Nonlinear (Dennis-Stein-Loday) HC.

$$CL_{m,n} = \left\{ \left\langle a_0, \dots, a_{j-1}, \begin{bmatrix} a_j^{(1)} \\ \vdots \\ a_j^{(m)} \end{bmatrix}, a_{j+1}, \dots, a_n \right| \right.$$

$$\left. 0 \le j \le n;\ 1 - a_0 \dots a_{j-1} a_j^{(k)} a_{j+1} \dots a_n \in GL(A) \right\}$$

$a_j \in M(A)$;

$(M(A) = \varinjlim M_n(A))$

Subject to:

★ $m=1$: $\langle a_0, \dots, [a_j], \dots, a_n \rangle$ all the same, den. by $\langle a_0, \dots, a_j, \dots, a_n \rangle$;

Also put $CL_{0,n} = \{*\}$

$CL_{*,\bullet}$ is a (simplicial)$\times$(cyclic) set (i.e. a cyclic object in simplicial sets):

The cyclic structure:

$$d_j\langle a_0,\dots,a_n\rangle = \langle a_0,\dots,a_j a_{j+1},\dots,a_n\rangle, \quad 0\le j<n;$$

☆ $$d_n\langle a_0,\dots,a_n\rangle = \langle a_n a_0, a_1,\dots,a_{n-1}\rangle;$$

$$T\langle a_0,\dots,a_n\rangle = \langle a_n, a_0,\dots,a_{n-1}\rangle$$

where a_i may be $a_i\in \mathcal{M}(A)$ or $\begin{bmatrix} a_i^{(1)}\\ \vdots \\ a_i^{(m)}\end{bmatrix}$ and

$$a\begin{bmatrix} a^{(1)}\\ \vdots \\ a^{(m)}\end{bmatrix} = \begin{bmatrix} aa^{(1)}\\ \vdots \\ aa^{(m)}\end{bmatrix}; \qquad \begin{bmatrix} a^{(1)}\\ \vdots \\ a^{(m)}\end{bmatrix}\cdot a = \begin{bmatrix} a^{(1)}a\\ \vdots \\ a^{(m)}a\end{bmatrix}$$

The simplicial structure:

$$d_k\left\langle a_0,\dots,\begin{bmatrix} a_j^{(1)}\\ \vdots \\ a_j^{(m)}\end{bmatrix},\dots,a_n\right\rangle = \left\langle a_0,\dots,\begin{bmatrix} a_j^{(2)}\\ \vdots \\ a_j^{(m)}\end{bmatrix},\dots,a_n\right\rangle, \quad k=0;$$

$$\left\langle a_0,\dots,\begin{bmatrix} a_j^{(1)}\\ \vdots \\ a_j^{(k)}+a_j^{(k+1)}\\ \vdots \\ a_j^{(m)}\end{bmatrix},\dots,a_n\right\rangle, \quad 1\le k<m; \qquad \left\langle a_0,\dots,\begin{bmatrix} a_j^{(1)}\\ \vdots \\ a_j^{(m-1)}\end{bmatrix},\dots,a_n\right\rangle \quad k=m$$

Here

$$a' \underset{j}{+} a'' = a' + a'' - a' \cdot a_{j+1} \cdots a_n a_0 \cdots a_{j-1} \cdot a''$$

Rmk An n-tuple $a_0, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ defines a monoid

$$\{a' \in M(A) \mid 1 - a_0 \cdots a_{j-1} \cdot a \cdot a_{j+1} \cdots a_n \in GL(A)\}$$

with the operation $\underset{j}{+}$; this monoid maps to $GL(A)$:

$$a \mapsto 1 - a_0 \cdots a_{j-1} a a_{j+1} \cdots a_n$$

Formulas ☆ turn it into a cyclic object in monoids.
If one disregards ☆ then $CL_{*,\bullet}$ (as a bisimplicial set) is the nerve of the simplicial category: $n \mapsto \coprod_{j;a_0,\dots,\hat{a}_j,\dots,a_n} \mathrm{Monoid}_{a_0,\dots,\hat{a}_j,\dots,a_n}$

Conjecturally:

$$\text{Maps}(\Delta^*\times\Delta^\cdot, |\ldots|)$$
$$\|$$
$$\underset{\Lambda^{op}}{\text{map hocolim}}\; CL_{*,\cdot}(A) \longrightarrow \mathrm{Sing}\,|BGL_*(A\{\Delta^\cdot\})|$$

(Recall: $A\{\Delta^n\} = A\{t_0,\dots,t_n\}/(\sum_j t_j - 1)$

a simplicial algebra. The R.H.S. computes $K_\cdot(A)$).

See: Notes on Dennis-Stein symbols.

Also note:

$$CL_{*,\cdot}(A) \longleftarrow BGL(A)$$

(maps to the $n=0$ part)

Most probably: extends to $BGL(A)^+$.

Indeed: $\pi_1 |CL_{*,\cdot}(A)|$ is Abelian.

$$\|$$
$GL(A)/\sim$; $1-ab \sim 1-ba$ (if belong to GL)

$$\langle E^a_{12}, E^b_{21}\rangle \underset{d_1}{\overset{d_0}{\rightrightarrows}} \begin{matrix} E^{ab}_{11} \\ E^{ba}_{22}\end{matrix}$$

$1-E^{ab}_{11} = e^{1-ab}_{11}$; $1-E^{ba}_{22} = e^{1-ba}_{22}$

Put $b=1$: $e^g_{11} \sim e^g_{22}$, $\forall g\in A^\times$

Or: for $g\in GL_n(A)$,

the image of $g_{i_1\dots i_n} \in GL(A)$ in π_1 does not depend

on $i_1 < \dots < i_n$. $\Rightarrow GL(A)/\sim$ is commutative.

Similarly: π_1 should act trivially on π_n.

(by the same reason, basically obvious)

If we believe in this:

$$HCL_i(A) := \pi_{i+1}\left(\operatorname{hocolim}_{\Lambda^{op}} CL_{*,\cdot}(A)\right) =: KL_i(A)$$

$$KL_i(A) \rightleftarrows K_i(A)$$

Mutually inverse on K_1.

What about on K_2? (Does this give a new proof of the Matsumoto theorem?)

In general?

Conclusion for now:

$$HCL_\bullet(A) = KL_{\bullet+1}(A)$$

Also Hochschild, negative/periodic cyclic version. Conjecturally

$$KL_\cdot(A) \rightleftarrows K_\cdot(A)$$

Should define explicitly

$$KL_\bullet(A) \to HC^-_\bullet(A)$$

And when A is pronilpotent, over $\mathbb{Q}$:

$$KL_\bullet(A) \to HC_{\bullet-1}(A)$$

When A is commutative, both are known; the latter given by explicit formulas using polylogarithms (later).

Also: seems well-suited for K-theory of cluster algebras.