{ij}$ are w.e.).

Define

$$\mathbb{K}^w(\mathcal{C}) = \Omega |\underbrace{N_\cdot wS_*\mathcal{C}}_{\text{bisimplicial set}}|$$

$$\pi_1 N_\cdot S_* \mathcal{C}_\cdot = K_0(w\mathcal{C})$$

gen. by (A), $A \in ob\,\mathcal{C}$

$(A_0) = (A_1)$ for $A_0 \xrightarrow{\sim} A_1$

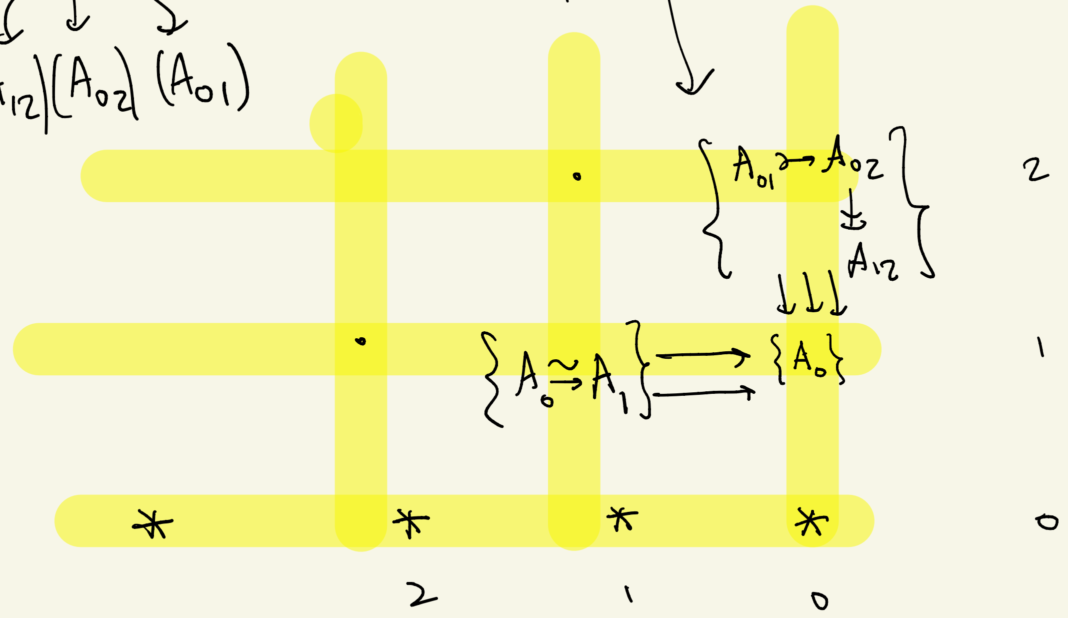
$(A_{01}) - (A_{02}) + (A_{12}) = 0$

for

$N_\cdot wS_*\mathcal{C}$:

$$\begin{pmatrix} A_{01} \rightarrowtail A_{02} \\ \downarrow \\ A_{12} \end{pmatrix} \quad \mapsto \quad (A_{12})\ (A_{02})\ (A_{01})$$

$$(A_0 \xrightarrow{\sim} A_1) \mapsto (A_1),\ (A_0)$$



<u>Note</u>: $BwC \longrightarrow \Omega BwS.C$ (first horizontal row)

<u>Compare</u>: $BGL \longrightarrow BGL^+$

<u>But also</u>:

$$Bw\underline{S.C} \to \Omega Bw S.\underline{S.C}$$

$$BwC \to \Omega BwS.C \to \Omega^2 BwS.S.C \to \cdots$$

Get a <u>spectrum</u>

<u>Theorem</u> Starting with $\Omega BwS.\mathcal{C}$, all $\to$ are homot.equs

Follows from

<u>The additivity theorem</u>

$$\mathbb{K}^w(\mathcal{F}_1^+\mathcal{C}) \xrightarrow[(s,q)]{\sim} \mathbb{K}^w(\mathcal{C}) \vee \mathbb{K}(\mathcal{C})$$

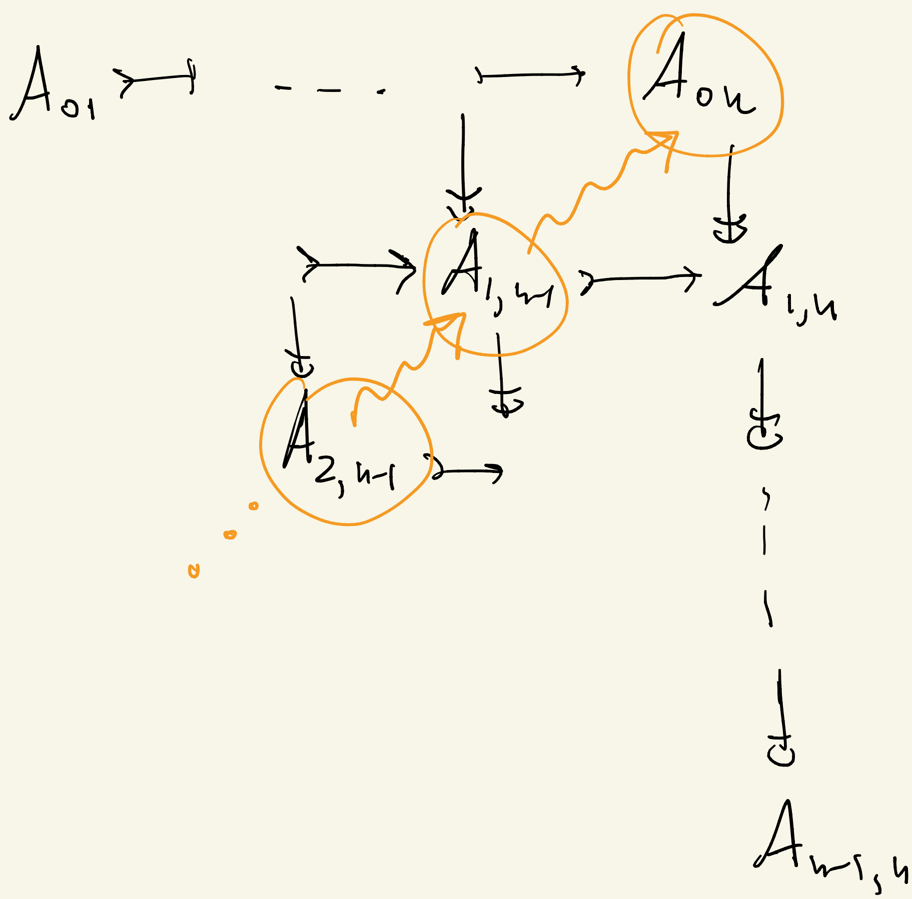
(Similar to the Q construction).

i.e. $$K_\cdot^w(\mathcal{F}_1^+\mathcal{C}) \xrightarrow{\sim} K_\cdot^w(\mathcal{C}) \oplus K_\cdot^w(\mathcal{C})$$

Pf Later; along the lines of the proof for Q.

S vs Q

<u>Note:</u> from an object of $S_n C$ we do get a simplex in $N_\bullet Q$



This does not <u>quite</u> agree with d_i, s_i. Namely:

$$\begin{tikzcd} A_{01} \arrow[r, tail] & A_{02} \arrow[d, two heads] & \\ & A_{12} \arrow[r, tail] & A_{02} \end{tikzcd}$$

$$
\begin{tikzcd}
A_{03} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d] & A_{01} \arrow[d] \\
& A_{12} \arrow[r, tail] \arrow[ur, squiggly] & A_{13} \arrow[d] \\
& & A_{23}
\end{tikzcd}
\quad \overset{d_0}{\longmapsto} \quad
\begin{tikzcd}
A_{12} \arrow[r, tail] & A_{13} \arrow[d] \\
& A_{23}
\end{tikzcd}
$$

To repair <u>this</u>:

$$
\begin{tikzcd}
\operatorname{Ob} S_{2n+1}\mathcal{C} \arrow[d] \\
N_n Q\mathcal{C}
\end{tikzcd}
$$

But <u>new</u> d_i, s_i on $S_{2\bullet-1}$:

d_i drop two rows/columns at once.

$$
\begin{tikzcd}
A_{2',1'} \arrow[r, tail] & A_{2',0'} \arrow[r, tail] \arrow[d] & A_{2',0} \arrow[r, tail] \arrow[d] & A_{2',1} \arrow[r, tail] \arrow[d] & A_{2',2} \arrow[d] \\
& A_{1',0'} \arrow[r, tail] & A_{1',0} \arrow[r, tail] \arrow[d] & A_{1',1} \arrow[r, tail] \arrow[d] \arrow[ur, squiggly] & A_{1',2} \arrow[d] \\
& & A_{0',0} \arrow[r, tail] \arrow[ur, squiggly] & A_{0',1} \arrow[r, tail] \arrow[d] & A_{0',2} \arrow[d] \\
& & & A_{0,1} \arrow[r, tail] & A_{0,2} \arrow[d] \\
& & & & A_{1,2}
\end{tikzcd}
$$

$[5] = (2' < 1' < 0' < 0 < 1 < 2)$

d_i^{new}: drop i', i both.

$\cap$

$N_2 Q(\mathcal{C})$

We get

$$\begin{tikzcd}
\vdots & \vdots \\
\mathrm{ob}\, S_5 C \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & N_2 QC \arrow[d, shift right=2, "d_0"'] \arrow[d] \arrow[d, shift left=2, "d_2"] \\
\mathrm{ob}\, S_3 C \arrow[r] \arrow[d, shift right=1, "d_0^{new}"'] \arrow[d, shift left=1, "d_1^{new}"] & N_1 QC \arrow[d, shift right=1, "d_0"'] \arrow[d, shift left=1, "d_1"] \\
\mathrm{ob}\, S_1 C \arrow[r] & N_0 QC
\end{tikzcd}$$

Now: 1) $(S_{2\cdot+1} C, d_\cdot^{new}, s_\cdot^{new})$ is the first subdivision of $S_\cdot C$; is hom. eq.

2) Both $\mathrm{ob}\, S_{2\cdot+1} C$ and $N_\cdot QC$ are categories; morphisms are

$$A_{ij} \xrightarrow{\sim} A'_{ij} \qquad \text{resp.} \qquad A_i \xrightarrow{\sim} A'_i$$

making all diagrams commute.
(Rmk: isoms in C are same as in QC).
Get a simplicial functor on simplicial categories.

Prop. This is a homotopy equivalence.

Pf later.

$$|\mathrm{Nerve}_* iS_\cdot \mathcal{C}| \simeq |\mathrm{Nerve}_* iS_{2\cdot+1}\mathcal{C}| \simeq |\mathrm{Nerve}_* iQ\mathcal{C}|$$

isoms in categ $S_{2\cdot+1}$

Finally: $|N_* N_\cdot iQ\mathcal{C}| \simeq |N_\cdot Q\mathcal{C}|$

General statement: $\mathcal{A} \subset \mathcal{B}$ subcategory

$$N_p^{\mathcal{A}} N_q^{\mathcal{B}} =: N_{pq} = \left\{ \begin{array}{ccccc} X_{00} \xrightarrow{a_{01}} & X_{01} \xrightarrow{a_{02}} & \cdots & \xrightarrow{a_{0p}} X_{0p} \\ \downarrow b_{10} & & & \downarrow b_{1p} \\ X_{10} \xrightarrow{a_{11}} & X_{11} \xrightarrow{a_{12}} & \cdots & \xrightarrow{a_{1p}} X_{1p} \\ \downarrow & \downarrow & & \downarrow \\ \vdots & \vdots & & \vdots \\ \downarrow b_{q0} & \downarrow & & \downarrow b_{qp} \\ X_{q0} \xrightarrow{a_{q1}} & X_{q1} \xrightarrow{a_{q2}} & \cdots & \xrightarrow{a_{qp}} X_{qp} \end{array} \right\} \quad \begin{array}{l} a_{ij} \in \mathcal{A} \\ b_{ij} \in \mathcal{B} \\ X_{ij} \in \mathrm{ob}(\mathcal{B}) \end{array}$$

Then $|N_\cdot \mathcal{B}| \xrightarrow{\sim} |N_*^{\mathcal{A}} N_\cdot^{\mathcal{B}}|$

(The swallowing lemma) Pf: $\mathcal{B} \to N_p^{\mathcal{A}}$ (a functor btwn simplicial categories) induces equivalence on $\mathcal{B}(-)$, $\forall p$.

In fact there is a pair of adjoint functors:

$\mathcal{B} \rightleftarrows N_p^{\mathcal{A}}$; $x_0 \overset{L}{\mapsto} (x_0 \xrightarrow{=} \cdots \xrightarrow{=} x_0)$; $\quad x_0 \overset{R}{\leftarrow\!\shortmid} (x_0 \to \cdots \to x_p)$