Some exercises

1. Twistor construction. We regard $\mathbb{P}^1$ as the union of two affine charts $\mathbb{C}_z$ and $\mathbb{C}_{z'}$, with $z'=1/z$ on the intersection, and we set $S^1=\{|z=1\}=\{|z'=1|\}$. Let $\sigma:\mathbb{P}^1\to\overline{\mathbb{P}}^1$ be the anti-holomorphic involution $z\mapsto -1/\overline{z}$.

(1) Let $\mathscr{H}$ be a holomorphic vector bundle on $\mathbb{C}_z$.

- Show that $\sigma^*\overline{\mathscr{H}}$ is a holomorphic vector bundle on $\mathbb{C}_{z'}$.

(2) Let $\mathscr{C}:\mathscr{H}_{|S^1}\otimes_{\mathscr{O}_{S^1}}\sigma^*\overline{\mathscr{H}}_{|S^1}\to\mathscr{O}_{S^1}$ be $\mathscr{O}_{S^1}$-linear inducing an isomorphism $\mathscr{H}^\vee_{|S^1}\simeq\sigma^*\overline{\mathscr{H}}_{|S^1}$. Then $\mathscr{C}$ defines a holomorphic bundle $\widetilde{\mathscr{H}}$ on $\mathbb{P}^1$ by gluing $\mathscr{H}^\vee$ and $\sigma^*\overline{\mathscr{H}}$ along the previous isomorphism. Assume that $\mathscr{H}$ is equipped with a meromorphic connection ∇ having a pole at $z=0$ only.

- Show that $\sigma^*\overline{\mathscr{H}}$ has a meromorphic connection having a pole at $z'=0$ only.
- Show that if $\mathscr{C}$ is compatible with the connections, then the connection ∇ on $\mathscr{H}^\vee$ and that on $\sigma^*\overline{\mathscr{H}}$ are compatible and define a meromorphic connection ∇ on $\widetilde{\mathscr{H}}$ with pole at $0,\infty$ only.
- In such a case, show that $\mathscr{C}$ is uniquely determined from its restriction to the local system $\mathscr{L}=\ker\nabla$, which is a non-degenerate pairing $C:\mathscr{L}_{|S^1}\otimes_{\mathbb{C}_{S^1}}\iota^{-1}\overline{\mathscr{L}}_{|S^1}\to\mathbb{C}_{S^1}$, where ι is the involution $z\mapsto -z$ (note that, for $z\in S^1$, $\sigma(z)=\iota(z)$).

Remark. Given $(\mathscr{H},\nabla)$ and a non-degenerate pairing $C:\mathscr{L}_{|S^1}\otimes_{\mathbb{C}_{S^1}}\iota^{-1}\overline{\mathscr{L}}_{|S^1}\to\mathbb{C}_{S^1}$ as above, it is difficult to check whether $\widetilde{\mathscr{H}}$ is trivial, or to compute the Birkhoff-Grothendieck decomposition of $\widetilde{\mathscr{H}}$, as this reduces to a transcendental question.

(3) Assume that we are given $(\mathscr{H},\mathscr{C})$ as above. Show that $\widetilde{\mathscr{H}}\simeq\sigma^*\overline{\widetilde{\mathscr{H}}}$. Conclude that, if $(\mathscr{H},\mathscr{C})$ is a *pure twistor of weight* 0, that is, if $\widetilde{\mathscr{H}}$ is the trivial bundle, then $H:=\Gamma(\mathbb{P}^1,\widetilde{\mathscr{H}})$ is equipped with a nondegenerate sesquilinear form.

2. Elementary $\mathbb{C}(\!(z)\!)$-vector spaces with connection. Let R be a finite dimensional $\mathbb{C}(\!(z)\!)$-vector space equipped with a connection ∇ having a *regular singularity*, i.e., there exists a basis of R in which $\nabla=d+A\mathrm{d}z/z$, A a constant matrix.

(1) Let $\varphi\in\mathbb{C}(\!(z)\!)$. Show that $\nabla+\mathrm{d}\varphi\,\mathrm{Id}$ is a connection which only depends on $\varphi \bmod \mathbb{C}[\![z]\!]$, that is, if $\varphi,\psi\in\mathbb{C}(\!(z)\!)$ are such that $\varphi-\psi\in\mathbb{C}[\![z]\!]$, then $(R,\nabla+\mathrm{d}\varphi\,\mathrm{Id})\simeq(R,\nabla+\mathrm{d}\psi\,\mathrm{Id})$.

(2) Show that if $\varphi\neq 0$ in $\mathbb{C}(\!(z)\!)/\mathbb{C}[\![z]\!]$, then $\ker\nabla=0$. Applying this to End, show the converse to the implication above.

(3) Let u be a new variable, let $\rho\in u\mathbb{C}[\![u]\!]$ with valuation $v_u(\rho)=p\geqslant 1$, and set $z=\rho(u)$. Show that $\mathbb{C}(\!(u)\!)$ is a $\mathbb{C}(\!(z)\!)$-vector space. Let R be a n-dimensional $\mathbb{C}(\!(u)\!)$-vector space. Show that R is a finite dimensional $\mathbb{C}(\!(z)\!)$-vector space and compute its dimension. It is denoted by ρ_*R.

(4) Assume R has a connection ∇ (w.r.t. to u). Show that $\nabla_{\partial_z} := \rho'(u)^{-1}\nabla_{\partial_u}$ defines a derivation of R as a $\mathbb{C}(\!(z)\!)$-vector space. Then (R, ∇_{∂_z}) is denoted $\rho_+(R, \nabla_{\partial_u})$.

(5) Let S be a m-dimensional $\mathbb{C}(\!(z)\!)$-vector space with a connection ∇ (w.r.t. z) and set $\rho^*S = \mathbb{C}(\!(u)\!) \otimes_{\mathbb{C}(\!(z)\!)} S$. Show that the formula $\nabla_{\partial_u}(1 \otimes s) = \rho'(u) \otimes \nabla_{\partial_z} s$ defines a connection on ρ^*S (w.r.t. u). It is denoted $\rho^+(S, \nabla)$.

(6) Let $\lambda \in u\mathbb{C}[\![u]\!]$ with $v_u(\lambda) = 1$. Compute $\lambda^+(S, \mathrm{d} + \mathrm{d}\psi \operatorname{Id} + A\mathrm{d}z/z)$ and $\lambda_+(R, \mathrm{d} + \mathrm{d}\varphi \operatorname{Id} + A\mathrm{d}u/u)$, $\varphi \in \mathbb{C}(\!(u)\!)$, $\psi \in \mathbb{C}(\!(z)\!)$ and A a constant matrix.

(7) Let (R, ∇) and (R', ∇') be two $\mathbb{C}(\!(u)\!)$-vector spaces with *regular* connection, and let $\lambda \in u\mathbb{C}[\![u]\!]$ with $v_u(\lambda) = 1$. Show that $\lambda_+(R, \nabla + \mathrm{d}\varphi \operatorname{Id}) \simeq (R', \nabla' + \mathrm{d}\psi \operatorname{Id})$ iff $\psi \circ \lambda \equiv \varphi \bmod \mathbb{C}[\![u]\!]$ and $(R, \nabla) \simeq (R', \nabla')$. (Hint: use the series $\rho(u)$ such that $\lambda \circ \rho = 1$ and show that $\lambda_+ = \rho^+$.)