The Quillen + construction

X path connected; $N \trianglelefteq \pi_1(X)$ N perfect

$$X \xrightarrow{f} X^+$$

$$\pi_1(X) \underset{\text{proj}}{\twoheadrightarrow} \pi_1(X^+) = \pi_1(X)/N$$

$$f_*: H_\bullet(X, f^*L) \xrightarrow{\sim} H_\bullet(X^+, L)$$

$\forall$ loc syst L on X^+

Construction: i) $X \longrightarrow X_1$

$$\begin{array}{ccc} \hat{X} & \longrightarrow & \tilde{X}_1 \\ \downarrow & & \downarrow {\scriptstyle \pi_1(X)/N} \\ X & \longrightarrow & X_1 \end{array}$$

$\hat{X} = {}^{N}X/N$

$\pi_1(X_1) = \pi_1(X)/N$

just kill all (classes γ_α of) loops in N by 2-cells e_α.

$$\tilde{X} \longrightarrow \tilde{X}_1$$

$$\begin{array}{ccccccc}
\pi_2 \hat{X} & \longrightarrow & \pi_2(\tilde{X}_1) & \longrightarrow & \pi_2(\tilde{X}_1, \hat{X}) & \longrightarrow & \pi_1(\hat{X}) = N \\
\downarrow & & \downarrow \cong \text{ (Hurewicz Thm)} & & \downarrow & & \downarrow \\
H_2 \hat{X} & \longrightarrow & H_2(\tilde{X}_1) & \twoheadrightarrow & H_2(\tilde{X}_1, \hat{X}) & \longrightarrow & H_1(\hat{X}) = N^{ab} = 0
\end{array}$$

So: the images of 2-cells e_α in $H_2(\tilde{X}_1, \hat{X})$ come from $\pi_2(\tilde{X}_1)$.
Kill their image in $\pi_2(X_1)$ by 3-cells b_α. We get X^+.

Now (roughly):

$$C_3(\tilde{X}^+, \hat{X}) \xrightarrow[\cong]{\partial} C_2(\tilde{X}^+, \hat{X})$$

basis: $b_\alpha \mapsto$ basis: e_α

Therefore

$$C_\bullet(X^+, X; \mathbb{Z}[\pi_1(X)/N]) \cong 0$$

$$C_\bullet(X, f^*\mathbb{Z}[\pi_1(X)/N]) \xrightarrow{\simeq} C_\bullet(X^+, \mathbb{Z}[\pi_1(X)/N])$$

$$\Downarrow$$

$$C_\bullet(X, f^*L) \xrightarrow{\simeq} C_\bullet(X^+, L) \quad \forall L \text{ on } X^+$$

Thm

$$\begin{array}{ccc} X & \longrightarrow & X^+ \\ & {\scriptstyle f}\searrow & \swarrow {\scriptstyle \exists! \text{ up to homotopy}} \\ & Y & \end{array}$$

$$\begin{array}{ccc} f_*: \pi_1 X & \longrightarrow & \pi_1 Y \\ \cup\uparrow & & \uparrow \\ N & \longrightarrow & \{e\} \end{array}$$

Pf By obstruction theory:

1) obstruction to lift f to the 2-skeleton:

Can extend f

$$\begin{array}{ccc} \pi_1 X & \longrightarrow & \pi_1 X^+ \\ & \searrow & \swarrow \exists \text{ — in our case, YES} \\ & \pi_1 Y & \end{array}$$

2) Given an extension to the n-skeleton X^+_n:

obstruction to extending to X^+_{n+1} is

in $H^{n+1}(X^+, X; \pi_n Y)$ — loc syst on X via $\pi_1 X \to \pi_1 Y$

In our case: <u>all vanish.</u>

Rmk Easier case of obstruction theory:

$$\begin{array}{ccccc} Y_\bullet & \longleftarrow & A_\bullet & \hookrightarrow & X_\bullet \\ & \nwarrow & - - ? - - & \swarrow & \end{array}$$

complexes; assuming $X_{<0} = 0$; and

$$\begin{array}{ccccc} Y_i & \longleftarrow & A_i & \hookrightarrow & X_i \\ & \nwarrow & - - \exists - - & \swarrow & \end{array}$$

Consequtive obstructions in

$H^{n+1}(X_\bullet, A_\bullet; H_n(Y_\bullet))$

Homotopy fiber

$$\begin{array}{ccccc} F(R) & \to & BGL(R) & \to & BGL(R)^+ \\ \| & & \uparrow & & \uparrow \\ F(R) & \to & \widetilde{BGL(R)} & \to & \widetilde{BGL(R)^+} \end{array}$$

$\pi_1(X)/N$

$$\begin{array}{ccc} H_2 F & & \\ \| & & \\ 0 & 0 & \\ \| & \| & \\ H_0(\widetilde{BGL}^+, H_1 F) & H_1(\widetilde{BGL}^+, H_1 F) & H_2(\widetilde{BGL}^+, H_1 F) \\ H_0(BGL(R)^+, H_0(F(R))) & H_1(BGL(R)^+, H_0(F(R))) & H_2(\widetilde{BGL}^+, H_0 F) \\ & \cong \mathbb{Z} & \cong \mathbb{Z} \\ \uparrow 2 & \uparrow 2 & \uparrow 2 \\ H_0(\widetilde{BGL(R)}) & H_1(\widetilde{BGL(R)}) & H_2(\widetilde{BGL}) \end{array}$$

We conclude: $\tilde{H}_\bullet(F(R)) \simeq 0$.

Let $G = \pi_1(F(R))$. Another spectral sequence:

$$H_i(G, H_j(\widetilde{F(R)})) \Rightarrow H_{i+j}(F(R))$$

E_∞ is acyclic. $\Rightarrow H_1 G = H_2 G = 0$

$$\begin{array}{ccc} H_3(G, H_0\tilde{F}) & & \\ H_2(G, H_0\tilde{F}) & H_2(G, H_1\tilde{F}) & H_2(G, H_2\tilde{F}) \\ H_1(G, H_0\tilde{F}) & H_1(G, H_1\tilde{F}) & H_1(G, H_2\tilde{F}) \\ H_0(G, H_0(\widetilde{F(R)})) & H_0(G, H_1(\widetilde{F(R)})) & H_0(G, H_2(\widetilde{F(R)})) \\ & 0 & \end{array}$$

$(H_1\tilde{F} = 0$ since $\pi_1\tilde{F} = 0)$

$$F(R) \longrightarrow BGL \longrightarrow BGL^+$$

$$0 \to \pi_2 BGL^+ \to \pi_1 F(R) \to \pi_1 BGL \twoheadrightarrow \pi_1 BGL^+$$

$$\pi_2(BGL(A)^+) \rightarrowtail G \twoheadrightarrow E(A)$$

<u>AND</u>: $H_1 G = H_2 G = 0$

$$\Downarrow$$

<u>universal</u> central extension of $E(A)$

<u>Also</u>: $H_3(G) \xrightarrow{\sim} H_0(G, H_2 \tilde{F})$

$$H_0(G, H_2 \tilde{F}) \cong \pi_2 \tilde{F} \cong H_0(\pi_1 F, \pi_2 F)$$

$$H_3(G) \downarrow \pi_2 F(R)$$

$$K_3(R) \simeq \pi_2 F(R) \simeq H_0(\pi_1 F, \pi_2 F)$$

<u>fact</u>: $\pi_1 F \circlearrowright \pi_2 F$ trivially

b/c $\pi_1(F)$ acts trivially on $\ker(\pi_n F \to \pi_n E)$

$$F \to E \to B$$

fibration

<u>Cor</u>: $K_3(R) \simeq H_3(St(R))$

Thm $BGL(A)^+$ is a homotopy commutative homotopy associative H-space

the operation:

$$a \oplus b = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} & \cdots \\ & b_{11} & & b_{12} & & \\ a_{21} & & a_{22} & & & \\ & b_{21} & & b_{22} & & \\ \vdots & & & & \ddots & \end{bmatrix}$$

Problem: $(a \oplus b) \oplus c$ differ from $a \oplus (b \oplus c)$

$a \oplus b$ differ from $b \oplus a$

by conjugation. But

Conjugation acts homotopically trivially on $BGL(A)^+$.

Main step in the proof: Define an almost conjugation to be a homom $GL(A) \to GL(A)$

given by

$$u_\bullet(a)_{kl} = \begin{cases} a_{ij} & \text{if } k=u(i),\ l=u(j) \\ \delta_{ij} & \text{otherwise} \end{cases}$$

for a given embedding $u : \mathbb{N} \hookrightarrow \mathbb{N}$.

(If u is a bijection, this is a conjugation by a permutation matrix).

Claim: An almost conjugation is homotopic to identity on $BGL(A)^+$ (to which it obviously extends).

Key step for that: Conjugation by an element of a group acts trivially on homology $H_\cdot(G)$. Explicit homotopy:

$$(g_1, \dots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i (\sigma g_1 \sigma^{-1}, \dots, \sigma g_i \sigma^{-1}, \sigma, g_{i+1}, \dots, g_n)$$

From this: $u_\bullet : BE(A)^+ \to BE(A)^+$ is a homotopy equivalence because:

- it induces isom on $H_\cdot$ (Hurewicz)
- $E(A)^+$ is simply-connected

Also: $BE(A)^+ \underset{\text{claim}}{\simeq} BGL(A)^+$

We deduce:

$$\widetilde{BGL(A)^+} \xrightarrow[u_\circ]{\sim} \widetilde{BGL(A)^+}$$

$\pi_1(BGL(A)^+)$-equivariant; therefore

$$u_\circ : BGL(A)^+ \xrightarrow[\text{h.e.}]{\sim} BGL(A)^+$$

and from <u>that</u>: $u_\circ \simeq \mathrm{id}$.

<u>Rmk</u> Conjugation (e.g. by permutations) acts homotopically trivially e.g. on

$$BGL(A\{\Delta^\bullet\})$$