Matrices: X_{ij}, $i,j \geq 0$

Dennis-Stein symbols

$a,b \in A$; $1-ab \in A^\times \Longleftrightarrow 1-ba \in A^\times \Longleftrightarrow \begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix} \in GL_2(A)$

In fact:

$$\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} = \begin{bmatrix} 1-ab & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1-ba \end{bmatrix}$$

Get paths:

$$\begin{tikzcd} \begin{bmatrix} 1-ab & 0 \\ 0 & 1 \end{bmatrix} \arrow[r, squiggly, dash, "\langle a,b\rangle_l"'] & \begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix} \arrow[r, squiggly, dash, "\langle a,b\rangle_r"'] & \begin{bmatrix} 1 & 0 \\ 0 & 1-ba \end{bmatrix} \end{tikzcd}$$

Rmk Paths in what sense? | a path $\sim$ a sequence of multiplying by elem. matr.

1) In $GL(A\{\Delta^*\})$: $A\{\Delta^n\} = A\{t_1, \ldots, t_n\} / (\Sigma t_i - 1)$

Simplicial group;

$$\langle a,b\rangle_l = \begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ bt & 1 \end{bmatrix}, \text{ etc.}$$

Or:

2) In the two-sided Volodin construction:

$W_{mn} \ni (\gamma_1, \ldots, \gamma_m; g; \delta_1, \ldots, \delta_n)$

$\exists \sigma: \gamma_i \in T^\sigma, \forall i$; $\exists \tau: \delta_j \in T^\tau, \exists \tau$; $g \in GL(A$

$\langle a,b\rangle$:

$$\begin{bmatrix} 1-ab & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\langle a,b\rangle_\ell} \begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix} \xrightarrow{\langle a,b\rangle_r} \begin{bmatrix} 1 & 0 \\ 0 & 1-ba \end{bmatrix}$$

Now we get two paths

$$\begin{tikzcd}
\begin{bmatrix} (1-ab)(1-ac) & 0 \\ 0 & 1 \end{bmatrix} \arrow[rr, bend left, "\langle a,b\rangle\langle a,c\rangle"] \arrow[rr, bend right, "\langle a,b+c-bac\rangle"'] \arrow[rr, phantom, "\text{Claim: \underline{homotopic}}"] & & \begin{bmatrix} 1 & 0 \\ 0 & (1-ba)(1-ca) \end{bmatrix}
\end{tikzcd}$$

<u>What we will show:</u>

$$\begin{tikzcd}
& \begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ -c & 1 \end{bmatrix} \arrow[dr, "\langle a,b\rangle_r\langle a,c\rangle_r"] & \\
\begin{bmatrix} (1-ab)(1-ac) & 0 \\ 0 & 1 \end{bmatrix} \arrow[ur, "\langle a,b\rangle_\ell\langle a,c\rangle_\ell"] \arrow[dr, "\langle a,b+c-bac\rangle_\ell"'] \arrow[rr, phantom, "\text{homotopic}"] & & \begin{bmatrix} 1 & 0 \\ 0 & (1-ba)(1-ca) \end{bmatrix} \\
& \begin{bmatrix} 1 & -a \\ -(b+c-bac) & 1 \end{bmatrix} \arrow[ur, "\langle a,b+c-bac\rangle_r"'] &
\end{tikzcd}$$

Proof We have

Notation: $X_{ij} = a$; e^a_{ij} = matrix X: $X_{kl} = \delta_{kl}$, $(k,l) \neq (i,j)$

$$e_{01}(a,b) = \begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix} \qquad i,j,\ldots \geq 0$$

$$\begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ -c & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1-ab & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1-ac & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (1-ab)(1-ac) & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b(1-ac) & 1 \end{bmatrix}\begin{bmatrix} 1 & -a(1-ca)^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -a \\ -(b+c-bac) & 1 \end{bmatrix} \cdot \mathrm{Ad}_{\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}}\left(\begin{bmatrix} 1 & -a(1-ca)^{-1} \\ 0 & 1 \end{bmatrix}\right)$$

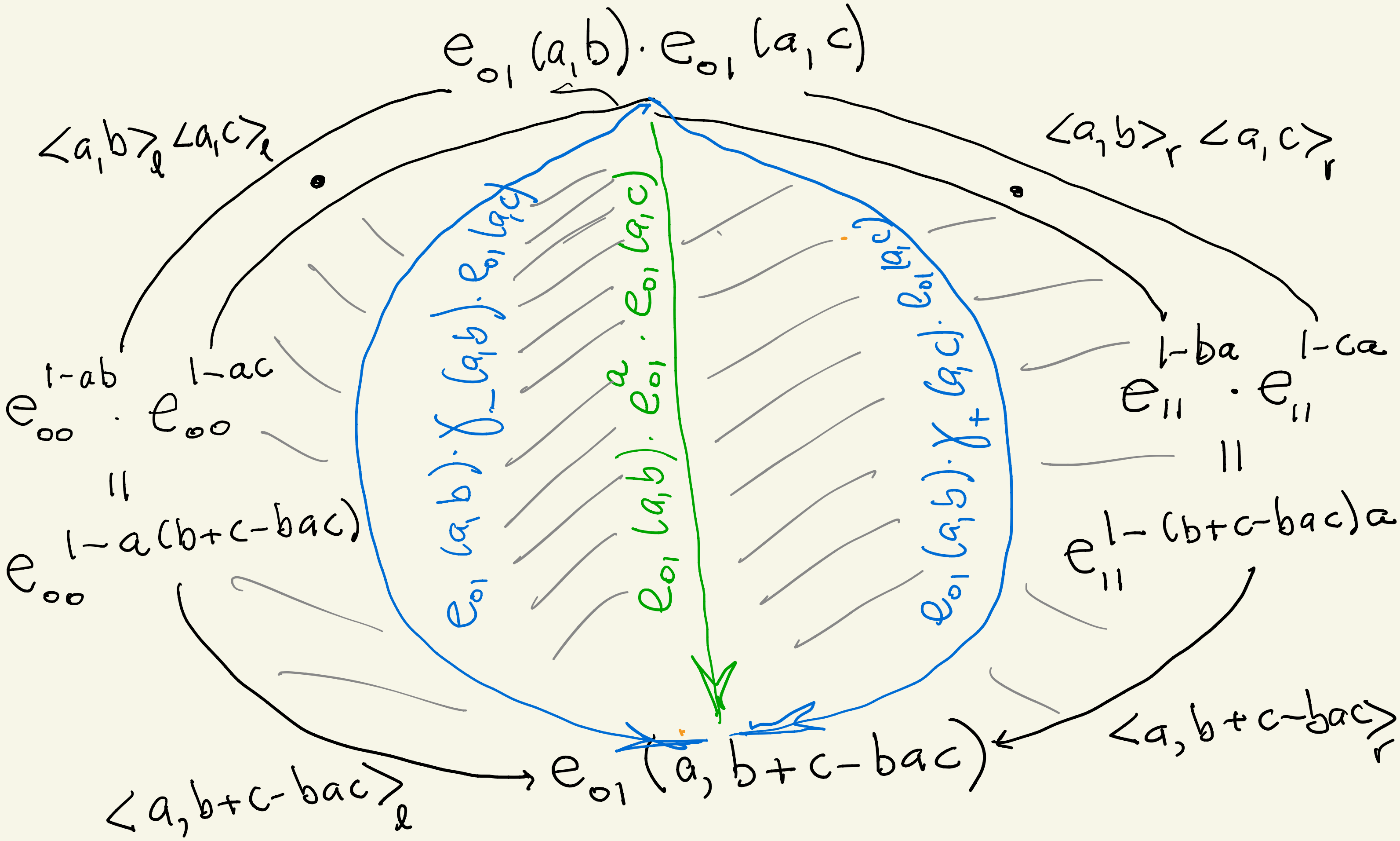
$$\langle a,b\rangle_l \langle a,c\rangle_l = \langle a, b+c-bac\rangle_l \cdot X_+(a,c)$$

$$\begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ -c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1-ba \end{bmatrix}\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1-ca \end{bmatrix}\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & -(1-ab)^{-1}a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -(1-ba)c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1-ba)(1-ca) \end{bmatrix}\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$

$$= \mathrm{Ad}_{\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}}\left(\begin{bmatrix} 1 & -a(1-ba)^{-1} \\ 0 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 & -a \\ -(b+c-bac) & 1 \end{bmatrix}$$

$$\langle a,b\rangle_r \langle a,c\rangle_r \doteq X_-(a,b) \cdot \langle a, b+c-bac\rangle_r$$



Here $\gamma_-(a,b) = \mathrm{Ad}^{-1}_{e_{01}(a,b)} X_-(a,b)$

$$\gamma_+(a,c) = \mathrm{Ad}^{-1}_{e_{01}(a,c)} X_+(a,c)$$

<u>Claim</u>: Both $\gamma_-(a,b), \gamma_+(a,c)$ are paths $1 \to e_{01}^a$.

<u>Pf</u> $\begin{bmatrix} 1 & -a \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -a \\ -c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -b & 1-ba \end{bmatrix}\begin{bmatrix} 1 & -a \\ -c & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a \\ -b-c+bac & 1 \end{bmatrix}$

<u>Claim</u>: Both paths $1 \to e_{01}^a$ are homotopic to the standard path.

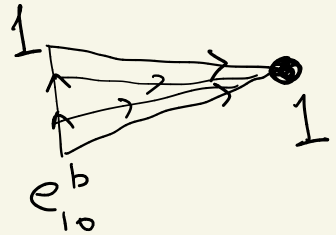
We use notation: $e_{ij}^{\underline{a}}$ if e_{ij}^a is an edge in the path, and e_{ij}^a if it is used as a factor in a matrix such as $e_{01}(a,b)$.

$$\gamma_-(a,b) = \mathrm{Ad}^{-1}_{e_{10}^{-b}\cdot e_{11}^{1-ba}\cdot e_{01}^{-a}}\, \mathrm{Ad}_{e_{10}^{-\underline{b}}}\left(e_{01}^{\underline{a(1-ba)^{-1}}}\right)$$

$$= \mathrm{Ad}_{e_{01}^{a}\, e_{11}^{(1-ba)^{-1}}\, e_{10}^{b-\underline{b}}}\left(e_{01}^{\underline{a(1-ba)^{-1}}}\right) = \mathrm{Ad}_{e_{01}^{a}}\left(\mathrm{Ad}_{e_{10}^{\tilde b-\underline{\tilde b}}}\left(e_{01}^{\underline{a}}\right)\right)$$

where $\tilde b = (1-ba)^{-1} b$

Finally, (for any $\tilde b$) $\mathrm{Ad}_{e_{10}^{\tilde b-\underline{\tilde b}}}\left(e_{01}^{\underline{a}}\right)$ is homotopic to $e_{01}^{\underline{a}}$: just deform $e_{10}^{\tilde b-\underline{\tilde b}}$ to 1 within paths



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Dennis-Stein symbols in K_2.

When $ab = ba$ and $1-ab \in A^\times$:

$$e_{00}^{1-ab} \text{———} e_{11}^{1-ba} \text{———} e_{00}^{1-ba}$$

gives a loop in $GL(A)$, therefore an elt $\langle a,b\rangle \in K_2(A)$. We get from the above:

$$\langle a,b\rangle\langle a,c\rangle = \langle a, b+c-bac\rangle$$

as well as $\langle a,b\rangle\langle b,a\rangle = 1.$

Also (next): $\langle ca,b\rangle\langle a,bc\rangle^{-1}\langle ab,c\rangle = 1$

Next:

$$e_{012}(a,b,c) := \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & -b \\ -c & 0 & 1 \end{bmatrix}$$

Also,

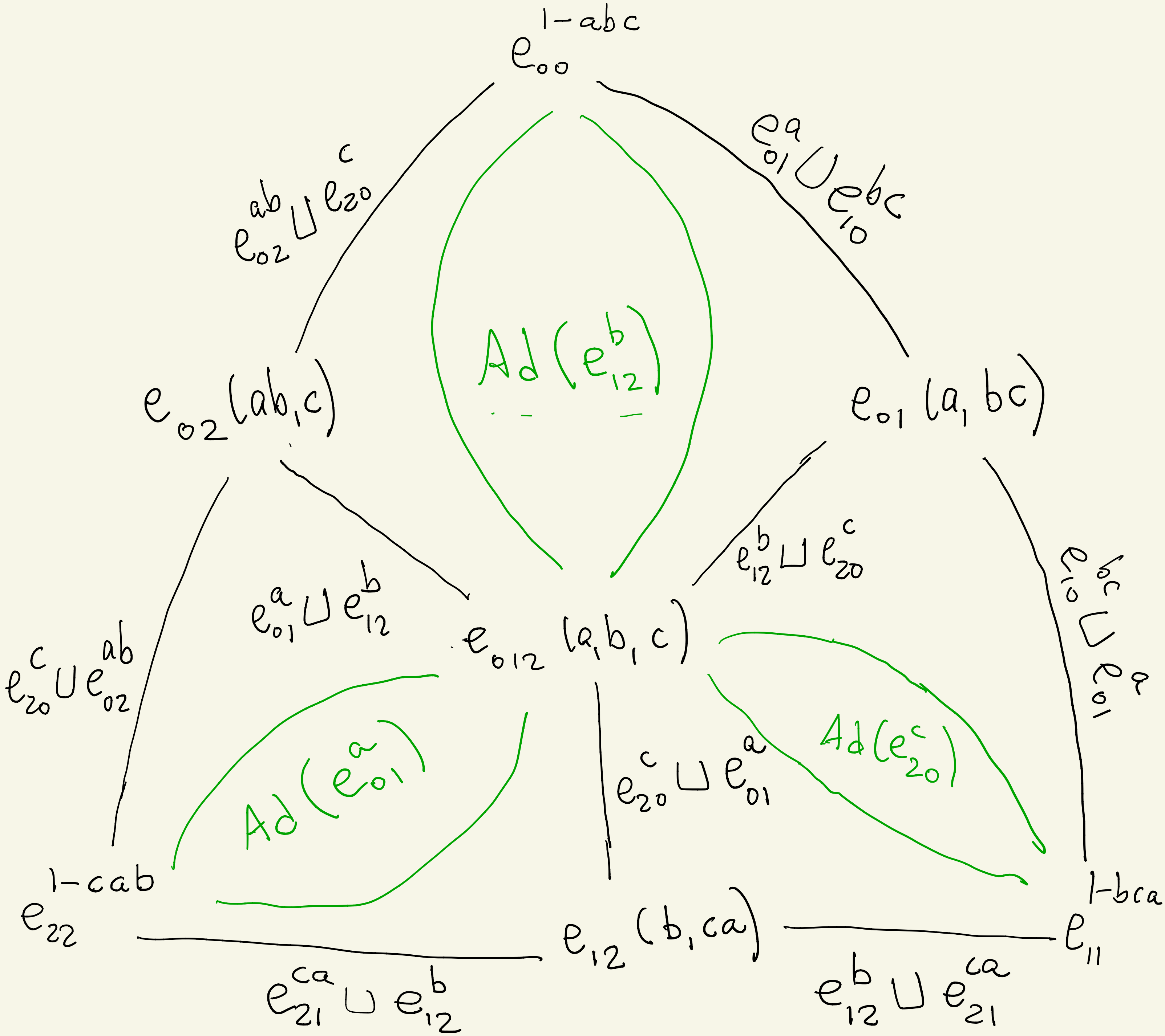
$$e_{ij}(a,b) := \begin{array}{cc} \begin{bmatrix} \ddots & & & & & & \\ & 1 & & & & & \\ & & 1 & \cdots & -a & & \\ & & \vdots & \ddots & \vdots & & \\ & & -b & \cdots & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{bmatrix} & \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \\ \begin{matrix} \qquad\qquad & i & \qquad\quad & j & \qquad\qquad \end{matrix} & \end{array}$$

Then

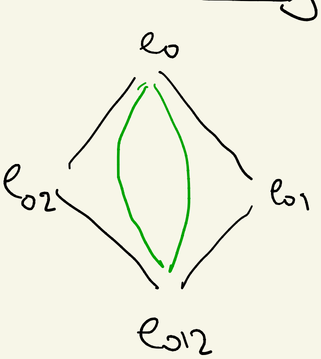
$$e_{12}^{b} \cdot e_{012}(a,b,c) \cdot e_{20}^{c} = e_{01}(a,bc)$$

$$e_{20}^{c} \cdot e_{012}(a,b,c) \cdot e_{01}^{a} = e_{12}(b,ca)$$

$$e_{01}^{a} \cdot e_{012}(a,b,c) \cdot e_{12}^{b} = e_{02}(ab,c)$$



<u>Meaning:</u>



$$e_{02}^{ab}\, e_{01}^{a} \sqcup e_{12}^{b}\, e_{20}^{c} =$$
$$e_{12}^{-b}\left(e_{01}^{a}\, e_{12}^{b} \sqcup e_{20}^{c}\, e_{10}^{bc}\right) e_{12}^{b}$$

Comparison to Steinberg symbols

$$e_{21}^{b}\, e_{12}^{-a} \begin{bmatrix} 1-ab & 0 \\ 0 & 1 \end{bmatrix} \cdot e_{21}^{-b}\, e_{12}^{a} : \text{path } e_{11}^{1-ab} \rightsquigarrow e_{22}^{1-ba}$$

$$e_{21}^{b}\, e_{12}^{-a} \begin{bmatrix} 1-ab & 0 \\ 0 & 1 \end{bmatrix} e_{21}^{-b}\, e_{12}^{a} \begin{bmatrix} (1-ab)^{-1} & 0 \\ 0 & 1 \end{bmatrix} : 1 \rightsquigarrow e_{11}^{(1-ab)^{-1}} \cdot e_{22}^{1-ab}$$

$$\Vert$$

$$e_{21}^{b}\, e_{12}^{-a}\, e_{21}^{-b(1-ab)^{-1}}\, e_{12}^{(1-ab)a}$$

Therefore $\langle a,b \rangle_{DS} = x_{21}^{b}\, x_{12}^{-a}\, x_{21}^{-b(1-ab)^{-1}}\, x_{12}^{(1-ab)a} \cdot h_{12}(1-ab)$

If $b \in A^\times$: $\quad \ker\left(\widehat{St}(A) \to E(A)\right)$

$$\langle a,b \rangle_{DS} = x_{12}^{b^{-1}} \cdot x_{12}^{-b^{-1}} \cdot x_{21}^{b} \cdot x_{12}^{-b^{-1}} \cdot \cancel{x_{12}^{b^{-1}-a} \cdot x_{12}^{-b^{-1}(1-ab)}} \cdot x_{12}^{b^{-1}(1-ab)}\, x_{21}^{-b(1-ab)^{-1}}\, x_{12}^{b^{-1}(1-ab)}$$

$$x_{12}^{-b^{-1}(1-ab)+(1-ab)a} \cdot h_{12}(1-ab)$$

$$= \mathrm{Ad}\left(x_{12}^{b^{-1}}\right)\left[w_{12}(-b^{-1}) \cdot w_{12}(b^{-1}(1-ab)) \cdot x_{12}^{(1-ab)b^{-1}(-1+ab)} \cdot h_{12}(1-ab) \cdot x_{12}^{b^{-1}}\right]$$

$$= \mathrm{Ad}\left(x_{12}^{b^{-1}}\right)\left[w_{12}(-b^{-1})\, w_{12}(b^{-1}(1-ab))\, \cancel{x_{12}^{(1-ab)^2(-b^{-1}+b^{-1})}} \cdot h_{12}(1-ab)\right]$$

$$= \mathrm{Ad}\left(x_{12}^{b^{-1}}\right)\left[w_{12}(-b^{-1})\, w_{12}(b^{-1}(1-ab))\, h_{12}(1-ab)\right]$$

$$\Vert$$

$$\left[h_{12}(-b^{-1})\, h_{12}(-b^{-1}(1-ab))^{-1}\, h_{12}(1-ab)\right]$$

Up to some sign issues, to be fixed:

$$\langle a,b\rangle_{DS} = \{b, 1-ab\}^{-1} = \{a, 1-ab\}$$

Note that we recover the properties of $\langle\ \rangle_{DS}$ from the properties of $\{,\}$:

$$\langle ab,c\rangle\langle a,bc\rangle^{-1}\langle ca,b\rangle = \{ab,1-abc\}\{a,1-abc\}^{-1}\{ca,1-abc\}$$

$$=\{abc,1-abc\}=1$$

$$\langle a,b\rangle\langle b,a\rangle = \{a,1-ab\}\{b,1-ab\} = \{ab,1-ab\}=1$$

$$\langle a,b\rangle\langle a,c\rangle = \{a,1-ab\}\{a,1-ac\} = \{a,1-a(b+c-bac)\}$$

$$=\langle a, b+c-bac\rangle$$

$v = -b^{-1} \qquad u = 1-ab$

$$h_{12}(v)\cdot h_{12}(uv)^{-1}\cdot h_{12}(u)$$

$$= \cancel{Ad\, h_{12}(v)}\Big)\left[h_{12}(uv)^{-1}\cdot h_{12}(u)\, h_{12}(v)\right]$$

$$\|$$

$$\cancel{Ad^{-1}\, h_{12}(uv)}\left[h_{12}(u)\, h_{12}(v)\, h_{12}(uv)^{-1}\right]$$

$$\|$$

$$\left[h_{12}(uv)\, h_{12}(v)^{-1} h_{12}(u)^{-1}\right]^{-1}$$

$$\|$$

$\langle a, b\rangle = \{1-ab, -b\} \qquad \{u, v\}^{-1} = \{1-ab, -b\}$

$\langle a, bc\rangle = \{1-abc, -bc\} \qquad + \{1-ab, -a\}$

$\langle ab, c\rangle = \{1-abc, -c\} \qquad \overline{\{1-ab, ab\} = 1}$

$\langle ca, b\rangle = \{1-abc, -b\}$