Recent developments in mathematical Quantum Chaos, II

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Recap: Quantum chaos of eigenfunctions

Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

of the Laplacian on a (mainly compact) Riemannian manifold (M, g). Suppose:
The geodesic flow $G^t : S^*_g M \to S^*_g M$ is ergodic, or Anosov or some other notion of "chaotic".

Problem How are eigenfunctions distributed in 'phase space' $S^*_g M$ (the unit cosphere bundle w.r.t. g) as $\lambda_j \to \infty$?

Semi-classical notation

It is often clearer to put $\hbar = \lambda_j^{-1}$ and denote the eigenfunctions by φ_j or just $\varphi_\hbar$. The eigenvalue problem is

$$\hbar^2 \Delta \varphi_\hbar = \varphi_\hbar.$$

Phase space distribution of eigenfunctions

It is measured by the matrix elements

$$\rho_j(A) = \langle A\varphi_j, \varphi_j \rangle$$

where $A \in \Psi^0(M)$ is a pseudo-differential operator of order zero. These talks are about the limits of $\rho_j(A)$ as $\lambda_j \to \infty \iff \hbar \to 0$. The weak limits of the ρ_j are invariant measures for the geodesic flow $g^t : S^*M \to S^*M$. What are the possible limit measures ν_0 (quantum limits).

Outline of talk II

- Entropy lower bounds for quantum limit measures (Anantharaman-Nonnenmacher) in the Anosov case.
- (If time permits) Applications of $\langle A\varphi_j, \varphi_j\rangle$ to nodal (zero) sets;
- Other important topics: variance sums, L^p norms of eigenfunctions (cf. Lecture Notes)

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Riemannian manifolds with Anosov geodesic flow

A geodesic flow g^t is called *Anosov* on on S_g^*M if the tangent bundle TS_g^*M splits into g^t invariant sub-bundles $E^u(\rho) \oplus E^s(\rho) \oplus \mathbb{R} X_H(\rho)$ where E^u is the unstable subspace and E^s the stable subspace. They satisfy: $\exists \lambda > 0$ s.th.

$$||dg^t v|| \leq Ce^{-\lambda t}||v||, \quad \forall v \in E^s, t \geq 0,$$

$$||dg^t v|| \leq Ce^{\lambda t}||v||, \quad \forall v \in E^u, t \leq 0.$$

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Facts about Anosov flows

The sub-bundles are integrable and give stable, resp. unstable foliations W^s, W^u. The leaves through x are denoted by $W^s(x), W^u(x)$.

Thus, the geodesic flow contracts everything exponentially fast along the stable leaves and expands everything exponentially fast along the unstable leaves.

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Compact hyperbolic manifolds

For most of this talk, we assume $(M, g) = \mathcal{H}^n/\Gamma$ is a compact hyperbolic surface (constant curvature -1). This simplifies all the formulae but does not simplify the proofs.

Lower bound on entropy of quantum limits

Anantharaman and Anantharaman-Nonnenmacher have proved two types of lower bounds on entropies of quantum limits:

- A lower bound on the topological entropy of the support of the quantum limit;
- A lower bound on the KS (Kolmogorov-Sinai) entropy of the quantum limit;
- G. Rivière has recently proved a sharp conjectured lower bound for negatively curved surfaces.

Definition of KS entropy h_μ

It is easiest to define local entropy of an invariant measure μ for g^t: Define the (Bowen) metric

$$d_T(\rho, \rho') = \max_{-T/2 \leq t \leq T/2} d(g^t \rho, g^t \rho'), \quad \rho, \rho' \in S^* M.$$

Let $B_T(\rho, r)$ be the Bowen ball of radius r around ρ wrt d_T.

Define the local entropy on an invariant measure μ for f by

$$h_\mu(\rho) = \lim_{\delta \to 0} \limsup_{T \to \infty} \frac{-\log \mu(B_T(\rho, \delta))}{T}.$$

Brin-Katok: the limit exists and

$$h_{KS}(\mu) = \int_{S^* M} h_\mu(\rho) d\mu(\rho).$$

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Bowen balls and geodesic tubes for compact hyperbolic surfaces

In this Anosov case, the Bowen ball $B_T(\rho, \epsilon_0)$ is the tubular neighborhood of radius $\epsilon_0 e^{-T/2}$ around the geodesic segment $[g^{-\epsilon_0}(\rho), g^{\epsilon_0}(\rho)]$. In the transverse direction to the orbit of ρ, it is a ball (or cube) of radius $\epsilon_0 e^{-T/2}$ in the $W^u - W^s$ directions.

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Entropy computations

1. $h_{KS}(\mu_\gamma) = 0$ (periodic orbit measure: Indeed, $\mu_\gamma(B_T(\rho, \epsilon)) = \epsilon$ for all T if $\rho \in \gamma$: the measure does not see transverse balls, along segments along γ. So, $-\frac{1}{T} \log \mu_\gamma(B_T(\rho, \epsilon)) = -\frac{\log \epsilon}{T} \to 0$ as $T \to \infty$ for $\rho \in \gamma$. Recall that $h_{KS}(\mu_\gamma) = \int_\gamma h_{\mu_\gamma}(\rho) ds(\rho)$.

2. Liouville measure μ_L: For a compact hyperbolic manifold of dimension d, $h_{KS}(\mu_L) = d - 1$. Indeed, for any ρ, $\mu_L(B_T(\rho, \epsilon)) = \epsilon e^{-(d-1)T}$ (the volume of the transverse cube to the orbit through ρ.)

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Topological entropy of an invariant set

Let $F \subset \Sigma = \{1, \ldots, \ell\}^{\mathbb{Z}}$ be a shift-invariant subset. Then, by definition,

$$h_{top}(F) \le \lambda \iff \forall \delta > 0,\ \exists C > 0$$

$$F \text{ can be covered by at most } Ce^{n(\lambda+\delta)}$$

$$\text{cylinders of length } n,\ \ \forall n.$$

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Anantharaman h_{top} bound

Theorem

Let ν_0 be a quantum limit measure. Then

- $h_{KS}(\nu_0) > 0$; *In fact, a positive proportion of the ergodic components of ν_0 must have entropy arbitrarily close to $\frac{\Lambda}{2}$.*
- $h_{top}(supp\, \nu_0) \geq \frac{\Lambda}{2}$ *($\Lambda = d - 1$ in constant curvature -1.)*

I.e. suppν_0 *cannot* be covered by $Ce^{n(\frac{\Lambda}{2}(1-\delta))}$ cylinder sets of length n.

Corollary

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Anantharaman-Nonnenmacher KS entropy bound in constant curvature

Theorem

Let ν_0 be a semiclassical measure associated to the eigenfunctions of the Laplacian on a compact hyperbolic manifold M. Then

$$h_{KS}(\mu) \geq \frac{d-1}{2}. \qquad (1)$$

They proved a similar result in the general Anosov case. They conjectured a sharp lower bound which has recently been proved by G. Rivière in dimension two.

Lindenstrauss-Soundararajan

THEOREM

Let $X = \Gamma\backslash\mathcal{H}^2$ be a compact arithmetic hyperbolic surface, or a finite area arithmetic hyperbolic surface defined by a congruence subgroup. Let $\{\varphi_j\}$ be an orthonormal basis of Hecke eigenfunctions (they are cuspidal L^2 eigenfunctions in the finite area case). Then this Hecke basis is QUE, i.e. its only quantum limit is Liouville.

Luo, Sarnak and Jakobson proved QUE for Eisenstein series in this case.

The proof is based on the special symmetries of joint eigenfunctions of the ring of Hecke operators on an arithmetic surface like $\Gamma = SL(2,\mathbb{Z})$. It does not use the relation between geodesic flow and the wave group.

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Back to Anantharaman's lower bound on $h_{top}(supp\nu_0)$

We will sketch the proof of Anantharman's lower bound when $F = \gamma$ is a closed (hyperbolic) orbit for the geodesic flow, and (M, g) is a compact hyperbolic manifold.

Thus, $h_{top}(\gamma) = 0$.

We want to prove that $\nu_0(\gamma) < 1$ for any quantum limit ν_0.

Symbolic coding and sequence space $\Sigma = \{1, \ldots, \ell\}^{\mathbb{Z}}$

Fix a partition $\{M_k\}$ of M and a corresponding partition T^*M_k of T^*M. Let P_{α_k} be the characteristic function of T^*M_k.

Let $\Sigma = \{1, \ldots, \ell\}^{\mathbb{Z}}$ where ℓ is the number of elements of the partition $\mathcal{P}^{0)}$.
Symbolic coding map: $v \in S^*M \to I(v) = (\alpha_k) \in \Sigma$ so that $g^n v \in P_{\alpha_j}$ for all $n \in \mathbb{Z}$.

The time one map g^1 conjugates to the shift $\sigma((\alpha_j)) = ((\alpha_{j+1}))$ on admissible sequences, i.e. sequences in the image of the coding map. But we mainly use Σ for quantum constructions.

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Cylinder sets

A cylinder set $[\alpha_0, \ldots, \alpha_{n-1}]$ of length n is the subset of Σ of sequences with the given segment in the $(0, \ldots, n-1)$ positions.

$$[\alpha_0, \ldots, \alpha_n] = \{\beta \in \Sigma : \beta_j = \alpha_j, \;\; j = 0, \ldots, n-1\}.$$

The set of such cylinder sets of length n is denoted Σ_n.

Classical measure of cylinder sets

An invariant measure ν_0 for g^t on S^*M corresponds to a shift-invariant measure on Σ.

$$\mu_0([\alpha_0, \dots, \alpha_n]) = \nu_0(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \cap \cdots \cap g^{-n+1} P_{\alpha_{n-1}}).$$

The measure $\nu_0(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \cap \cdots \cap g^{-n+1} P_{\alpha_{n-1}})$ is the probability wrt ν_0 that an orbit successively visits $P_{\alpha_0}, P_{\alpha_1}, \cdots$.

The entropy measures the exponential decay rate of these probabilities for large times.

Such a cylinder set is something like a small Bowen ball.

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KS entropy of an invariant measure μ

Given a partition

$$\mathcal{P} = (P_1, \ldots, P_k)$$

of $S_g^* M$, define the Shannon entropy of the partition by $h(\mu, \mathcal{P}) = \sum_{j=1}^{k} \mu(P_j) \log \mu(P_j)$.

Under iterates of the time one map g of the geodesic flow, the partition is refined to

$$\mathcal{P}_\alpha^n = \{P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \cap \cdots \cap g^{-n+1} P_{\alpha_{n-1}}\}.$$

One defines $h_n(\mu, \mathcal{P})$ to be the Shannon entropy of this partition and then defines

$$h_{KS}(\mu, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} h_n(\mu, \mathcal{P}).$$

Then $h_{KS}(\mu) = \sup_{\mathcal{P}} h_{KS}(\mu, \mathcal{P})$.

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Quantized cylinder set operators

On the quantum level, one quantizes the partition to define a smooth quantum partition of unity $\hat{P}_k$ by smoothing out the characteristic functions of M_k.

Given a cylinder set $[\alpha_0, \dots, \alpha_{n-1}]$, define the quantum cylinder set operator on $L^2(M)$

$$\hat{P}_\alpha = \hat{P}_{\alpha_{n-1}}(n-1)\hat{P}_{\alpha_{n-2}}(n-2)\cdots\hat{P}_{\alpha_0}, \tag{2}$$

where

$$\hat{P}(k) = U^{*k}\hat{P}U^k. \tag{3}$$

Here, $U = e^{i\sqrt{\Delta}}$ is the propagator at unit time (or in the semi-classical framework, $U = e^{i\hbar\Delta/2}$).

New Idea I: Quantum measures of cylinder sets

Definition: *Let $\varphi_\hbar$ be an eigenfunction of Δ. Define the associated quantum "measure" of cylinder sets $\mathcal{C} = [\alpha_0, \ldots, \alpha_{n-1}] \in \Sigma_n$ by*

$$\mu_\hbar([\alpha_0, \ldots, \alpha_{n-1}]) = \langle \hat{P}_{\alpha_{n-1}}(n-1) \cdots \hat{P}_{\alpha_0}(0)\varphi_\hbar, \varphi_\hbar \rangle. \qquad (4)$$

$\mu_\hbar([\alpha_0, \ldots, \alpha_{n-1}])$ *is the probability amplitude that the quantum particle in state $\varphi_\hbar$ the phase space visits the regions $P_{\alpha_0}, P_{\alpha_1}, \ldots, P_{\alpha_{n-1}}$ at times $0, 1, \ldots, n-1$.*

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Quantum measures are shift invariant

One has

$$\mu_\hbar([\alpha_0, \ldots, \alpha_{n-1}]) = \mu_\hbar(\sigma^{-1}[\alpha_0, \ldots, \alpha_{n-1}]).$$

Proof: This means

$$\langle U^{-(n-2)} P_{\alpha_{n-1}} U^{n-2} \cdots U P_{\alpha_0} U^{-1} \varphi_\hbar, \varphi_\hbar \rangle$$

$$= \langle U^{-(n-1)} P_{\alpha_{n-1}} U^{n-1} \cdots U P_{\alpha_0} \varphi_\hbar, \varphi_\hbar \rangle$$

which follows because $\rho_h(A) = \langle A\varphi_\hbar, \varphi_\hbar \rangle$ is an invariant state.

More on quantum measures of cylinder sets

The quantum measures are not "classical dynamical". They are not measures and they are not being defined in terms of symbols or sets in S^*M.

One "quantizes" the cylinder set $\mathcal{C} = [\alpha_0, \ldots, \alpha_{n-1}]$ as the operator

$$\hat{\mathcal{C}} = U^{-(n-1)} \hat{P}_{\alpha_{n-1}} U \hat{P}_{\alpha_{n-2}} \cdots U P_{\alpha_0} \tag{5}$$

Then

$$[\alpha_0, \ldots, \alpha_{n-1}] \to \mu_\hbar([\alpha_0, \ldots, \alpha_{n-1}]) = \langle \hat{\mathcal{C}} \varphi_\hbar, \varphi_\hbar \rangle \tag{6}$$

is a linear functional $\mu_\hbar$ on the span of the cylinder functions on Σ.

Big Problem: $\mu_\hbar$ are not positive measures since $\hat{\mathcal{C}}_\hbar$ is not a positive operator.

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Then

$$[\alpha_0, \ldots, \alpha_{n-1}] \to \mu_\hbar([\alpha_0, \ldots, \alpha_{n-1}]) = \langle \hat{\mathcal{C}} \varphi_\hbar, \varphi_\hbar \rangle \tag{6}$$

is a linear functional $\mu_\hbar$ on the span of the cylinder functions on Σ.

Big Problem: $\mu_\hbar$ are not positive measures since $\hat{\mathcal{C}}_\hbar$ is not a positive operator.

Classical vs quantum cylinders

It is important to note that

$$\hat{\mathcal{C}} = U^{-(n-1)}\hat{P}_{\alpha_{n-1}}U\hat{P}_{\alpha_{n-2}}\cdots UP_{\alpha_0} \tag{7}$$

is NOT the quantization of (the characteristic function of)

$$P_{\alpha_0} \cap g^{-1}P_{\alpha_1} \cap \cdots \cap g^{-n+1}P_{\alpha_{n-1}}.$$

Quantization is not a homomorphism.

New Idea II: Dispersive hyperbolic estimate in constant curvature

Theorem

(Anantharaman) Let $\mathcal{C}$ be a cylinder set of length n. Then

$$|\mu_\hbar(\mathcal{C})| \leq C_\beta(\hbar^{-1}e^{-n})^{d/2}$$

$$\textit{uniformly for } \; n \leq \beta\,|\log\hbar|$$

for any β.

Note that the estimate is only good if $n \geq |\log\hbar|$.

Similar (but more complicated) estimate exists for any Anosov geodesic flow. This is one of Anantharaman's key technical contributions. No analogue estimate exists at present for Bowen balls.

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Theorem for $F = \gamma$, (M, g) hyperbolic

For each n, γ may be covered by one n-cylinder

$$W_n(\gamma) = \{[\alpha_{-n/2}, \ldots, \alpha_0, \ldots, \alpha_{n/2}]\},$$

namely the one specified by the indices of the cells P_α that γ passes through starting from some fixed $x_0 \in \gamma$. Fix n.

A novel time average for $N \geq \vartheta |\log \hbar|$,

Let us temporarily pretend that $\mu_\hbar$ is also a positive measure. Then we would have

$$
\begin{aligned}
|\mu_\hbar(W_n)| &= |\frac{1}{N-n} \sum_{k=0}^{N-n-1} \mu_\hbar(\sigma^{-k} W_n)| \\
&= \left| \mu_\hbar \left(\frac{1}{N-n} \sum_{k=0}^{N-n-1} \mathbf{1}_{\sigma^{-k} W_n} \right) \right| \\
&\leq \sum_{\mathcal{C} \in \Sigma_N(W_n,\tau)} \mu_\hbar(\mathcal{C}) + \tau \sum_{\mathcal{C} \notin \Sigma_N(W_n,\tau)} \mu_\hbar(\mathcal{C}) \\
&= \mu_\hbar(\Sigma_N(W_n,\tau)) + \tau \mu_\hbar(\Sigma_N(W_n,\tau)^c) \\
&\leq (1-\tau)(1-\vartheta)) + \tau < 1.
\end{aligned}
$$

($\Sigma_N(W_n, \tau)$ on next page; uses main estimate)

$\Sigma_N(W_n, \tau)$

Define $\Sigma_N(W_n, \tau)$ to be the set of N- cylinders $[\alpha_0, \ldots, \alpha_{N-1}]$ such that

$$\frac{\#\{j \in [0, N-n] : [\alpha_j, \ldots, \alpha_{j+n-1}] \in W_n\}}{N-n+1} \geq \tau.$$

They correspond to orbits that spend a proportion $\geq \tau$ of their time in W_n (i.e. $e^{-n/2}$ close to γ.)

By the ergodic theorem, the orbits Liouville almost all $v \in S^*M$ spend a time proportion in W_n equal to the Liouville measure of $W_n \sim e^{-n(d-1)}$. So $\Sigma_N(W_n, \tau)$ has small Liouville measure.

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End of heuristic proof

Since W_n is fixed, the weak convergence $\mu_\hbar \to \mu_0$ implies,

$$|\mu_0(W_n)| \quad \leq \quad (1-\tau)\,(1-\vartheta)) + \tau < 1. \tag{8}$$

Since $\gamma \subset W_n$, the same estimate applies to γ. So $\mu_0(\gamma) < 1$. μ_0 is the transfer to Σ of ν_0, so this ends the heuristic proof.

Applications of QE to nodal sets of eigenfunctions

These talks have been devoted to finding limits of the states $\rho_j(A) = \langle A\varphi_j, \varphi_j \rangle$. Such states are fundamental in quantum mechanics.

But they are not so standard in classical PDE or geometric analysis. One application is to determining the structure as $\lambda_j \to \infty$ of the zero set

$$Z_{\varphi_j} = \{x \in M : \varphi_j(x) = 0\}.$$

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Distribution of nodal hypersurfaces

By the distribution of zeros we mean how they wind around on the manifold. If $U \subset M$ is a nice open set, we want to determine the total hypersurface volume $\mathcal{H}^{n-1}(Z_{\varphi_j} \cap U)$ as $\lambda_j \to \infty$, or more generally the integral of $f \in C(M)$ over the nodal set,

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1}. \tag{9}$$

Problem: How does $\frac{1}{\lambda_j} \langle [Z_{\varphi_j}], f \rangle$ behave as $\lambda_j \to \infty$.

Random wave model predicts $\sim \int_M f dV_g$.

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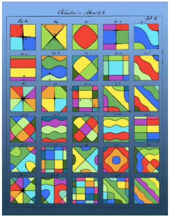
Chladni diagrams: Integrable case



Volumes of nodal hypersurfaces

Even for $f \equiv 1$ there is no asymptotic formula in the ergodic case. The best result to date on volumes of nodal hypersurfaces on general real analytic Riemannian manifolds is:

THEOREM

(Donnelly-Fefferman, Inv. Math. 1988) Suppose that (M, g) *is real analytic. Then*

$$c_1 \lambda \leq \mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \leq C_2 \lambda.$$

Yau conjectured this result for C^∞ metrics in 1982; it is still open in the non-analytic case.

Complex nodal hypersurfaces

Eigenfunctions of eigenvalue λ^2 are the analogues on manifolds of polynomials of degree λ. It is simpler to study complex zeros of analytic continuations of polynomials to $\mathbb{C}^m$.

Similarly, it is easier to determine the distribution law of complex nodal hypersurfaces

$$Z_{\varphi_j^{\mathbb{C}}} = \{\zeta \in M_{\mathbb{C}} : \varphi_j^{\mathbb{C}}(\zeta) = 0\},$$

where $\varphi_j^{\mathbb{C}}$ is the analytic continuation of φ_j to the complexification $M_{\mathbb{C}}$ of M. The metric induces a natural Kaehler form $\omega_g = \partial\bar{\partial}\rho$ on $M_{\mathbb{C}}$ (Guillemin, Lempert-Szoke).

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Complex nodal sets in the ergodic case

Theorem

(Z, '07) Assume (M, g) is real analytic and that the geodesic flow of (M, g) is ergodic. Let $Z_{\varphi^{\mathbb{C}}_{\lambda_j}}$ be tne complex zero set in $M_{\mathbb{C}}$ of $\varphi^{\mathbb{C}}_{\lambda}$. Then for almost all λ_j, and all f

$$\frac{1}{\lambda_j} \int_{Z_{\varphi^{\mathbb{C}}_{\lambda_j}}} f \omega_g^{n-1} \to \frac{i}{\pi} \int_{M_{\mathbb{C}}} f \bar{\partial} \partial \sqrt{\rho} \wedge \omega_g^{n-1}$$

(Also results of Hezari ('08) in the Schrödinger case and Toth-Z ('09) in the boundary case.)
As usual in quantum ergodicity, we may have to delete a possible subsequence of exceptional eigenvalues.

QE and Poincaré-Lelong

Main ideas of proof:

1. The delta function on the complex nodal set is given by $[Z_j] = \partial\bar{\partial} \log |\varphi_j^{\mathbb{C}}|^2$. (First use of complexification)

2. In $M_{\mathbb{C}}$, the (properly normalized) $|\varphi_j^{\mathbb{C}}|^2$ are quantum ergodic if the geodesic flow is ergodic.

3. In the complex domain, $\log |\varphi_j^{\mathbb{C}}|^2$ converges strongly if the (properly normalized) $|\varphi_j^{\mathbb{C}}|^2$ converges weakly. (Second use of complexification).

4. Use ergodicity of g^t to determine these weak limits.

Final thoughts

Although much progress has been been made recently, there is a long way to go before we understand:

- Whether any (M, g) are QUE. E.g. compact hyperbolic surfaces. In view of cat maps, we need to use properties of Δ-eigenfunctions that are not properties of "quantum cap map" eigenfunctions. E.g. boundary values of eigenfunctions on the ideal boundary (automorphic distributions)
- Multiplicities of eigenvalues–all but hopeless to improve $O(\frac{\lambda^{n-1}}{\log \lambda})$ bound (sharp in cat map setting.)
- The extent to which eigenfunctions of chaotic systems resemble random waves (e.g. in norms, nodal domains, critical point sets).

Limit distribution of zeros is singular along zero section

- The Kaehler structure (Guillemin, Lempert-Szoke) on the cotangent bundle is $\overline{\partial}\partial\rho$. But the limit current is $\overline{\partial}\partial\sqrt{\rho}$. The latter is singular along $M = \{\xi = 0\}$; its highest exterior power is the delta-function on the real M.
- One might be able to use the singularity to determine the limit distribution of the real zero sets.
- One hopes to use the same techniques to count critical points of eigenfunction and determine their distribution law.