An Introduction to Loose Legendrians in High Dimensions

Kylerec seminar

Wenyuan Li

Northwestern
Symplectic & contact manifolds

**Definition**

A symplectic manifold is a pair \((M, \omega)\) where \(\omega \in \Omega^2(M)\) is a closed 2-form such that \(\omega \wedge n \neq 0\) (where \(\dim M = 2n\)).
Definition

A symplectic manifold is a pair \((M, \omega)\) where \(\omega \in \Omega^2(M)\) is a closed 2-form such that \(\omega \wedge^n \neq 0\) (where \(\dim M = 2n\)).

Definition

A (cooriented) contact manifold is a pair \((Y, \xi)\) where \(\xi = \ker \alpha\) is a hyperplane distribution such that \(\alpha \wedge (d\alpha)^\wedge^n \neq 0\) (where \(\dim Y = 2n + 1\)).
Definition
A symplectic manifold is a pair \((M, \omega)\) where \(\omega \in \Omega^2(M)\) is a closed 2-form such that \(\omega^n \neq 0\) (where \(\dim M = 2n\)).

Definition
A (cooriented) contact manifold is a pair \((Y, \xi)\) where \(\xi = \ker \alpha\) is a hyperplane distribution such that \(\alpha \wedge (d\alpha)^n \neq 0\) (where \(\dim Y = 2n + 1\)).

Definition
An isotropic submanifold \(\Lambda\) in \((M, \omega)\) (resp. \((Y, \alpha)\)) is a submanifold such that \(\omega|_\Lambda = 0\) (resp. \(\alpha|_\Lambda = 0\)). When \(\dim \Lambda = n\), it is called a Lagrangian (resp. Legendrian) submanifold.
**Theorem**

*For any point in a symplectic (resp. contact) manifold, there is a neighbourhood that is symplectomorphic (resp. contactomorphic) to an open subset in $\mathbb{R}^{2n}_{\text{std}}$ (resp. $\mathbb{R}^{2n+1}_{\text{std}}$).*

**Remark**

There are also standard neighbourhood theorems for symplectic/contact submanifolds, isotropic/Lagrangian/Legendrian submanifolds, etc.

There are no 'local invariants' in symplectic/contact topology. They are more 'flexible' (more like topology).
Theorem

For any point in a symplectic (resp. contact) manifold, there is a neighbourhood that is symplectomorphic (resp. contactomorphic) to an open subset in $\mathbb{R}_{std}^{2n}$ (resp. $\mathbb{R}_{std}^{2n+1}$).

Remark

There are also standard neighbourhood theorems for symplectic/contact submanifolds, isotropic/Lagrangian/Legendrian submanifolds, etc.
Darboux & Weinstein theorems

**Theorem**

For any point in a symplectic (resp. contact) manifold, there is a neighbourhood that is symplectomorphic (resp. contactomorphic) to an open subset in $\mathbb{R}_{std}^{2n}$ (resp. $\mathbb{R}_{std}^{2n+1}$).

**Remark**

There are also standard neighbourhood theorems for symplectic/contact submanifolds, isotropic/Lagrangian/Legendrian submanifolds, etc.

There are no ‘local invariants’ in symplectic/contact topology. They are more ‘flexible’ (more like topology).
Question: Shall we expect any global flexibility behaviour?

Yes. There are some global flexibility results called $h$-principles. Question again: What is an $h$-principle?

Roughly speaking, $h$-principle enables one to reduce a symplectic/contact topology problem to an algebraic topology problem.
• **Question:** Shall we expect any global flexibility behaviour?

• **Answer:** Yes. There are some global flexibility results called $h$-principles.

Roughly speaking, $h$-principle enables one to reduce a symplectic/contact topology problem to an algebraic topology problem.
**Question:** Shall we expect any global flexibility behaviour?

**Answer:** Yes. There are some global flexibility results called $h$-principles.

**Question again:** What is an $h$-principle?
**Question:** Shall we expect any global flexibility behaviour?

**Answer:** Yes. There are some global flexibility results called $h$-principles.

**Question again:** What is an $h$-principle?

**Answer:** Roughly speaking, $h$-principle enables one to reduce a symplectic/contact topology problem to an algebraic topology problem.
An example in smooth topology

**Proposition**

For any \( n \in \mathbb{Z} \), there is an immersion \( f : S^1 \to \mathbb{R}^2 \) with \( \text{rot}(Df) = n \) (existence).

Any immersions \( f_0, f_1 : S^1 \to \mathbb{R}^2 \) such that \( \text{rot}(Df_0) = \text{rot}(Df_1) \) are homotopic through immersions (uniqueness).
An example in smooth topology

Proposition

For any $n \in \mathbb{Z}$, there is an immersion $f : S^1 \to \mathbb{R}^2$ with $\text{rot}(Df) = n$ (existence).

Any immersions $f_0, f_1 : S^1 \to \mathbb{R}^2$ such that $\text{rot}(Df_0) = \text{rot}(Df_1)$ are homotopic through immersions (uniqueness).

We are solving the classification problem in the following steps:
An example in smooth topology

Proposition

For any $n \in \mathbb{Z}$, there is an immersion $f : S^1 \to \mathbb{R}^2$ with $\text{rot}(Df) = n$ (existence).

Any immersions $f_0, f_1 : S^1 \to \mathbb{R}^2$ such that $\text{rot}(Df_0) = \text{rot}(Df_1)$ are homotopic through immersions (uniqueness).

We are solving the classification problem in the following steps:

1. decouple the map $f$ and its derivative $df$, extract the information separately and get necessary conditions;
Proposition

For any $n \in \mathbb{Z}$, there is an immersion $f : S^1 \to \mathbb{R}^2$ with $\text{rot}(Df) = n$ (existence).

Any immersions $f_0, f_1 : S^1 \to \mathbb{R}^2$ such that $\text{rot}(Df_0) = \text{rot}(Df_1)$ are homotopic through immersions (uniqueness).

We are solving the classification problem in the following steps:

1. decouple the map $f$ and its derivative $df$, extract the information separately and get necessary conditions;
2. show that this is enough.
What is an $h$-principle?

Suppose we want to find a map $f : M \to N$ such that

$$P(f, Df, D^2f, \ldots, D^k f) = 0 \text{ or } P(f, Df, \ldots, D^k f) \neq 0.$$
What is an $h$-principle?

Suppose we want to find a map $f : M \to N$ such that

$$P(f, Df, D^2f, \ldots, D^k f) = 0 \text{ or } P(f, Df, \ldots, D^k f) \neq 0.$$ 

1. First we try to find a ‘formal’ solution $(F_0, F_1, \ldots, F_k)$ such that

$$P(F_0, F_1, \ldots, F_k) = 0 \text{ or } P(F_0, F_1, \ldots, F_k) \neq 0.$$
What is an \( h \)-principle?

Suppose we want to find a map \( f : M \to N \) such that

\[
P(f, Df, D^2f, \ldots, D^k f) = 0 \text{ or } P(f, Df, \ldots, D^k f) \neq 0.
\]

1. First we try to find a 'formal' solution \((F_0, F_1, \ldots, F_k)\) such that

\[
P(F_0, F_1, \ldots, F_k) = 0 \text{ or } P(F_0, F_1, \ldots, F_k) \neq 0.
\]

2. Next we try to show that \((F_0, F_1, \ldots, F_k)\) is homotopic to a genuine solution \((f, Df, \ldots, D^k f)\) (in the jet bundle).
What is an \(h\)-principle?

Suppose we want to find a map \(f : M \rightarrow N\) such that

\[
P(f, Df, D^2f, ..., D^k f) = 0 \text{ or } P(f, Df, ..., D^k f) \neq 0.
\]

1. First we try to find a ‘formal’ solution \((F_0, F_1, ..., F_k)\) such that

\[
P(F_0, F_1, ..., F_k) = 0 \text{ or } P(F_0, F_1, ..., F_k) \neq 0.
\]

2. Next we try to show that \((F_0, F_1, ..., F_k)\) is homotopic to a genuine solution \((f, Df, ..., D^k f)\) (in the jet bundle).

If step 2 works, then we say that \(h\)-principle holds in this case.
formal solutions in symplectic/contact topology

What are formal solutions in the symplectic/contact world?
What are formal solutions in the symplectic/contact world?

**Definition**

An almost symplectic manifold is a manifold $M$ with a nondegenerate 2-form $\omega$ (i.e. $\omega^n \neq 0$).
formal solutions in symplectic/contact topology

What are formal solutions in the symplectic/contact world?

**Definition**
An almost symplectic manifold is a manifold $M$ with a nondegenerate 2-form $\omega$ (i.e. $\omega^n \neq 0$).

**Definition**
An almost contact manifold is a manifold $Y$ with a 1-form $\alpha$ and a 2-form $\beta$ so that $\alpha \wedge \beta^n \neq 0$. 
What are formal solutions in the symplectic/contact world?

**Definition**

An almost symplectic manifold is a manifold $M$ with a nondegenerate 2-form $\omega$ (i.e. $\omega^{\wedge n} \neq 0$).

**Definition**

An almost contact manifold is a manifold $Y$ with a 1-form $\alpha$ and a 2-form $\beta$ so that $\alpha \wedge \beta^{\wedge n} \neq 0$.

**Definition**

An formal isotropic embedding $\Lambda \to M$ (resp. $\Lambda \to Y$) is a smooth embedding $f : \Lambda \to M$ and an isotropic bundle map $F : T\Lambda \to f^*TM$ (resp. $F : T\Lambda \to f^*\xi \hookrightarrow f^*TY$) that is homotopic to $Df$. 
Theorem (Gromov 1986 for the symplectic version; Eliashberg-Mishachev 2002 (probably) for the contact version)

Isotropic embeddings into symplectic/contact manifolds of dimension less than $n$ satisfy all $h$-principles.
Theorem (Gromov 1986 for the symplectic version; 
Eliashberg-Mishachev 2002 (probably) for the contact version)

Isotropic embeddings into symplectic/contact manifolds of dimension less than \( n \) satisfy all h-principles.

Theorem (Gromov 1971; independently by Lees 1976 for the 
Lagrangian version and Duchamp 1984 for the Legendrian version)

Lagrangian/Legendrian immersions into symplectic/contact manifolds satisfy all h-principles.
Theorem (Gromov 1986 for the symplectic version; Eliashberg-Mishachev 2002 (probably) for the contact version)

Isotropic embeddings into symplectic/contact manifolds of dimension less than \( n \) satisfy all h-principles.

Theorem (Gromov 1971; independently by Lees 1976 for the Lagrangian version and Duchamp 1984 for the Legendrian version)

Lagrangian/Legendrian immersions into symplectic/contact manifolds satisfy all h-principles.

Questions: What about Lagrangian/Legendrian embeddings?
$h$-principle for loose Legendrians

Not all Legendrian submanifolds are flexible, and there are invariants defined by pseudo-holomorphic curves (Legendrian contact homology). However, in dimension at least 5, there is a class of Legendrians that are flexible:

Theorem (Murphy 2012)

When the dimension of the contact manifold is at least 5, there is a class of Legendrian submanifolds called loose Legendrians that satisfy all $h$-principles.
Not all Legendrian submanifolds are flexible, and there are invariants defined by pseudo-holomorphic curves (Legendrian contact homology). However, in dimension at least 5, there is a class of Legendrians that are flexible:

**Theorem (Murphy 2012)**

*When the dimension of the contact manifold is at least 5, there is a class of Legendrian submanifolds called loose Legendrians that satisfy all $h$-principles.*
Consider the standard contact manifold \((\mathbb{R}^{2n+1}, \alpha_{\text{std}} = dz - \sum_{i=1}^{n} y_i dx_i)\). As along as \(x_i\) and \(z\) are known, we can compute \(y_i = \partial z / \partial x_i\).
How do we characterize Legendrians?

Consider the standard contact manifold $(\mathbb{R}^{2n+1}, \alpha_{\text{std}} = dz - \sum_{i=1}^{n} y_i dx_i)$. As along as $x_i$ and $z$ are known, we can compute $y_i = \partial z / \partial x$.

**Definition**

The front projection in the contact manifold $\mathbb{R}^{2n+1}_{\text{std}}$ is

$$
\pi_{\text{front}} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{n+1}; (x_i, y_i, z) \mapsto (x_i, z).
$$
How do we characterize Legendrians?

Consider the standard contact manifold \((\mathbb{R}^{2n+1}, \alpha_{\text{std}} = dz - \sum_{i=1}^{n} y_i dx_i)\). As long as \(x_i\) and \(z\) are known, we can compute \(y_i = \partial z / \partial x_i\).

**Definition**

The front projection in the contact manifold \(\mathbb{R}^{2n+1}_{\text{std}}\) is

\[
\pi_{\text{front}} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{n+1}; (x_i, y_i, z) \mapsto (x_i, z).
\]

There cannot be vertical tangencies, but there can be singularities like cusps \((x^2, 3x, 2x^3)\) in \(\mathbb{R}^3\).
Definition of loose Legendrians

**Definition**

Let $I^3 \subset \mathbb{R}^3$ be a cube of side length 1, $\Lambda_0 \subset I^3$ be a Legendrian curve whose front projection is a zig-zag and is equal to \{(x, y, z)| y = z = 0\} near $\partial I^3$. Let $\rho > 1$. $D_\rho = \{(q, p) \in \mathbb{R}^{2(n-1)}| ||p|| \leq \rho, |q| \leq \rho\}$. $Z_\rho = \{(q, p)| p = 0, |q| \leq \rho\}$. Then a standard loose chart is

$$(I^3 \times D_\rho, \Lambda_0 \times Z_\rho).$$

A Legendrian embedding $\Lambda \hookrightarrow (Y, \xi)$ is loose if there exists a loose chart.
Definition of loose Legendrians

**Definition**

Let $I^3 \subset \mathbb{R}^3$ be a cube of side length 1, $\Lambda_0 \subset I^3$ be a Legendrian curve whose front projection is a zig-zag and is equal to $\{(x, y, z)|y = z = 0\}$ near $\partial I^3$. Let $\rho > 1$. $D_\rho = \{(q, p) \in \mathbb{R}^{2(n-1)}||p| \leq \rho, |q| \leq \rho\}$. $Z_\rho = \{(q, p)|p = 0, |q| \leq \rho\}$. Then a standard loose chart is

$$(I^3 \times D_\rho, \Lambda_0 \times Z_\rho).$$

A Legendrian embedding $\Lambda \hookrightarrow (Y, \xi)$ is loose if there exists a loose chart.
Definition of loose Legendrians

Here are two basic questions:

1. Why do we need a zig-zag?

2. Why do we require $\rho > 1$ for the contact neighbourhood $I^3 \times D^\rho$?

3. Why doesn't it work in dimension 3?
Definition of loose Legendrians

Here are two basic questions:

1. Why do we need a zig-zag?
2. Why do we require $\rho > 1$ for the contact neighbourhood $I_3 \times D_\rho$?
3. Why doesn't it work in dimension 3?
Here are two basic questions:

1. Why do we need a zig-zag?
Definition of loose Legendrians

Here are two basic questions:

1. Why do we need a zig-zag?
2. Why do we require $\rho > 1$ for the contact neighbourhood $I^3 \times D_\rho$?
Definition of loose Legendrians

Here are two basic questions:

1. Why do we need a zig-zag?
2. Why do we require $\rho > 1$ for the contact neighbourhood $I^3 \times D_{\rho}$?
3. Why doesn’t it work in dimension 3?
Why do we need a zig-zag?

Given a formal Legendrian embedding on the left, how one would try to get a genuine Legendrian embedding that is close to it.
Why do we need a zig-zag?

Given a formal Legendrian embedding on the left, how one would try to get a genuine Legendrian embedding that is close to it.

Fuchs-Tabachnikov (1997) showed that one can get Legendrian links in dimension 3 by adding zig-zags (called stablizations).
Why do we need a zig-zag?

Given a formal Legendrian embedding on the left, how one would try to get a genuine Legendrian embedding that is close to it.

Fuchs-Tabachnikov (1997) showed that one can get Legendrian links in dimension 3 by adding zig-zags (called stablizations).
Why do we need a zig-zag?

In smooth topology, Eliashberg-Mishachev (2011) considered embeddings that have zig-zag singularities, called wrinkled embeddings, and proved an $h$-principle.

Remark However, this construction in dimension 3 changes the formal data, i.e. the genuine Legendrian on the right is not homotopic to the formal Legendrian on the left in the jet bundle.
Why do we need a zig-zag?

In smooth topology, Eliashberg-Mishachev (2011) considered embeddings that have zig-zag singularities, called wrinkled embeddings, and proved an $h$-principle.
Why do we need a zig-zag?

In smooth topology, Eliashberg-Mishachev (2011) considered embeddings that have zig-zag singularities, called wrinkled embeddings, and proved an $h$-principle.

Remark

However, this construction in dimension 3 changes the formal data, i.e. the genuine Legendrian on the right is not homotopic to the formal Legendrian on the left in the jet bundle.
Why do we require $\rho > 1$?

- What we need is not a single loose chart, but in fact arbitrary loose charts.
Why do we require $\rho > 1$?

- What we need is not a single loose chart, but in fact arbitrary loose charts.
- Wrinkled embeddings have embryos (which are singularities in the Legendrian) that need to be resolved. For each embryo, there is one loose chart after resolution.
Why do we require $\rho > 1$?

- What we need is not a single loose chart, but in fact arbitrary loose charts.
- Wrinkled embeddings have embryos (which are singularities in the Legendrian) that need to be resolved. For each embryo, there is one loose chart after resolution.
Why do we require $\rho > 1$?

**Figure:** Shrinking the loose chart to get a long tube containing arbitrarily many loose charts. The red region contains many loose charts.
Properties of loose Legendrians

- Its Legendrian contact homology is 0.
Properties of loose Legendrians

- Its Legendrian contact homology is 0.
- **Question:** If the Legendrian contact homology is 0, is the Legendrian necessarily loose?
Its Legendrian contact homology is 0.

**Question:** If the Legendrian contact homology is 0, is the Legendrian necessarily loose?

Murphy-Siegel and Lazarev-Sylvan essentially showed that for some contact manifolds (that are boundaries of certain Liouville manifolds), this is not true. However, the question is still open in many important cases, e.g. for Legendrians in $\mathbb{R}^{2n+1}_{\text{std}}$ or $S^{2n+1}_{\text{std}}$. 
Consider the loose Legendrian sphere $\Lambda \subset S_{\text{std}}^{2n+1}$ (that is formally isotopic to the unknotted sphere). By attaching a handle $D^n \times D^n$ to $\Lambda$, we get a Liouville (Weinstein) manifold that is diffeomorphic to $T^*S^n$ (by $h$-principle) but has zero wrapped Fukaya category.
Applications to Liouville manifolds

- Consider the loose Legendrian sphere $\Lambda \subset S^{2n+1}_{\text{std}}$ (that is formally isotopic to the unknotted sphere). By attaching a handle $D^n \times D^n$ to $\Lambda$, we get a Liouville (Weinstein) manifold that is diffeomorphic to $T^* S^n$ (by $h$-principle) but has zero wrapped Fukaya category.

- Using Kirby calculus (for Weinstein manifolds), Casals-Murphy and Lazarev can simplify the handlebody decomposition of certain Liouville manifolds and get a number of cool results, e.g.

$$X_{1,b} = \left\{ xy^b + \sum_{i=1}^{n-1} z_i^2 = 1 \right\} \subset \mathbb{C}^{n+1}, \ b \geq 2$$

are all symplectomorphic and have 0 wrapped Fukaya categories/symplectic cohomologies.
Displacement energy

Given a Legendrian $\Lambda$, what is the $C^0$-norm of the Hamiltonian that is required to displace $\Lambda$, so that there are no Reeb chords between $\Lambda$ and $\varphi^1_H(\Lambda)$?
Displacement energy

- Given a Legendrian $\Lambda$, what is the $C^0$-norm of the Hamiltonian that is required to displace $\Lambda$, so that there are no Reeb chords between $\Lambda$ and $\varphi_h^1(\Lambda)$?

- Nakamura showed that, for loose Legendrians, the norm only depends on the loose chart. Dimitroglou Rizell and Sullivan’s work implies that loose Legendrians are the easiest to be displaced.
Ekholm-Eliashberg-Murphy-Smith used loose Legendrians to create exact Lagrangian immersions with few double points.