Conjugate fillings and Legendrian weaves

Tianjin University Geometry & Topology Seminar

Wenyuan Li

Northwestern
Legendrians links and Lagrangian fillings

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- $D^4$ has a symplectic structure $\omega_{st} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. This is an exact form $\omega_{st} = d\alpha_{st}$ where

$$\alpha_{st} = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2).$$

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- A Lagrangian in $D^4$ is a surface $L$ such that $\omega_{st}|_L = 0$ and an exact Lagrangian is a Lagrangian so that $\alpha_{st}|_L$ is exact. A Legendrian knot/link in $S^3$ is a knot/link $\Lambda$ such that $\alpha_{st}|_{\Lambda} = 0$. 
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- For a Legendrian knot/link in $S^3$, we can study exact Lagrangian surfaces in $D^4$ with boundary on the Legendrian knot/link, called Lagrangian fillings.
For a Legendrian in $J^1 M = T^* M \times \mathbb{R}_z$ with contact form $\alpha_{st} = dz - ydx$, the projection onto $M \times \mathbb{R}$ is called the front projection.
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Viewing $D^4$ as $T^* D^2$, when the Legendrian $\Lambda$ is contained in $S^* D^2$, the projection onto $D^2$ is also called front projection. When moreover $\Lambda \subset S^*_{y_2 < 0} D^2_{x_1, x_2} \cong J^1 D^1$, this recovers the standard front projection.
Recent results on Lagrangian fillings

One important question is to classify Lagrangian fillings of a given Legendrian link up to Hamiltonian isotopies relative to boundary. First of all, that means we need to find ways to construct and distinguish Lagrangian fillings.

- Pinching sequences of Reeb chords (Ekholm-Honda-Kalman '12);
- Conjugate Lagrangians of alternating Legendrians (Shende-Treumann-Williams-Zaslow '15);
- Lagrangian projections of free Legendrian weaves (Treumann-Zaslow '16, Casals-Zaslow '20).
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- Since 2020, infinitely many Lagrangian fillings have been constructed.
  
  1. Casals-Gao (Jan '20) showed infinite fillings using constructible sheaves (plus cluster algebra) for $(n, k)$-torus links for $n \geq 3, k \geq 6$ or $n, k \geq 4$;
  2. Casals-Zaslow (Jul '20) showed infinite fillings using constructible sheaves (plus cluster algebra) for a certain class of Legendrian links;
  3. Gao-Shen-Weng (Sep '20) showed infinite fillings using augmentation varieties (plus cluster algebra) for almost all positive braid closures;
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Comparison between different constructions

- One natural question is, however, how all 3 constructions of Lagrangian fillings are related.
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**Theorem (Hughes ’21)**

For max-tb Legendrian positive braid closures, there is a Hamiltonian isotopy from (1) pinching sequences of Reeb chords to certain (3) Lagrangian projections of Legendrian weaves.

**Theorem (Casals-L. 22’)**

For max-tb Legendrian positive braid closures (in fact a larger class), there is a Hamiltonian isotopy from (2) conjugate fillings of alternating Legendrians to certain (3) Lagrangian projections of Legendrian weaves.
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3. The boundary of the twisted ribbon is can be immersed into the plane, called the alternating strand diagram.
The alternating strand diagram lifts to the unit conormal bundle $S^* D^2$. This is called an alternating Legendrian.
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2. The conjugate surface admits an exact Lagrangian embedding into $T^*D^2$, whose projection onto $D^2$ covers the black/white regions. This is the conjugate Lagrangian filling of the alternating Legendrian link (Shende-Treumann-Williams-Zaslow ’15).
For any Legendrian (rainbow) closure of positive braid, there is a bicolored graph on $D^2$ whose alternating Legendrian is isomorphic to the positive braid closure.
Conjugate Lagrangians and sheaves

- The construction of conjugate Lagrangians and alternating Legendrians allows one to apply the techniques of constructible sheaves easily.
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The constructible sheaves we consider are sheaves on $D^2$ that are locally constant with respect to the stratification by the front projection $\pi(\Lambda)$ (with extra conditions). More precisely, they are sheaves with singular support on $\Lambda$.

We can consider microlocal rank 1 sheaves, which are constructible sheaves whose stalks jump by 1 when we cross the front projection.
Under some conditions, the moduli space of such sheaves forms a variety or a stack. A Lagrangian filling $L$ gives rise to an open toric chart $H^1(L; \mathbb{C}^\times)$ parametrizing rank 1 local systems on $L$, which is an open subset of the stack.
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For conjugate Lagrangian fillings, the corresponding sheaves are supported in the union of black and white regions with rank 1. Different conjugate Lagrangian fillings of the same Legendrian may give different toric charts.
Legendrian weaves

- Legendrian weaves are Legendrian surfaces in $J^1\Sigma = T^*\Sigma \times \mathbb{R}$. Their projection onto $\Sigma$ is a branched $n$-fold covering.
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The front projection of the Legendrian weave has $n$ sheets of different heights at a generic points. Different sheets may intersect along some line segments. We encode the information of the front by drawing these lines of intersection.
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- For a Legendrian weave in $J^1D^2$, the boundary is a Legendrian positive braid closure in $J^1S^1$.
- The Lagrangian projection of the weave into $T^*D^2$ is an immersed Lagrangian filling of the Legendrian link in $T^*D^2|_{S^1} \cong J^1S^1$. 
Recall that as $D^4 \cong T^* D^2$, the boundary $S^3 = T^* D^2|_{S^1} \cup S^* D^2$. The Legendrian boundary of a weave lives in $T^* D^2|_{S^1}$ instead of $S^* D^2$. We need to perturb the contact boundary into $S^* D^2$ in order to compare with conjugate Lagrangian fillings.
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When there are two transverse planes in the front projection, the corresponding flags satisfy a transverse condition.
One can project the sheaf on $D^2 \times \mathbb{R}$ with singular support on the weave onto a sheaf on $D^2$ with singular support on the link.
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- Different weaves may give different toric charts in the moduli space.
First, we consider the simple case of Legendrian \((2, k)\)-torus links (here we take \(k = 3\)).
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We claim that the following conjugate Lagrangian filling and Lagrangian projection of Legendrian weave are Hamiltonian isotopic.
We apply Reidemeister moves and to the Legendrian knot and guess the behaviour of the Lagrangian filling under these Reidemeister moves. If this is indeed the case, then the proof is completed.
We can summarize the local moves for Reidemeister moves as follows. This is the main technical result.
Here is a more complicated example converts a more general conjugate filling to a Legendrian weave. This is the building block for the conjugate filling and Legendrian weave associated to a positive braid closure.
Recall that constructible sheaves (of microlocal rank 1) with singular support on a Legendrian link $\Lambda$ forms a variety or a stack, and a Lagrangian filling defines an open toric chart $H^1(L; \mathbb{C}^\times)$.
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There exists a distinguished class of closed 1-cycles in $L$ that gives a basis of $H_1(L; \mathbb{Z})$. They give rise to standard coordinate functions in $H^1(L; \mathbb{C}^\times)$. Different Lagrangian fillings define different coordinates. They are related by a birational map.
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In many known cases, these moduli spaces are cluster varieties/stacks, meaning that it is covered (up to codimension 2) by toric charts, and the birational maps between toric charts are given by explicit combinatorial formulas.
Algebraic implications

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- There exists a distinguished class of closed 1-cycles in $L$ that gives a basis of $H_1(L; \mathbb{Z})$. They give rise to standard coordinate functions in $H^1(L; \mathbb{C}^\times)$. Different Lagrangian fillings define different coordinates. They are related by a birational map.
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- Conjugate fillings and Legendrian weaves give rise to different cluster coordinates. Our theorem implies that these coordinates are the same (due to Hamiltonian invariance of sheaf categories).
Thank you!