Star-exponentials on a complex symplectic manifold

(joint work with Pierre Schapira)

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Introduction and motivations

- Given a star-product $\star_\hbar$ on a symplectic manifold M, the star-exponential of $H\colon M \to \mathbb{R}$ is defined by the series:

$$Exp_{\star_\hbar}(\frac{tH}{i\hbar}) = \sum_{n\geq 0} \frac{1}{n!}(\frac{t}{i\hbar})^n H^{\star_\hbar n}$$

  (Here $\hbar$ is not a formal parameter but a positive real number.)
- It was introduced in [BFFLS 1978] as a tool to study the spectrum of observables without referring to an underlying Hilbert space.
- If $\hbar$ is a formal parameter, the star-exponential does not have any obvious meaning in the deformation quantization algebra $C^\infty(M)[[\hbar]]$. It would rather belong to $C^\infty(M)[[\hbar, \hbar^{-1}]]$.
- But $(C^\infty(M)[[\hbar, \hbar^{-1}]], \star_\hbar)$ is not an algebra.
- With P. Schapira, by using techniques from micolocal analysis, we have constructed an algebra of deformation quantization on the cotangent bundle of a complex manifold, containing the star-exponentials.

Example (Harmonic oscillator (BFFLS Ann. Phys. 1978))

$T^*\mathbb{R} = \mathbb{R}^2$ with Moyal product $\star_M$. Hamiltonian: $H(x,\xi) = \frac{1}{2}(\xi^2 + x^2)$

$$Exp_{\star_M}(\frac{tH}{i\hbar}) = \frac{1}{\cos(t/2)} \exp\big(\frac{(x^2+\xi^2)}{i\hbar}\tan(t/2)\big)$$

for $|t| < \pi$. The convergence is in $\mathcal{D}'(\mathbb{R}^2)$. It is a periodic distribution in t.

$$\frac{1}{\cos(t/2)} \exp\big(\frac{(x^2+\xi^2)}{i\hbar}\tan(t/2)\big) = \sum_{n\geq 0} \exp(-i(n+\frac{1}{2})t)\pi_n(x,\xi)$$

where

$$\pi_n(x,\xi) = 2(-1)^n \exp(-\frac{(x^2+\xi^2)}{\hbar}) L_n(\frac{2(x^2+\xi^2)}{\hbar}),$$

where the L_n's are the Laguerre polynomials.

$$H \star_M \pi_n = \hbar(n+1/2)\pi_n \qquad \pi_n \star_M \pi_{n'} = \delta_{nn'}\pi_n.$$

Example (Feynman Path Integral (GD LMP 1990))

- Normal star-product:

$$(f \star_N g)(\bar{z}, z) = fg + \sum_{n\geq 1} \frac{\hbar^n}{n!} \frac{\partial^n f}{\partial z^n} \frac{\partial^n g}{\bar{z}^n}.$$

- In the holomorphic representation of the CCR $[a, a^\dagger] = \hbar$
  $(af)(\bar{z}) = \hbar f'(\bar{z})$ $(a^\dagger f)(\bar{z}) = \bar{z} f(\bar{z})$
  the FPI takes the form (Faddeev, Les Houches, 1975):

$$\int \prod_s \frac{d\bar{\xi}_s d\xi_s}{2\pi i\hbar} \exp\left[\frac{1}{2}(\bar{z}\xi_t + z\bar{\xi}_0) - \frac{1}{\hbar}\int_0^t ds \frac{1}{2}(\bar{\xi}_s\dot{\xi}_s - \dot{\bar{\xi}}_s\xi_s) + H(\bar{\xi}_s, \xi_s)\right]$$

  integration is over paths $s \mapsto (\bar{\xi}_s, \xi_s)$ restricted to boundary conditions $\bar{\xi}_t = \bar{z}$ and $\xi_0 = z$.
- Heuristically:

$$''Exp_{\star_N}(\frac{tH}{i\hbar})(\bar{z}, z) = \exp(-\frac{1}{\hbar}\bar{z}z)\ FPI(t, H)(\bar{z}, z)''$$

The sheaf of microdifferential operators $\mathcal{E}_{T^*X}$

- Let X be a complex manifold.
- At the beginning of the 70's, Sato-Kashiwara-Kawai (and Louis Boutet de Monvel) have constructed the sheaf of microdifferential operators $\mathcal{E}_{T^*X}$.
- $\mathcal{E}_{T^*X}$ is a $\mathbb{C}^\times$-conic filtered sheaf of rings.
- Locally: $U \subset T^*X$, $(x,\xi) \in U$, a section $P \in \mathcal{E}_{T^*X}(U)$ is described by its total symbol:

$$\sigma_{\mathrm{tot}}(P)(x;\xi) = \sum_{-\infty<j\leq m} p_j(x;\xi), \qquad m \in \mathbb{Z}, \quad p_j \in \Gamma(U;\mathcal{O}_{T^*X}(j)).$$

- $\sigma_{\mathrm{tot}}(P)$ satisfies growth conditions (canonical estimates):

$$\begin{cases} \text{for any compact subset } K \text{ of } U \text{ there exist positive constants} \\ C,\varepsilon \text{ such that } \sup\limits_{(x;\xi)\in K} |p_j(x;\xi)| \leq C\varepsilon^{-j}(-j)! \text{ for all } j<0. \end{cases}$$

- If Q is an operator of total symbol $\sigma_{\mathrm{tot}}(Q)$, then the total symbol of the product $P \circ Q$ is given by the Leibniz product.

$$\sigma_{\mathrm{tot}}(P\circ Q) \quad = \quad \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_{\mathrm{tot}}(P)\partial_x^\alpha \sigma_{\mathrm{tot}}(Q).$$

The sheaf of microdifferential operators $\mathcal{E}_{T^*X}$

- Filtered by the order of operators: $\mathcal{E}_{T^*X} = \cup_{m\in\mathbb{Z}}\mathcal{E}_{T^*X}(m)$
- The associated graded sheaf of rings

$$\operatorname{gr}\mathcal{E}_{T^*X} \simeq \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_{T^*X}(j).$$

- Consider a *homogeneous* symplectic transformation

$$\varphi\colon T^*X \supset U \xrightarrow{\sim} V \subset T^*Y.$$

Then φ may be locally quantized as an isomorphism of filtered sheaves of rings

$$\Phi\colon \varphi_*\mathcal{E}_{T^*X}|_U \xrightarrow{\sim} \mathcal{E}_{T*Y}|_V.$$

Remark: This isomorphism exists locally and is not unique.

The field $\mathbf{k}$

Set $\widehat{\mathbf{k}} = \mathbb{C}[[\hbar, \hbar^{-1}]$. An element $a \in \widehat{\mathbf{k}}$ of order $m \in \mathbb{Z}$ is a formal series:

$$a = \sum_{-\infty < j \leq m} a_j \hbar^{-j}, \quad a_j \in \mathbb{C}.$$

One defines $\mathbf{k}$ as the subfield of $\widehat{\mathbf{k}}$ of series satisfying

there exist $C, \varepsilon > 0$ with $|a_j| \leq C\varepsilon^{-j}(-j)!$ for all $j < 0$.

The sheaf $\mathcal{W}_{T^*X}$

- If one forgets about the homogeneity of T^*X, there exists a *no more conic* filtered sheaf of **k**-algebras $\mathcal{W}_{T^*X}$.
- It is a special case of a more general construction (algebroid stacks) of deformation quantization of a complex symplectic manifold. [Kontsevich (LMP 2001) for the formal case, Polesello-Schapira (IMRN 2004) for the analytic case in the spirit of the construction by Kashiwara (1996) of the quantization for complex contact manifolds.]
- Introduce a new parameter $\hbar$ to replace homogeneity.
- The formal version of $\mathcal{W}_{T^*X}$ is similar to deformation quantization in the C^∞ setting.

The sheaf $\mathcal{W}_{T^*X}$

- Locally $\mathcal{W}_{T^*X}$ is described as follows.
  $U \subset T^*X$, a section $P \in \mathcal{W}_{T^*X}(U)$ has a total symbol

$$\sigma_{\mathrm{tot}}(P)(x;\xi) = \sum_{-\infty<j\le m} p_j(x;\xi)\hbar^{-j}, \qquad m \in \mathbb{Z}, \quad p_j \in \mathcal{O}_{T^*X}(U),$$

$$\begin{cases} \text{for any compact subset } K \text{ of } U \text{ there exist constants } C, \varepsilon > 0 \\ \text{such that } \sup\limits_{(x;\xi)\in K} |p_j(x,\xi)| \le C\varepsilon^{-j}(-j)! \text{ for all } j < 0. \end{cases}$$

- Its associated graded ring is

$$\mathrm{gr}\,\mathcal{W}_{T^*X} \simeq \mathcal{O}_{T^*X}[\hbar, \hbar^{-1}].$$

- The product is given by the Leibniz product:

$$\sigma_{\mathrm{tot}}(P) \star \sigma_{\mathrm{tot}}(Q) := \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_{\mathrm{tot}}(P) \cdot \partial_x^\alpha \sigma_{\mathrm{tot}}(Q).$$

The sheaf $\mathcal{W}_{T^*X}$

- A symplectic transformation

$$\varphi\colon T^*X \supset U \xrightarrow{\sim} V \subset T^*Y.$$

as in the case of $\mathcal{E}_{T^*X}$, φ can be locally quantized as an isomorphism of filtered sheaves of rings

$$\Phi\colon \varphi_*\mathcal{W}_{T^*X}|_U \xrightarrow{\sim} \mathcal{W}_{T*Y}|_V.$$

Again, this isomorphism exists locally and is not unique.

From $\mathcal{E}_{T^*X}$ to $\mathcal{W}_{T^*X}$

The sheaves $\mathcal{E}_{T^*X}$ and $\mathcal{W}_{T^*X}$ are linked as follows.
Let $t \in \mathbb{C}$ be the coordinate and define

$$\mathcal{E}_{T^*(X\times\mathbb{C}),\hat{t}} = \{P \in \mathcal{E}_{T^*(X\times\mathbb{C})};\ [P, \partial_t] = 0\}.$$

Set

$$T^*_{\tau\neq 0}(X \times \mathbb{C}) = \{(x, t; \xi, \tau); \tau \neq 0\}$$

and consider the map

$$\rho\colon\ T^*_{\tau\neq 0}(X \times \mathbb{C}) \to T^*X, \qquad \rho(x, t; \xi, \tau) = (x; \xi/\tau).$$

The ring $\mathcal{W}_{T^*X}$ on T^*X is given by

$$\mathcal{W}_{T^*X} := \rho_*(\mathcal{E}_{T^*(X\times\mathbb{C}),\hat{t}}|_{T^*_{\tau\neq 0}(X\times\mathbb{C})}).$$

(One should think of τ as being $\hbar^{-1}$.)

Outline

	commutative		noncommutative
	$\mathcal{O}_X^{s,\hbar}$	$\rightsquigarrow$	$\mathcal{W}_{T^*X}^s$ (resolvent)
Laplace	$\downarrow$		$\downarrow$
	$\mathcal{O}_X^{t,\hbar}$	$\rightsquigarrow$	$\mathcal{W}_{T^*X}^t$ (exponential)

- $\frac{1}{s-H} \in \mathcal{W}_{T^*X}^s$
- $\exp(\frac{tH}{\hbar}) \in \mathcal{W}_{T^*X}^t$
- $\frac{\partial}{\partial t}\Phi(t) = \frac{1}{\hbar}H\Phi(t), \quad \Phi(0) = 1$

The sheaf $\mathcal{O}_X^{s,\hbar}$

Definition ($\mathcal{O}_X^{\hbar}$)

We denote by $\mathcal{O}_X^{\hbar}$ the filtered sheaf of **k**-algebras whose sections of order m on an open set $U \subset X$ are series

$$f(x,\hbar) = \sum_{-\infty < j \leq m} f_j(x)\hbar^{-j}, \quad f_j \in \mathcal{O}_X(U),$$

satisfying:

$$\begin{cases} \text{for any compact subset } K \text{ of } U \text{ there exist positive} \\ \text{constants } C, \varepsilon \text{ such that } \sup_K |f_j| \leq C\varepsilon^{-j}(-j)! \text{ for all } j < 0. \end{cases}$$

Let $\mathbb{C}_s$ denote $\mathbb{C}$ with coordinate s. Let $a \colon \mathbb{C}_s \times X \to X$ be the projection. The sheaf $\mathcal{O}_X^{s,\hbar}$ is defined as the derived proper direct image:

Definition

$\mathcal{O}_X^{s,\hbar} := R^1 a_! \mathcal{O}_{\mathbb{C}_s \times X}^{\hbar}$

The sheaf $\mathcal{O}_X^{s,\hbar}$ – The convolution algebra $H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C})$

- For a compact subset K of $\mathbb{C}$, we identify the vector space $H_K^1(\mathbb{C};\mathcal{O}_\mathbb{C})$ with the quotient space $\Gamma(\mathbb{C}\setminus K;\mathcal{O}_\mathbb{C})/\Gamma(\mathbb{C};\mathcal{O}_\mathbb{C})$ and, if $f\in\Gamma(\mathbb{C}\setminus K;\mathcal{O}_\mathbb{C})$, we still denote by f its image in $H_K^1(\mathbb{C};\mathcal{O}_\mathbb{C})$ or in $H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C})$.
- Let K and L be compact subsets of $\mathbb{C}$, let $f\in\Gamma(\mathbb{C}\setminus K;\mathcal{O}_\mathbb{C})$ and $g\in\Gamma(\mathbb{C}\setminus L;\mathcal{O}_\mathbb{C})$.
- The convolution product $f*_c g$ is given by

$$(f*_c g)(z)=\frac{1}{2i\pi}\int_\gamma f(z-w)g(w)dw \tag{1}$$

  where γ is a counter clockwise oriented circle which contains L and $|z|$ is chosen big enough so that $z+K$ is outside of the disc bounded by γ.
- $(H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C}),*_c)$ is an abelian algebra.
- $$\frac{1}{z^{n+1}}*_c\frac{1}{z^{m+1}}=\frac{(n+m)!}{n!m!}\frac{1}{z^{n+m+1}}.$$

The sheaf $\mathcal{O}_X^{s,\hbar}$

- For U open, relatively compact in X, sections of order m defined on a neighborhood of $\bar{U}$, are described by:

$$f(s,x,\hbar) = \sum_{j \leq m} f_j(s,x)\hbar^{-j},$$

- $f_j(s,x)$ is holomorphic on $(\mathbb{C}_s \setminus K_0) \times U$, K_0 compact independent of j.
- $\forall K \subset (\mathbb{C}_s \setminus K_0) \times U$, we have canonical estimates.
- $\mathcal{O}_X^{s,\hbar}$ is a sheaf of filtered $\mathbf{k}$-modules.
- Extend the convolution product to $\mathcal{O}_X^{s,\hbar}$ as follows. For two sections $f(s,x,\hbar) = \sum_{-\infty < j \leq m} f_j(s,x)\hbar^{-j}$ and $g(s,x,\hbar) = \sum_{-\infty < j \leq m'} g_j(s,x)\hbar^{-j}$ of $\mathcal{O}_X^{s,\hbar}$, set:

$$\begin{cases} f(s,x,\hbar) *_c g(s,x,\hbar) = \sum_{-\infty < j \leq m+m'} h_j(s,x)\hbar^{-j}, \\ h_k(s,x) = \sum_{i+j=k} \frac{1}{2i\pi} \int_\gamma f_i(s-w,x) g_j(w,x) dw. \end{cases}$$

Theorem

The sheaf $\mathcal{O}_X^{s,\hbar}$ has a structure of a filtered abelian $\mathbf{k}$-algebra.

The sheaf $\mathcal{O}_X^{t,\hbar}$

- Locally, sections of order m are described by:

$$U \subset X, \quad f(t,x,\hbar) = \sum_{j\in\mathbb{Z}} f_j(t,x)\hbar^{-j}, \quad f_j \in \Gamma(U, \mathcal{O}_{\mathbb{C}_t \times X|_{t=0}})$$

- $\forall K \subset U$, $\exists \eta > 0$:
  - $f_j(t,x)$ is holomorphic around $\{|t| \leq \eta\} \times K$
  - $\exists C, \varepsilon > 0$, $\sup\limits_{x\in K, |t|\leq\eta} |f_j(t,x)| \leq C\varepsilon^{-j}(-j)!$ for all $j < 0$.
  - $\exists M, R > 0$, $\sup\limits_{x\in K} |f_j(t,x)| \leq M\frac{R^{j-m}}{(j-m)!}|t|^{j-m}, \quad \forall |t| \leq \eta \ \forall j \geq m$.

The sheaf $\mathcal{O}_X^{t,\hbar}$

Facts:

- $\hbar^{-1}\colon \mathcal{O}_X^{t,\hbar}(m) \xrightarrow{\sim} \mathcal{O}_X^{t,\hbar}(m+1)$.
- If $f \in \mathcal{O}_X^{t,\hbar}(m)$ and $g \in \mathcal{O}_X^{t,\hbar}(m')$, then $fg \in \mathcal{O}_X^{t,\hbar}(m+m')$.

Theorem

$\mathcal{O}_X^{t,\hbar}$ *is a sheaf of abelian filtered* $\mathbf{k}$*-algebras.*

- The sheaf $\mathcal{O}_X^{t,\hbar}$ does not admit a formal counterpart.

Laplace transform

- The sheaves $\mathcal{O}_X^{t,\hbar}$ and $\mathcal{O}_X^{s,\hbar}$ are related by a kind of Laplace transform.
- On an open set U of X, consider a section:

$$f(s,x,\hbar) \in \Gamma((\mathbb{C}_s \setminus K) \times U; \mathcal{O}^{\hbar}_{\mathbb{C}_s \times X}).$$

i.e.

$$f(s,x,\hbar) = \sum_{-\infty < j \leq m} f_j(s,x)\hbar^{-j},$$

- Define the Laplace transform $\mathcal{L}(f)$ of f by

$$\mathcal{L}(f)(t,x,\hbar) = \frac{1}{2i\pi}\int_\gamma f(s,x,\hbar)\exp(st\hbar^{-1})\, ds,$$

where γ is a counter clockwise oriented circle centered at 0 with radius $R \gg 0$.

- $$\mathcal{L}(s^{-n-1}) = \hbar^{-n}t^n/n!, \qquad \mathcal{L}(\frac{1}{s-1}) = \exp(t\hbar^{-1}).$$

Theorem

The Laplace transform induces a **k***-linear isomorphism of filtered* **k***-algebras*

$$\mathcal{L}\colon \mathcal{O}_X^{s,\hbar} \xrightarrow{\sim} \mathcal{O}_X^{t,\hbar}.$$

- Note: A formal version $\hat{\mathcal{O}}_X^{s,\hbar}$ of $\mathcal{O}_X^{s,\hbar}$ does exist, but the Laplace transform cannot be applied to that formal version.
- Take a sequence $\{c_j\}_{j\leq 0}$ in $\mathbb{C}$ and consider the section f of $\hat{\mathcal{O}}_X^{s,\hbar}$:

$$f(s,\hbar) = \sum_{j\leq 0} \frac{c_j}{(s-1)} \hbar^{-j}.$$

- Then, formally, the Laplace transform of f is given by

$$\mathcal{L}(f)(t,\hbar) = \sum_{j\leq 0} \sum_{n\geq 0} c_j \frac{t^n}{n!} \hbar^{-n-j},$$

- The coefficient of $\hbar^0$ is $\sum_{n\geq 0} c_{-n} \frac{t^n}{n!}$, which does not converge around $t=0$ in general.

The sheaf $\mathcal{W}^s_{T^*X}$

Denote by s the coordinate on $\mathbb{C}_s$. Let $\mathcal{W}_{\mathbb{C}_s \times T^*X}$ be the subsheaf of $\mathcal{W}_{T^*(\mathbb{C}\times X)}$ consisting of sections not depending on ∂_s:

$$\mathcal{W}_{\mathbb{C}_s \times T^*X} = \{P \in \mathcal{W}_{T^*(\mathbb{C}_s \times X)} \mid [P, s] = 0\}.$$

As for $\mathcal{O}^{s,\hbar}_X$, the sheaf $\mathcal{W}^s_{T^*X}$ is defined as a proper direct image of $\mathcal{W}_{\mathbb{C}_s \times T^*X}$ by the projection $a\colon \mathbb{C}_s \times X \to X$:

Definition

The sheaf of $\mathbf{k}$-modules $\mathcal{W}^s_{T^*X}$ on T^*X is given by

$$\mathcal{W}^s_{T^*X} := R^1 a_! \, \mathcal{W}_{\mathbb{C}_s \times T^*X}.$$

The sheaf $\mathcal{W}^s_{T^*X}$

Theorem

(i) *The sheaf $\mathcal{W}^s_{T^*X}$ is naturally endowed with a structure of a filtered* $\mathbf{k}$*-algebra and* $\operatorname{gr}\mathcal{W}^s_{T^*X} \simeq R^1 a_! \mathcal{O}_{\mathbb{C}_s\times T^*X}[\hbar,\hbar^{-1}]$.

(ii) *Consider two complex manifolds X and Y, two open subsets $U_X \subset T^*X$ and $U_Y \subset T^*Y$ and a symplectic isomorphism $\psi\colon U_X \xrightarrow{\sim} U_Y$. Then, locally, ψ may be quantized as an isomorphism of filtered* $\mathbf{k}$*-algebras $\Psi\colon \mathcal{W}^s_{T^*X} \xrightarrow{\sim} \mathcal{W}^s_{T^*Y}$ such that the isomorphism induced on the graded algebras coincides with the isomorphism $R^1 a_! \mathcal{O}_{\mathbb{C}_s\times T^*X}[\hbar,\hbar^{-1}] \xrightarrow{\sim} R^1 a_! \mathcal{O}_{\mathbb{C}_s\times T^*Y}[\hbar,\hbar^{-1}]$ induced by ψ.*

(iii) *Assume X is affine. There is an isomorphism of filtered sheaves of* $\mathbf{k}$*-modules (not of algebras), called the "total symbol" morphism:*

$$\sigma_{\mathrm{tot}}\colon \mathcal{W}^s_{T^*X} \xrightarrow{\sim} \mathcal{O}^{s,\hbar}_{T^*X}. \tag{3}$$

The total symbol of a product is given by the Leibniz formula with a convolution product in the s variable.

The sheaf $\mathcal{W}^s_{T^*X}$

- Assume X affine. For each Stein open subset W of T^*X and each relatively compact open subset $U \Subset W$, sections of order m are described by:

$$\sigma_{\mathrm{tot}}(P)(s,x;\xi,\hbar) = \sum_{-\infty<j\leq m} p_j(s,x;\xi)\hbar^{-j}$$

- $p_j \in \Gamma((\mathbb{C}_s \setminus K_0) \times U, \mathcal{O}_{\mathbb{C}_s \times T^*X})$
- p_j satisfies canonical estimates on $K \subset (\mathbb{C}_s \setminus K_0) \times U$.
- The symbolic calculus is given by:

$$\sigma_{\mathrm{tot}}(P \circ Q) = \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}\partial_\xi^\alpha \sigma_{\mathrm{tot}}(P) *_c \partial_x^\alpha \sigma_{\mathrm{tot}}(Q).$$

The sheaf $\mathcal{W}^t_{T^*X}$

- $\mathcal{W}^t_{T^*X}$ is a filtered sheaf of **k**-algebras (algebra of exponentials).
- Locally, a section P of order m of $\mathcal{W}^t_{T^*X}$ on a Stein open subset V of T^*X and an open subset $U \Subset V$, $\sigma_{\mathrm{tot}}(P)$ is written as a series: $\sigma_{\mathrm{tot}}(P)(t,x;\xi,\hbar) = \sum_{j\in\mathbb{Z}} p_j(t,x;\xi)\hbar^{-j}$
- $\forall K \subset U \subset T^*X$, $\exists \eta > 0$:
  - $p_j(t,x;\xi)$ is holomorphic around $\{|t| \leq \eta\} \times K$
  - $\exists C, \varepsilon > 0$, $\sup\limits_{(x;\xi)\in K, |t|\leq\eta} |p_j(t,x;\xi)| \leq C\varepsilon^{-j}(-j)!$ for all $j < 0$.
  - $\exists M, R > 0$, $\sup\limits_{(x;\xi)\in K} |p_j(t,x;\xi)| \leq M\frac{R^{j-m}}{(j-m)!}|t|^{j-m}, \quad \forall |t| \leq \eta \ \forall j \geq m$.
- Symbolic calculus: usual Leibniz product.
- $\mathcal{W}^t_{T^*X}$ contains $\mathcal{W}_{T^*X}$ as a subalgebra.

Exponential elements

- Consider a section P of $\mathcal{W}_{T^*X}(0)$ on an open subset U of T^*X.
- For each compact subset K of U, there exists $R > 0$ such that the section $s - P$ of $\mathcal{W}^s_{T^*X}$ defined on $\mathbb{C}_s \times U$ is invertible on $(\mathbb{C}_s \setminus D(0,R)) \times K$, ($D(0,R)$ closed disc of radius R.)
- $\dfrac{1}{s-P}$ defines an element of $\Gamma(U; \mathcal{W}^s_{T^*X})$.
- Expand $\dfrac{1}{s-P}$ as $\sum_{n\geq 0} \dfrac{P^n}{s^{n+1}}$ and apply Laplace transform.
- Denote by $\exp(t\hbar^{-1}P)$ the image by $\mathcal{L}$ of $\frac{1}{s-P}$.

Exponential elements

Theorem

For $P \in \mathcal{W}_{T^*X}(0)$ (order 0), there is a section $\exp(t\hbar^{-1}P) \in \mathcal{W}^t_{T^*X}$ such that, (when X is affine):

$$\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P)) = \sum_{n \geq 0} \frac{(t\hbar^{-1}\sigma_{\mathrm{tot}}(P))^{\star n}}{n!},$$

where the star-product $f^{\star n}$ means the product given by the Leibniz formula.