Regulator map

$A \supset I$ pronilpotent ideal

$C^{nl}_\bullet(A, I) =$ Ab subgroup of $\mathbb{Z}[CL_{*,\bullet}(A)]$ generated by

$\left\langle a_0, \ldots, \begin{bmatrix} a^{(1)}_j \\ \vdots \\ a^{(m)}_j \end{bmatrix}, \ldots, a_n \right\rangle$ such that: $\forall k$

at least one of $a_0, \ldots, a^{(k)}_j, \ldots, a_m$ is in I.

This is a bicomplex; $C^{nl}_\bullet(A, I)$ stands for the total complex.

$$CC^{nl}_\bullet(A, I) = C^{nl}_\bullet(A, I)((u)) / u\, C^{nl}_\bullet(A, I)[[u]]$$

(as usual); differential = (total diff) + uB

$$\mathbb{Z}_\bullet \quad \begin{tikzcd}
u^{-2}\{\langle a_0\rangle\} \arrow[r, "uB"] & {} \arrow[r, "uB"] \arrow[d, "b"', "\partial"] & \{\langle a_0,a_1,a_2\rangle, \langle \begin{bmatrix} a_0 \\ a_0' \end{bmatrix}, a_1\rangle, \dots\} \arrow[d, "b"', "\partial"] \\
 & u^{-1}\{\langle a_0\rangle\} \arrow[r, "uB"] & \{\langle a_0,a_1\rangle, \begin{bmatrix} a_0 \\ a_0' \end{bmatrix}\} \arrow[d, "b"', "\partial"] \\
2 & 1 & \{\langle a_0\rangle\} \\
 & & 0
\end{tikzcd}$$

regulator map:

when char $k=0$; A comm.

$$\begin{tikzcd}
CC_\bullet^{nil}(A,I) \arrow[d, "r"] \\
\Omega^\bullet_{A,I}((u))/u\Omega^\bullet[[u]]
\end{tikzcd}$$

ud

$$\begin{tikzcd}
u^{-2}\Omega^0_{A/I} \arrow[r, "ud"] & u^{-1}\Omega^1_{A,I} \arrow[r, "ud"] \arrow[d, "0"] & \Omega^2_{A/I} \\
3 & u^{-1}\Omega^0_{A/I} \arrow[r, "ud"] & \Omega^1_{A/I} \\
 & 2 & 1 \quad \Omega^0_{A,I} \\
 & & 0
\end{tikzcd}$$

$$\Omega^\bullet_{A,I} = \ker\left(\Omega^\bullet_{A/k} \to \Omega^\bullet_{(A/I)/k}\right)$$

Definition of the regulator map.

Put $\log_k(1-z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$, $k \geq 1$.

Define:

$$u^{-p}\langle a_0,\dots,a_n\rangle \longmapsto \frac{u^{-p}}{n!}\log_{p+1}(1-a_0\cdots a_n)\cdot \frac{da_1}{a_1}\cdots\frac{da_n}{a_n}$$

$$u^{-p}\left\langle a_0,\dots,\begin{bmatrix}a_j\\ a_j'\end{bmatrix},\dots,a_n\right\rangle \longmapsto 0,\quad j=0;$$

$$\downarrow$$

$$\pm ?\ \frac{u^{-1-p}}{n!}\Big(\log_{p+2}\big((1-a_0\cdots a_j\cdots a_n)\cdot(1-a_0\cdots a_j'\cdots a_n)\big) - \log_{p+2}(1-a_0\cdots a_j\cdots a_n) - \log_{p+2}(1-a_0\cdots a_j'\cdots a_n)\Big)\cdot\frac{da_1}{a_1}\cdots\widehat{\frac{da_j}{a_j}}\cdots\frac{da_n}{a_n}$$

$$u^{-p}\left\langle a_0,\dots,\begin{bmatrix}a_j^{(1)}\\ \vdots\\ a_j^{(m)}\end{bmatrix},\dots,a_n\right\rangle \longmapsto 0,\quad m>2.$$

Claim This is a morphism of complexes.

Ex 1) $\langle a_0,a_1,a_2\rangle \xmapsto{b} \langle a_0a_1,a_2\rangle - \langle a_0,a_1a_2\rangle + \langle a_2a_0,a_1\rangle$, $\quad \langle a_0,a_1,a_2\rangle \xrightarrow{\partial+uB} 0$

$$\mapsto \log(1-a_0a_1a_2)\left(\frac{da_2}{a_2}-\frac{d(a_1a_2)}{a_1a_2}+\frac{da_1}{a_1}\right)=0$$

$$\log(1-a_0a_1a_2)\frac{da_2}{a_2} \xmapsto[ud]{} 0 \quad (\mathrm{mod}\ u\dots)$$

2) $u^{-1}\langle a_0\rangle \xmapsto{b+\partial+uB} \langle 1,a_0\rangle \mapsto \log(1-a_0)\cdot\frac{da_0}{a_0}$

$u^{-1}\log_2(1-a_0) \xrightarrow{ud} \log(1-a_0)\cdot\frac{da_0}{a_0}$

3) $$\left\langle \begin{bmatrix} a_0 \\ a_0' \end{bmatrix} \right\rangle \xmapsto{b+uB+\partial} \langle a_0\rangle + \langle a_0'\rangle - \langle a_0 + a_0' - a_0 a_0'\rangle$$

$$\left\langle \begin{bmatrix} a_0 \\ a_0' \end{bmatrix} \right\rangle \longmapsto 0$$

$$\langle a_0\rangle + \langle a_0'\rangle - \langle a_0 + a_0' - a_0 a_0'\rangle \longmapsto \log(1-a_0) + \log(1-a_0') - \log\big((1-a_0)(1-a_0')\big) = 0$$

4) $$u^{-1}\left\langle \begin{bmatrix} a_0 \\ a_0' \end{bmatrix} \right\rangle \xmapsto{\cancel{b}+uB+\partial} \left\langle 1, \begin{bmatrix} a_0 \\ a_0' \end{bmatrix} \right\rangle - u^{-1}\big(\langle a_0\rangle + \langle a_0'\rangle - \langle a_0 + a_0' - a_0 a_0'\rangle\big)$$

$$u^{-1}\left\langle \begin{bmatrix} a_0 \\ a_0' \end{bmatrix} \right\rangle \longmapsto 0 \longmapsto$$

✓

$$u^{-1}\Big(\log_2\big((1-a_0)(1-a_0')\big) - \log_2(1-a_0) - \log_2(1-a_0')\Big)$$

$$- u^{-1}\Big(\log_2(1-a_0) + \log_2(1-a_1) - \log_2\big((1-a_0)(1-a_1)\big)\Big)$$

In general:

$$u^{-p}\langle a_0,\dots,a_n\rangle \xmapsto{b} \langle a_0a_1,\dots,a_n\rangle + \sum \pm \langle a_0,\dots,a_ja_{j+1},\dots,a_n\rangle \pm \langle a_na_0, a_1,\dots,a_{n-1}\rangle$$

$$u^{-p}\langle a_0,\dots,a_n\rangle \xmapsto{\partial} 0$$

$$u^{-p}\langle a_0,\dots,a_n\rangle \xmapsto{uB} u^{1-p}\sum \langle 1, a_j,\dots,a_0,\dots,a_{j-1}\rangle$$

$$\frac{u^{-p}}{(n-1)!}\log_{p+1}(1-a_0\cdots a_n)\left(\frac{da_n}{a_n}\cdots\frac{da_n}{a_n} + \cancel{\sum\pm\frac{da_1}{a_1}\cdots\frac{d(a_ja_{j+1})}{a_ja_{j+1}}}\cdots\frac{da_n}{a_n} \mp \frac{da_1}{a_1}\cdots\frac{da_{n-1}}{a_{n-1}}\right)$$

$$\frac{u^{-p+1}}{n!}\log_p(1-a_0\cdots a_n)\sum_j \frac{da_0}{a_0}\cdots\frac{da_n}{a_n}$$

$\parallel$ ✓

$$\frac{u^{-p}}{n!}\log_{p+1}(1-a_0\cdots a_n)\cdot\frac{da_1}{a_1}\cdots\frac{da_n}{a_n} \xmapsto{ud} \frac{u^{-p+1}}{n!}\cdot\frac{\log_p(1-a_0\cdots a_n)}{a_0\cdots a_n}\cdot d(a_0\cdots a_n)\cdot\frac{da_1}{a_1}\cdots\frac{da_n}{a_n}$$

Lemma $d\left(\log_{p+2}(1-a_0a_1\cdots c_j\cdots a_n)\,\frac{da_1}{a_1}\cdots\widehat{\frac{da_j}{a_j}}\cdots\frac{da_n}{a_n}\right)=$

$$=\left(\frac{da_0}{a_0}+\frac{dc_j}{c_j}\right)\cdot\log_{p+1}(1-a_0\cdots c_j\cdots a_n)\cdot\frac{da_1}{a_1}\cdots\wedge\cdots\frac{da_n}{a_n}$$

$$u^{-p}\left\langle a_0,\dots,\begin{bmatrix}a_j\\a_j'\end{bmatrix},\dots,a_n\right\rangle \xmapsto{uB} \sum \pm u^{+1-p}\left\langle 1, a_i,\dots,\begin{bmatrix}a_j\\a_j'\end{bmatrix},\dots,a_{i-1}\right\rangle$$

(n+1 term)

$$\Big\downarrow r$$

$$\xrightarrow{\partial}\quad \frac{(n+1)}{(n+1)!}u^{1-p}\Big[\log_{p+1}\big((1-a_0\cdots a_j\cdots)(1-a_0\cdots a_j'\cdots)\big)-$$
$$-\log_{p+1}(1-a_0\cdots a_j\cdots)-\log_{p+1}(1-a_0\cdots a_j'\cdots)\Big]\frac{da_0}{a_0}\cdots\widehat{\frac{da_j}{a_j}}\cdots\frac{da_n}{a_n}=$$

$$=\frac{u^{-p}}{n!}\Big[\log_{p+1}(1-a_0\cdots(a_j+a_j')\cdots a_n)-\log_{p+1}(1-\cdots a_j\cdots)-\log_{p+1}(1-\cdots a_j'\cdots)\Big]\cdot\frac{da_0}{a_0}\cdots\widehat{\frac{da_j}{a_j}}\cdots\frac{da_n}{a_n}$$

$$\xrightarrow{r}\quad u^{-p}\langle a_0,\dots,a_j,\dots,a_n\rangle+u^{-p}\langle a_0,\dots,a_j',\dots,a_n\rangle-u^{-p}\langle a_0,\dots,a_j+a_j',\dots,a_n\rangle$$

$$\Big\downarrow r$$

$$\frac{u^{-p}}{n!}\log_{p+1}(1-a_0\cdots a_j\cdots)\cdot\frac{da_1}{a_1}\cdots\frac{da_j}{a_j}\cdots\frac{da_n}{a_n}+\frac{u^{-p}}{n!}\log_{p+1}(1-\cdots a_j'\cdots)\frac{da_1}{a_1}\cdots\frac{da_j'}{a_j'}\cdots\frac{da_n}{a_n}$$
$$-\frac{u^{-p}}{n!}\log_{p+1}(1-a_0\cdots(a_j+a_j')\cdots a_n)\cdot\frac{da_1}{a_1}\cdots d\log(a_j+a_j')\cdots\frac{da_n}{a_n}$$

$$\rightarrow\ \frac{u^{-1-p}}{n!}\Big[\log_{p+2}(1-\cdots a_j\cdots)+\log_{p+2}(1-\cdots a_j'\cdots)-\log_{p+2}(1-\cdots(a_j+a_j')\cdots)\Big]\frac{da_1}{a_1}\cdots\widehat{\frac{da_j}{a_j}}\cdots\frac{da_n}{a_n}$$

$$\xrightarrow{ud}\ \square + \square$$

BY LEMMA

In addition:

$$u^{-p}\left\langle a_0,\dots,\begin{bmatrix}a_j\\a_j'\end{bmatrix},\dots,a_n\right\rangle \xmapsto{\ b\ } u^{-p}\Big(\left\langle a_0a_1,\dots,[\ \ ],\dots,a_n\right\rangle$$

$$+\sum\pm\left\langle a_0,\dots,a_ia_{i+1},\dots,[\ \ ],\dots\right\rangle \pm \left\langle a_0,\dots,\begin{bmatrix}a_{j-1}a_j\\a_{j-1}a_j'\end{bmatrix}\dots\right\rangle$$

$$\pm\left\langle a_0,\dots,\begin{bmatrix}a_ja_{j+1}\\a_j'a_{j+1}\end{bmatrix},\dots\right\rangle + \sum\pm\left\langle a_0,\dots,\begin{bmatrix}a_j\\a_j'\end{bmatrix},\dots,a_ka_{k+1}\dots\right\rangle$$

$$+\left\langle a_na_0,\dots,\begin{bmatrix}a_j\\a_{j+1}\end{bmatrix},\dots\right\rangle\Big)$$

$$\Big\downarrow r$$

$$\frac{u^{-p}}{n!}\Big[\log_{p+1}\big(1-\dots(a_j+a_j')\dots\big)-\log_{p+1}\big(1-\dots a_j\dots\big)-\log_{p+1}\big(1-\dots a_j'\dots\big)\Big].$$

$$\sum\pm\Big(\dots\frac{d(a_ia_{i+1})}{a_ia_{i+1}}\dots\frac{da_2}{a_2}\dots\widehat{\frac{da_j}{a_j}}\dots\Big)-\frac{da_1}{a_1}\dots\frac{da_{j-2}}{a_{j-2}}\widehat{\frac{da_j}{a_j}}\dots$$

$$\pm\dots\widehat{\frac{da_j}{a_j}}\cdot\frac{da_{j+2}}{a_{j+2}}\dots+\sum\pm\dots\widehat{\frac{da_j}{a_j}}\dots\frac{d(a_ka_{k+1})}{a_ka_{k+1}}\dots$$

$$\pm\dots\widehat{\frac{da_j}{a_j}}\dots\frac{da_{n-1}}{a_{n-1}}\Big)=0$$

$$0 \xleftarrow{\ r\ }$$

Next: $u^{-p}\left\langle\begin{bmatrix}a_0\\a_0'\end{bmatrix},a_1,\dots,a_n\right\rangle \xmapsto{\ uB\ } u^{-p}\sum\pm\left\langle 1,a_j,\dots,\begin{bmatrix}a_0\\a_0'\end{bmatrix},\dots,a_{j-1}\right\rangle$

$$\Big\downarrow r$$

$$\partial:\quad u^{-p}\Big(\langle a_0,\dots,a_n\rangle+\langle a_0',\dots,a_n\rangle-\langle a_0+a_0',\dots,a_n\rangle\Big)$$

$$\Big\uparrow r$$

$$u^{-p}\frac{(n+1)}{(n+1)!}\Big[\log_{p+1}\big(1-(a_0+a_0')a_1\dots a_n\big)-\log_{p+1}\big(1-a_0\dots a_n\big)-\log_{p+1}\big(1-a_0'\dots a_n\big)\Big]\cdot\frac{da_1}{a_1}\dots\frac{da_n}{a_n}$$

$$\xmapsto{\ b\ }\dots\xmapsto{\ r\ }0$$

And finally:

$$u^{-p}\Big\langle \cdots \begin{bmatrix} a_j \\ a_j' \\ a_j'' \end{bmatrix} \cdots \Big\rangle \overset{b+uB}{\longmapsto} \cdots \overset{r}{\longmapsto} 0$$

$$\Big\downarrow \partial$$

$$u^{-p}\Big(\Big\langle \cdots \begin{bmatrix} a_j' \\ a_j'' \end{bmatrix} \cdots \Big\rangle - \Big\langle \cdots \begin{bmatrix} a_j + a_j' \\ a_j'' \end{bmatrix} \cdots \Big\rangle +$$

$$+ \Big\langle \cdots \begin{bmatrix} a_j \\ a_j' + a_j'' \end{bmatrix} \cdots \Big\rangle - \Big\langle \cdots \begin{bmatrix} a_j \\ a_j' \end{bmatrix} \cdots \Big\rangle$$

$$\Big\downarrow r$$

$$0$$

∎

Next: Dennis/ch trace map.

$$\langle a \rangle \longmapsto (1-a)^{-1} \otimes a \in C_1(A)$$

$$\langle a_0, a_1 \rangle \longmapsto (1-a_0a_1)^{-1} \otimes a_0 \otimes a_1 - (1-a_1a_0)^{-1} \otimes a_1 \otimes a_0$$

$$\Big\downarrow b$$

$$(1-a_0a_1)^{-1}a_0 \otimes a_1 + a_1(1-a_0a_1)^{-1} \otimes a_0 - (1-a_0a_1)^{-1} \otimes a_0a_1$$
$$- (1-a_1a_0)^{-1}a_1 \otimes a_0 - a_0(1-a_1a_0)^{-1} \otimes a_1 + (1-a_1a_0)^{-1} \otimes a_1a_0$$

The two edges of the Dennis trace map can be easily found:

$$\langle a_0, a_1, a_2 \rangle \longmapsto (1-a_0a_1a_2)^{-1} \otimes a_0 \otimes a_1 \otimes a_2 + (1-a_1a_2a_0)^{-1} \otimes a_1 \otimes a_2 \otimes a_0 + (1-a_2a_0a_1)^{-1} \otimes a_2 \otimes a_0 \otimes a_1$$

$$\langle a_0, \dots, a_n \rangle \mapsto \sum (-1)^{nj} (1-a_j \dots a_0 \dots a_{j-1})^{-1} \otimes a_j \otimes \dots \otimes a_{j-1}$$

$$\begin{array}{ccc} \left\langle \begin{bmatrix} a_0 \\ a_0' \end{bmatrix} \right\rangle & \longmapsto & (1-a_0')^{-1}(1-a_0)^{-1} \otimes a_0 \otimes a_0' \\ \downarrow \partial & & \downarrow b \\ \langle a_0 \rangle + \langle a_0' \rangle - \langle a_0 + a_0' - a_0a_0' \rangle & \mapsto & (1-a_0)^{-1} \otimes a_0 + (1-a_0')^{-1} \otimes a_0' \\ & & [(1-a_0)(1-a_0')]^{-1} \otimes (a_0+a_0'-a_0a_0') \end{array}$$

$$\left\langle \begin{bmatrix} a_0^{(1)} \\ \vdots \\ a_0^{(m)} \end{bmatrix} \right\rangle \longmapsto (1-a_0^{(m)})^{-1} \dots (1-a_0^{(1)})^{-1} \otimes a_0^{(1)} \otimes \dots \otimes a_0^{(m)}$$

(the actual Dennis trace)

The general formula is harder to guess.

Its composition with HKR is easier:

$$\langle a_0, \dots, a_n \rangle \longmapsto \frac{1}{(n-1)!} (1-a_0 \dots a_n)^{-1} da_0 \dots da_n$$

$$\langle \dots \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \dots \rangle \longmapsto 0$$

$$d_i^{-1} \dots \longmapsto 0$$

Ex. Loday-friendly cluster coordinates on a NC torus

$yx = q\,xy$ $\quad x, y$ invertible $\quad k = \mathbb{C}[q^{\pm 1}]$

$$\mathcal{A} = k\langle x^{\pm 1}, y^{\pm 1}\rangle / (yx - qxy)$$

$a = x \qquad b = x^{-1}(1-y) \qquad z = x^{-1}y^{-1}(x+y-1)$

$c = y^{-1}(1-x)$

$y = 1 - ab;\ z = 1 - bc;$

$a + c - bac = 1;$

$1 - ba = q(1-ab);$

$1 - cb = q(1-bc)$

$1 - ab = y$

$1 - ba = 1 - x^{-1}(1-y)x =$

$= 1 - (1 - qy) = qy$

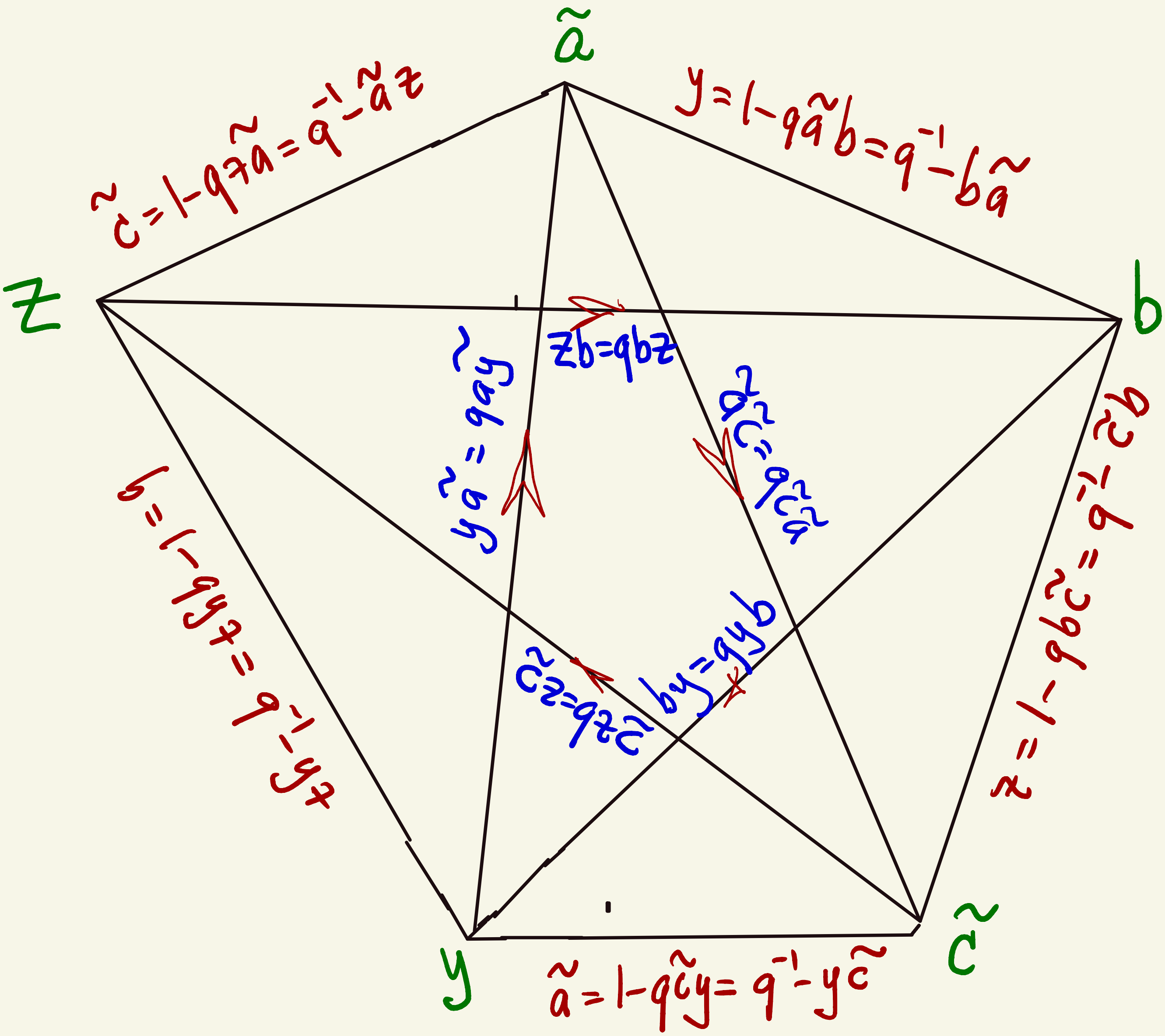
$a + c - bac = a + (1-ba)c = a + qyc =$

$= \cancel{x} + q(1-x) =$

From there: $\langle a, b\rangle \in K_2^{rel}$, i.e.

$$\pi_2\left(\mathrm{cofibre}\left(K_2(k) \to K_2(\mathcal{A}\right)\right)$$

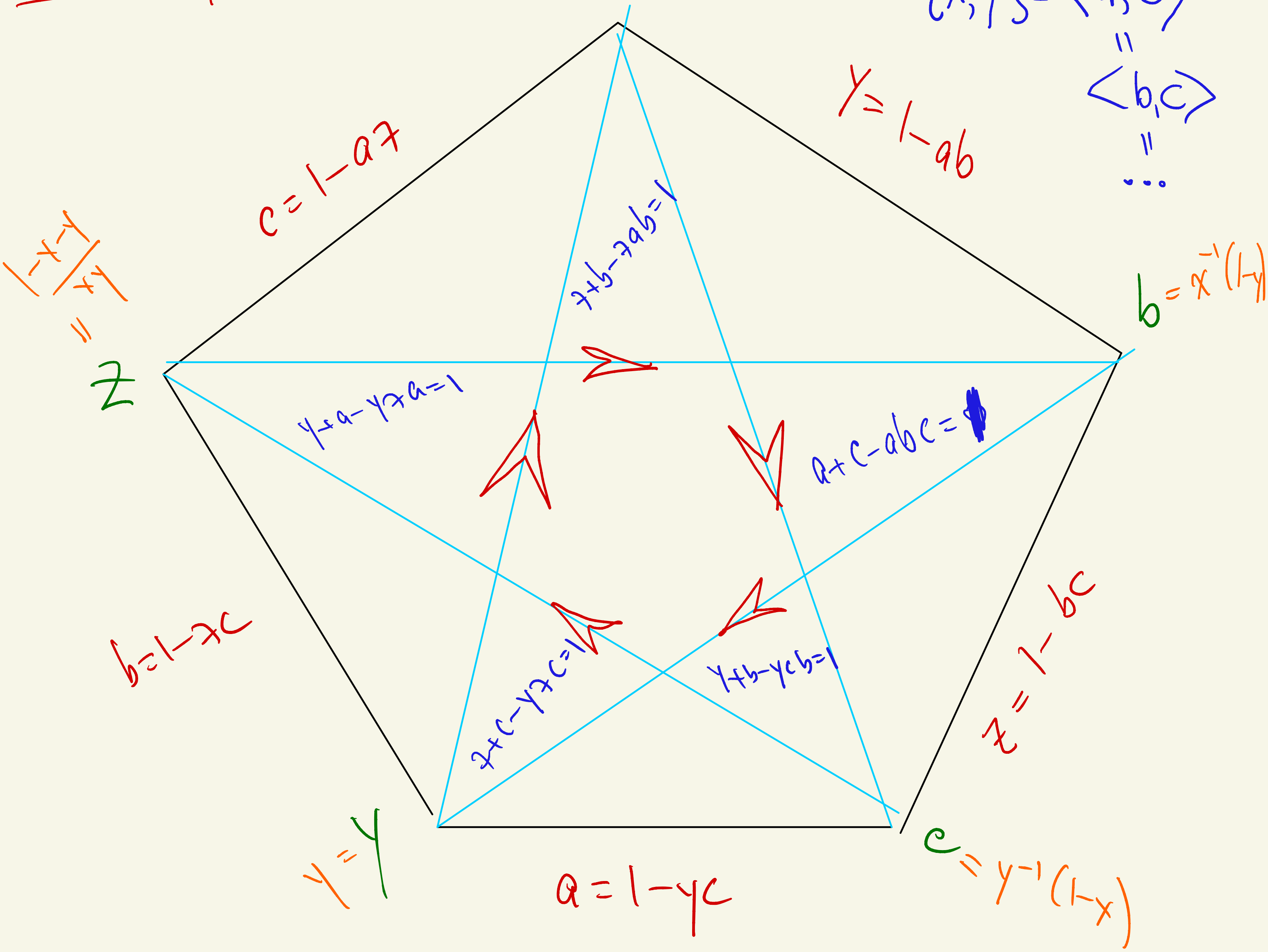
Also: $\tilde{a} = q^{-1}a;\ \tilde{c} = q^{-1}c$



These are $\mathbb{Z}/5$-invariant relations on $\mathcal{A}$. Since $\langle a, b\rangle = \langle b, c\rangle$, we have $\langle \tilde{a}, b\rangle = \langle b, \tilde{c}\rangle = \langle \tilde{c}, y\rangle = \langle y, z\rangle = \langle z, \tilde{a}\rangle$ in K_2^{rel}!

Classically:

$\{x,y\} = \langle a,b\rangle = \langle b,c\rangle = \dots$



Cluster transformation T:

$$\langle a,b\rangle + \langle c,b\rangle = \langle a+c-abc, b\rangle$$

$$y \Longrightarrow x \Longrightarrow \frac{1-x}{y}$$

$$T^5 = 1 \qquad \{Tx, Ty\} = \{x,y\}$$