COUNTING NODAL LINES WHICH TOUCH THE BOUNDARY OF AN ANALYTIC DOMAIN

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Abstract. We consider the zeros on the boundary $\partial\Omega$ of a Neumann eigenfunction φ_{λ_j} of a real analytic plane domain Ω. We prove that the number of its boundary zeros is $O(\lambda_j)$ where $-\Delta\varphi_{\lambda_j} = \lambda_j^2 \varphi_{\lambda_j}$. We also prove that the number of boundary critical points of either a Neumann or Dirichlet eigenfunction is $O(\lambda_j)$. It follows that the number of nodal lines of φ_{λ_j} (components of the nodal set) which touch the boundary is of order λ_j. This upper bound is of the same order of magnitude as the length of the total nodal line, but is the square root of the Courant bound on the number of nodal components in the interior. More generally, the results are proved for piecewise analytic domains.

1. Introduction

This article is concerned with the high energy asymptotics of nodal lines of Neumann (resp. Dirichlet) eigenfunctions φ_{λ_j} on piecewise real analytic plane domains $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta\varphi_{\lambda_j} = \lambda_j^2 \varphi_{\lambda_j} \ \text{ in } \ \Omega, \\ \\ \partial_\nu \varphi_{\lambda_j} = 0 \ \text{ (resp. } \varphi_{\lambda_j} = 0) \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

Here, ∂_ν is the interior unit normal. We refer to λ_j as the 'frequency' and note that the Laplace eigenvalue is λ_j^2 (unlike [DF, L] and some other references). We denote by $\{\varphi_{\lambda_j}\}$ an orthonormal basis of eigenfunctions of the boundary value problem corresponding to the eigenvalues $\lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \cdots$ enumerated according to multiplicity. The nodal set

$$\mathcal{N}_{\varphi_{\lambda_j}} = \{x \in \Omega : \varphi_{\lambda_j}(x) = 0\}$$

is a curve (possibly with self-intersections at the *singular points*) which intersects the boundary in the set $\mathcal{N}_{\varphi_{\lambda_j}} \cap \partial\Omega$ of boundary nodal points. The motivating problem of this article is the following: how many nodal lines (i.e. components of the nodal set) touch the boundary? Since the boundary lies in the nodal set for Dirichlet boundary conditions, we remove it from the nodal set before counting components. Henceforth, the number of components of the nodal set in the Dirichlet case means the number of components of $\mathcal{N}_{\varphi_{\lambda_j}} \backslash \partial\Omega$.

In the following, and henceforth, $C_\Omega > 0$ denotes a positive constant depending only on the domain Ω.

Theorem 1. *Let Ω be a piecewise analytic domain and let $n_{\partial\Omega}(\lambda_j)$ be the number of components of the nodal set of the jth Neumann or Dirichlet eigenfunction which intersect $\partial\Omega$. Then there exists C_Ω such that $n_{\partial\Omega}(\lambda_j) \leq C_\Omega \lambda_j$.*

Date: Sept 24, 2007.
Research partially supported by NSERC grant # OGP0170280 and a William Dawson Fellowship.
Research partially supported by NSF grant # DMS-0603850.

For generic piecewise analytic plane domains, zero is a regular value of all the eigenfunctions φ_{λ_j}, i.e. $\nabla \varphi_{\lambda_j} \neq 0$ on $\mathcal{N}_{\varphi_{\lambda_j}}$ [U]; we then call the nodal set regular. Each regular nodal set decomposes into a disjoint union of connected components which are homeomorphic either to circles contained in the interior Ω^o of Ω or to intervals intersecting the boundary in two points. We term the former 'closed nodal loops' and the latter 'open nodal lines'. Thus, we are counting open nodal lines. Such open nodal lines might 'percolate' in the sense of [Ze] (i.e. become infinitely long in the scaling limit $\Omega \to \lambda_j \Omega$), or they might form λ_j^{-1} -'small' half-loops at the boundary.

For the Neumann problem, the boundary nodal points are the same as the zeros of the boundary values $\varphi_{\lambda_j}|_{\partial\Omega}$ of the eigenfunctions. The number of boundary nodal points is thus twice the number of open nodal lines. Hence we can count open nodal lines by counting boundary nodal points. In the Neumann case, our result follows from:

Theorem **2.** *Suppose that $\Omega \subset \mathbb{R}^2$ is a piecewise real analytic plane domain. Then the number $n(\lambda_j) = \# \mathcal{N}_{\varphi_{\lambda_j}} \cap \partial\Omega$ of zeros of the boundary values $\varphi_{\lambda_j}|_{\partial\Omega}$ of the jth Neumann eigenfunction satisfies $n(\lambda_j) \leq C_\Omega \lambda_j$, for some $C_\Omega > 0$.*

This is a more precise version of Theorem 1 in cases such as integrable billiard domains (rectangles, discs, ellipses) where the entire nodal set is connected due to the large grid of self-intersection points of the nodal set. The analogous result in the Dirichlet case is stated in Corollary 4. Counting boundary nodal points of eigenfunctions has obvious similarities to measuring the length of the interior nodal line, and our results show that the order of magnitude is the same. We recall that S. T. Yau conjectured that in all dimensions, the hypersurface volume should satisfy $c\lambda_j \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_{\lambda_j}}) \leq C\lambda_j$ for some positive constants c, C depending only on (M, g) [Y1, Y2]. The lower bound was proved in dimension two for smooth domains by Brüning-Gomes [BG] and both the upper and lower bounds were proved in all dimensions for analytic (M, g) by Donnelly-Fefferman [DF, DF2] (see also [L]). For general C^∞ Riemannian manifolds (M, g) in dimensions ≥ 3 there are at present only exponential bounds [HHL, HS]. Our methods involve analytic continuation to the complex as in [DF, DF2, L], and it is not clear how to extend them to C^∞ domains.

In comparison to the order $O(\lambda_j)$ of the number of boundary nodal points, the total number of connected components of $\mathcal{N}_{\varphi_{\lambda_j}}$ has the upper bound $O(\lambda_j^2)$ by the Courant nodal domain theorem. Here, and henceforth, the notation $A(\lambda) = O(g(\lambda))$ means that there exists a constant $C_\Omega > 0$ independent of λ such that $A(\lambda) \leq C_\Omega g(\lambda)$. Only in very rare cases is it known whether this upper bound is achieved (in terms of order of magnitude). When the upper bound is achieved, the number of open nodal lines in dimension 2 is of one lower order in λ_j than the number of closed nodal loops. This effect is known from numerical experiments of eigenfunctions and random waves [BGS, FGS]. The only rigorous result we know is the recent proof in [NS] that the average number of nodal components of a random spherical harmonic is of order of magnitude λ_j^2. In special cases, the number of connected components can be much smaller than the Courant bound, e.g. two or three for arbitrarily high eigenvalues [Lew].

Our methods also yield estimates on the number of critical points of φ_{λ_j} which occur on the boundary. We denote the boundary critical set by

$$\mathcal{C}_{\varphi_{\lambda_j}} = \{q \in \partial\Omega : (d\varphi_{\lambda_j})(q) = 0\}.$$

In the case of Neumann eigenfunctions, $q \in \mathcal{C}_{\varphi_{\lambda_j}} \iff d(\varphi_{\lambda_j}|_{\partial\Omega}(q)) = 0$ since the normal derivative automatically equals zero on the boundary, while in the Dirichlet case $q \in \mathcal{C}_{\varphi_{\lambda_j}} \iff \partial_\nu \varphi_{\lambda_j}(q) = 0$ since the boundary is a level set.

A direct parallel to Yau's conjecture for interior critical points of generic analytic metrics would be the Bézout bound $\#\mathcal{C}_{\varphi_{\lambda_j}} \leq C_\Omega \lambda_j^n$ in dimension n, and λ_j^{n-1} for boundary critical points. However, this bound is unstable since the critical point sets do not even have to be discrete when the eigenfunctions have degenerate critical points, and the true count in the discrete case might reflect the size of the determinant of the Hessian at the critical point. Note that there is no non-trivial lower bound on the number of interior critical points [JN]. Related complications occur for boundary critical points. For instance, the radial eigenfunctions on the disc are constant on the boundary; thus, boundary critical point sets need not be isolated. We therefore need to add a non-degeneracy condition on the derivative $\partial_t(\varphi_{\lambda_j}|_{\partial\Omega})$ to ensure that its zeros are isolated and can be counted by our methods.

We phrase the condition in terms of the Pompeiu problem and Schiffer conjecture, which asserts the disc is the only smooth plane domain possessing a Neumann eigenfunction which is constant on the boundary. In [Ber]) it is proved that the disc is the only bounded simply connected plane domain possessing an infinite sequence of such Neumann eigenfunctions. We say that the Neumann problem for a bounded domain has the *asymptotic Schiffer property* if there exists $C > 0$ such that, for all Neumann eigenfunctions φ_{λ_j} with sufficiently large λ_j,

$$\frac{\|\partial_t \varphi_{\lambda_j}\|_{L^2(\partial\Omega)}}{\|\varphi_{\lambda_j}\|_{L^2(\partial\Omega)}} \geq e^{-C\lambda_j}. \tag{1.2}$$

Here, the L^2 norms refer to the restrictions of the eigenfunction to $\partial\Omega$. It seems plausible that Berenstein's result might extend to this asymptotic Schiffer property, i.e. that the disc is the only example where (1.2) fails for an infinite sequence.

Theorem **3.** *Let $\Omega \subset \mathbb{R}^2$ be piecewise real analytic. Suppose that $\varphi_{\lambda_j}|_{\partial\Omega}$ satisfies the asymptotic Schiffer condition (1.2) in the Neumann case. Then the number of $n_{crit}(\lambda_j) = \#\mathcal{C}_{\varphi_{\lambda_j}}$ of critical points of a Neumann or Dirichlet eigenfunction φ_{λ_j} which lie on $\partial\Omega$ satisfies $n_{crit}(\lambda_j) \leq C_\Omega \lambda_j$ for some $C_\Omega > 0$*

This is apparently the first general result on the asymptotic number of critical points of eigenfunctions as $\lambda_j \to \infty$. Our results on boundary critical points give some positive evidence for the Bézout upper bound in the analytic case, and simple examples such as the disc show that there does not exist a non-trivial lower bound (see §2). In general, counting critical points is subtler than counting zeros or singular points (i.e. points where $\varphi_{\lambda_j}(x) = d\varphi_{\lambda_j}(x) = 0$; see [HS, HHL]).

In the case of Dirichlet eigenfunctions, endpoints of open nodal lines are always boundary critical points, since they must be singular points of φ_{λ_j}. Hence, an upper bound for $n_{\text{crit}}(\lambda_j)$ also gives an upper bound for the number of open nodal lines.

Corollary **4.** *Suppose that $\Omega \subset \mathbb{R}^2$ is a piecewise real analytic plane domain. Let $n_{\partial\Omega}(\lambda_j)$ be the number of open nodal lines of the jth Dirichlet eigenfunction, i.e. connected components of $\{\varphi_{\lambda_j} = 0\} \subset \Omega^o$ whose closure intersects $\partial\Omega$. Then there exists $C_\Omega > 0$ such that $n_{\partial\Omega}(\lambda_j) \leq C_\Omega \lambda_j$.*

The question may arise why we are concerned with piecewise analytic domains $\Omega^2 \subset \mathbb{R}^2$. By this, we mean a compact domain with piecewise analytic boundary, i.e. $\partial\Omega$ is a union of a finite number of piecewise analytic curves which intersect only at their common endpoints (cf. [HZ]). Our interest in such domains is due to the fact that many important types of domains in classical and quantum billiards, such as the Bunimovich stadium or Sinai billiard, are only piecewise analytic. Their nodal sets have been the subject of a number of numerical studies (e.g. [BGS, FGS]). More generally, it is of interest to study the nodal distributions of piecewise analytic Euclidean plane domains with ergodic billiards, which can never be fully analytic.

The results stated above are corollaries of one basic result concerning the *complex zeros and critical points* of analytic continuations of Cauchy data of eigenfunctions. When $\partial\Omega \in C^\omega$, the eigenfunctions can be holomorphically continued to an open tube domain in $\mathbb{C}^2$ projecting over an open neighborhood W in $\mathbb{R}^2$ of Ω which is independent of the eigenvalue. We denote by $\Omega_{\mathbb{C}} \subset \mathbb{C}^2$ the points $\zeta = x + i\xi \in \mathbb{C}^2$ with $x \in \Omega$. Then $\varphi_{\lambda_j}(x)$ extends to a holomorphic function $\varphi^{\mathbb{C}}_{\lambda_j}(\zeta)$ where $x \in W$ and where $|\xi| \leq \epsilon_0$ for some $\epsilon_0 > 0$. We mainly use the complexifications to obtain upper bounds on real zeros, so are not concerned with the maximal ϵ_0, i.e. the 'radius of the Grauert tube' around $\partial\Omega$, and do not include the radius in our notation for the complexification.

Assuming $\partial\Omega$ real analytic, we define the (interior) complex nodal set by

$$\mathcal{N}^{\mathbb{C}}_{\varphi_{\lambda_j}} = \{\zeta \in \Omega_{\mathbb{C}} : \varphi^{\mathbb{C}}_{\lambda_j}(\zeta) = 0\},$$

and the (interior) complex critical point set by

$$\mathcal{C}^{\mathbb{C}}_{\varphi_{\lambda_j}} = \{\zeta \in \Omega_{\mathbb{C}} : d\varphi^{\mathbb{C}}_{\lambda_j}(\zeta) = 0\}.$$

We are mainly interested in the restriction of $\varphi^{\mathbb{C}}_{\lambda_j}$ to the complexification $(\partial\Omega)_{\mathbb{C}}$ of the boundary, i.e. the open complex curve in $\mathbb{C}^2$ obtained by analytically continuing a real analytic parameterization $Q : S^1 \to \partial\Omega$. The map Q admits a holomorphic extension to an annulus $A(\epsilon)$ (see (3.2)) around the parameterizing circle S^1 and its image $Q_{\mathbb{C}}(A(\epsilon)) \subset \mathbb{C}^2$ is an annulus in the complexification of the boundary; it is analogous to a Grauert tube around the real analytic boundary in the sense of [GS1, LS1]. We then define the boundary complex nodal set by

$$\mathcal{N}^{\partial\Omega_{\mathbb{C}}}_{\varphi_{\lambda_j}} = \{\zeta \in \partial\Omega_{\mathbb{C}} : \varphi^{\mathbb{C}}_{\lambda_j}(\zeta) = 0\},$$

and the (boundary) complex critical point set by

$$\mathcal{C}^{\partial\Omega_{\mathbb{C}}}_{\varphi_{\lambda_j}} = \{\zeta \in \partial\Omega_{\mathbb{C}} : d\varphi^{\mathbb{C}}_{\lambda_j}(\zeta) = 0\}.$$

More generally, we may assume $\partial\Omega$ is piecewise real analytic and holomorphically extend eigenfunctions to the analytic continuations of the real analytic boundary arcs. The radii of these analytic continuations of course shrink to zero at the corners.

THEOREM **5.** *Suppose that $\Omega \subset \mathbb{R}^2$ is a piecewise real analytic plane domain, and denote by $(\partial\Omega)_{\mathbb{C}}$ the union of the complexifications of its real analytic boundary components.*

(1) *Let $n(\lambda_j, \partial\Omega_{\mathbb{C}}) = \#\mathcal{N}^{\partial\Omega_{\mathbb{C}}}_{\varphi_{\lambda_j}}$. Then there exists a constant $C_\Omega > 0$ independent of the radius of $(\partial\Omega)_{\mathbb{C}}$ such that $n(\lambda_j, \partial\Omega_{\mathbb{C}}) \leq C_\Omega \lambda_j$.*

(2) *Suppose that the Neumann eigenfunctions satisfy (1.2) and let* $n_{crit}(\lambda_j, \partial\Omega_{\mathbb{C}}) = \#\mathcal{C}_{\varphi_{\lambda_j}}^{\partial\Omega_{\mathbb{C}}}$. *Then there exists* $C_\Omega > 0$ *independent of the radius of* $(\partial\Omega)_{\mathbb{C}}$ *such that* $n_{crit}(\lambda_j, \partial\Omega_{\mathbb{C}}) \leq C_\Omega \lambda_j$.

The theorems on real nodal lines and critical points follow from the fact that real zeros and critical points are also complex zeros and critical points, hence

$$n(\lambda_j) \leq n(\lambda_j, \partial\Omega_{\mathbb{C}}); \quad n_{\text{crit}}(\lambda_j) \leq n_{\text{crit}}(\lambda_j, \partial\Omega_{\mathbb{C}}). \tag{1.3}$$

All of the results are sharp, and are already obtained for certain sequences of eigenfunctions on a disc (see §2). If the condition (1.2) is not satisfied, the boundary value of φ_{λ_j} must equal a constant C_j modulo an error of the form $o(e^{-C\lambda_j})$. It is very likely that this forces the boundary values to be constant, but it would take us too far afield in this article to prove it.

Although our main interest is in counting open nodal lines, the method of proof of Theorem 5 generalizes from $\partial\Omega$ to a rather large class of real analytic curves $C \subset \Omega$, even when $\partial\Omega$ is not real analytic. Let us call a real analytic curve C a *good* curve if there exists a constant $a > 0$ so that for all λ_j sufficiently large,

$$\frac{\|\varphi_{\lambda_j}\|_{L^2(\partial\Omega)}}{\|\varphi_{\lambda_j}\|_{L^2(C)}} \leq e^{a\lambda_j}. \tag{1.4}$$

Here, the L^2 norms refer to the restrictions of the eigenfunction to C and to $\partial\Omega$. The following result deals with the case where $C \subset \partial\Omega$ is an *interior* real-analytic curve. The real curve C may then be holomorphically continued to a complex curve $C_{\mathbb{C}} \subset \mathbb{C}^2$ obtained by analytically continuing a real analytic parametrization of C.

THEOREM **6.** *Suppose that* $\Omega \subset \mathbb{R}^2$ *is a* C^∞ *plane domain, and let* $C \subset \Omega$ *be a good interior real analytic curve in the sense of (1.4). Let* $n(\lambda_j, C) = \#\mathcal{N}_{\varphi_{\lambda_j}} \cap C$ *be the number of intersection points of the nodal set of the* j*-th Neumann (or Dirichlet) eigenfunction with* C. *Then there exists* $A_{C,\Omega} > 0$ *depending only on* C, Ω *such that* $n(\lambda_j, C) \leq A_{C,\Omega}\lambda_j$.

Although the upper bounds are sharp for some domains, we do not present necessary or sufficient conditions on a domain that the bounds on zeros or critical points are achieved on that domain for some sequence of eigenfunctions. We do not know any domain for which they are not achieved, but there are few domains where the bounds can be explicitly tested. The boundary (or rather its unit ball bundle) is naturally viewed as a kind of quantum 'cross section' of the wave group [HZ]. The growth rate of the modulus and zeros of Cauchy data of complexified eigenfunctions depend on what kind of 'cross section' the boundary provides. In work in progress, we show that at least for some piecewise analytic domains with ergodic billiards, the the number of complex zeros of $\varphi_{\lambda_j}^{\mathbb{C}}|_{\partial\Omega_{\mathbb{C}}}$ is $\sim C\lambda_j$. It seems that this asymptotic reflects the fact that the boundary is a representative cross section in this case.

We note that some of the methods and results of this paper are restricted to dimension two. In higher dimensions, zeros of the Cauchy data are not isolated and we would have to count numbers of components of the boundary nodal set. This seems inaccessible at present.

The organization of this article is as follow: In §3, we use the layer potential representations of Cauchy data of eigenfunctions, or equivalently the representation in terms of the Calderon projector, to analytically continue eigenfunctions. The analytic continuation of the layer potential representation has previously been studied by Vekua [V], Garabedian [G], and in the form we need by Millar [M1, M2, M2]. The analytic continuation is somewhat subtle due to the presence of logarithms in the layer potentials, and does not appear to be well-known; so we present complete details (which are sometimes sketchy in the original articles) in Appendix 8. In §4, we relate growth of complex zeros to growth of the log modulus of the complexified eigenfunctions. In §5 - §6, we prove the main results. The complexified layer potential representation is used to obtain an upper bound on the growth rate of the complexified eigenfunctions $\varphi_{\lambda_j}^{\mathbb{C}}$ in a fixed complex tube around the boundary as $\lambda_j \to \infty$. The estimate is simpler for interior curves (§5) since on the boundary the analytic continuation involves a Volterra operator that must be inverted. Almost the same method gives analogous results on critical points; for the sake of brevity, the argument is only sketched in §7. Finally, for the benefit of the reader, in Appendix 9 we collect all the special functions formulas for Bessel and Hankel functions that are used in §3 and §5 - §6.

We would like to thank P. Ebenfelt, L. Ehrenpreis, C. Epstein, P. Koosis and M. Zworski for informative discussions of analytic continuations of eigenfunctions and logarithmic integrals.

2. Examples

We begin with examples illustrating the issues we face. Eigenfunctions are only computable in (quantum) completely integrable cases, and at present the only known examples are the unit disc, ellipses and rectangles. It is a classical conjecture of Birhoff that ellipses are the only smooth Euclidean plane domains with integrable billiards, so one does not expect further explicitly computable examples. In addition, one can construct approximate eigenfunctions, or quasi-modes for many further domains [BB]; it is plausible, although it is not proved here, that our results extend to real analytic quasi-modes.

2.1. **The unit disc D.** The unit disc shows that there do not exist universal lower bounds on the number of boundary nodal points in terms of the frequency λ_j.

The standard orthonormal basis of real valued Neumann eigefunctions is given in polar coordinates by $\varphi_{m,n}(r,\theta) = C_{m,n} \sin m\theta J_m(j'_{m,n} r)$, (resp. $C_{m,n} \cos m\theta J_m(j'_{m,n} r)$) where $j'_{m,n}$ is the nth critical point of the Bessel function J_m and where $C_{m,n}$ is the normalizing constant. The Δ-eigenvalue is $\lambda^2_{m,n} = (j'_{m,n})^2$. The parameter m is referred to as the angular momentum. Dirichlet eigenfunctions have a similar form with $j'_{m,n}$ replaced by the nth zero $j_{m,n}$ of J_m. Nodal loops correspond to zeros of the radial factor while open nodal lines correspond to zeros of the angular factor.

If we fix m and let $\lambda_{m,n} \to \infty$ we obtain a sequence of eigenfunctions of bounded angular momentum but high energy. In the sin case (e.g.), the open nodal lines consist of the union of rays $C_m = \{\theta = \frac{2\pi k}{m}, k = 0, \dots, \frac{m-1}{m}\}$ through the mth roots of unity. Hence, for each m there exist sequences of eigenfunctions with $\lambda \to \infty$ but with m open nodal lines; hence, there exists no lower bound on the number of nodal lines touching the boundary in terms of the energy. This example also shows that there cannot exist a general unconditional result counting intersections of nodal lines with interior curves, since $\varphi_{m,n}|_{C_m} \equiv 0$ and hence the 'number' of nodal points on the interior curve C_m is infinite. In particular, C_m is not 'good' in the sense of (1.4).

At the opposite extreme are the whispering gallery modes which concentrate along the boundary. These are eigenfunctions of maximal angular momentum (with given energy), and $\lambda_m \sim m$. As discussed in [BB], they are asymptotically given by the real and imaginary parts of $e^{i\lambda_m s} Ai_p(\rho^{-1/3}\lambda_m^{2/3} y)$. Here, $Ai_p(y) := Ai(-t_p + y)$ where Ai is the Airy function and $\{-t_p\}$ are its negative zeros. Also, s is arc-length along ∂D, ρ is a normalizing constant and $y = 1 - r$. Whispering gallery modes saturate the upper bound on the number of open nodal lines.

2.2. **An ellipse.** A new feature in comparison to the disc is the existence of Gaussian beams along the minor axis, which is a stable elliptic bouncing ball orbit. The modes oscillate along the minor exis and have Gaussian decay in the tranvserve direction; thus, they are highly localized along the minor axis. Such bouncing ball modes do not exist in the disc and occur when m is fixed and $n \to \infty$.

We express an ellipse in the form $x^2 + \frac{y^2}{1-a^2} = 1, \quad 0 \le a < 1$, with foci at $(x,y) = (\pm a, 0)$. We define elliptical coordinates (φ, ρ) by $(x,y) = (a\cos\varphi\cosh\rho, a\sin\varphi\sinh\rho)$. Here, $0 \le \rho \le \rho_{\max} = \cosh^{-1} a^{-1}, \quad 0 \le \varphi \le 2\pi$. The lines $\rho = const$ are confocal ellipses and the lines $\varphi = const$ are confocal hyperbolae. The foci occur at $\varphi = 0, \pi$ while the origin occurs at $\rho = 0, \varphi = \frac{\pi}{2}$.

The eigenvalue problem separates into a pair of Mathieu equations,

$$\begin{cases} \partial_\varphi^2 G_{m,n} - c^2\cos^2\varphi G_{m,n} = -\lambda_{m,n}^2 G_{m,n} \\ \\ \partial_\rho^2 F - c^2\cosh^2\rho F_{m,n} = \lambda_{m,n}^2 F_{m,n} \end{cases} \tag{2.1}$$

where c is a certain parameter. The eigenfunctions have the form $\Psi_{m,n}(\varphi,\rho) = C_{m,n}F_{m,n}(\rho)\cdot G_{m,n}(\varphi)$ where, $F_{m,n}(\rho) = Ce_m(\rho, \frac{k_n c}{2})$ and $G_{m,n}(\varphi) = ce_m(\varphi, \frac{k_n c}{2})$ (and their sin analogues). Here, ce_m, Ce_m are special Mathieu functions (cf. [C] (3.10)-(3.2)). The Neumann or Dirichlet boundary conditions determine the eigenvalue parameters $k_n c$. The nodal lines are of course given by $\{G_{m,n} = 0\} \cup \{F_{m,n} = 0\}$. For more details and computer graphics of elliptic bouncing ball modes we refer to [C]; the original work was done by Keller-Rubinow.

The eigenfunctions which are the Gaussian beams are characterized as follows: Since the minor axis is

$$I = \{(\rho,\varphi) \in [0,\rho_{\max}] \times [0,2\pi];\ \varphi = \frac{\pi}{2}\}.$$

one looks for eigenfunctions with mass concentrated along this interval. Consider the extremal energy levels that satisfy

$$c^2\lambda_{m,n}^{-2} = 1 + O(\lambda_{m,n}^{-1}).$$

One rewrites (2.1) in the form:

$$\begin{cases} -\lambda_{m,n}^{-2}\partial_\varphi^2 G_{m,n} + (\lambda_{m,n}^{-2}c^2\cos^2\varphi - 1)G_{m,n} = 0 \\ \\ -\lambda_{m,n}^{-2}\partial_\rho^2 F_{m,n} + (\lambda_{m,n}^{-2}c^2\cosh^2\rho + 1)F_{m,n} = 0 \end{cases} \tag{2.2}$$

Given the choice of energy level,

$$\lambda_{m,n}^{-2}c^2\cos^2\varphi - 1 = \cos^2\varphi - 1 + O(\lambda_{m,n}^{-1}) = -\sin^2(\varphi) + O(\lambda_{m,n}^{-1}).$$

The potential $V_1(\varphi) = \cos^2\varphi - 1$ has a nondegenerate minimum at $\varphi = \frac{\pi}{2}$. So, the solutions $G = G_{m,n}$ to the first equation in (2.2) are asymptotic to ground state Hermite functions.

More precisely,

$$G_{m,n}(\varphi;\lambda_{m,n}) = c_{m,n}(\lambda_{m,n}) e^{-\lambda_{m,n}\cos^2\varphi}(1+O(\lambda_{m,n}^{-1})). \tag{2.3}$$

In the second equation in (2.2) the potential is $V_2(\rho) = \cosh^2\rho + 1 + O(\lambda_{m,n}^{-1}) > 0$ for $\lambda_{m,n} \geq \lambda_0$ sufficiently large. In this case the solution has purely oscillatory asymptotics:

$$F_{m,n}(\rho;\lambda_{m,n}) = e^{i\lambda_{m,n}\int_0^\rho \sqrt{\cosh^2 x+1}dx} a_+(\rho;\lambda_{m,n}) + e^{-i\lambda_{m,n}\int_0^\rho \sqrt{\cosh^2 x+1}dx} a_-(\rho;\lambda_{m,n}) \tag{2.4}$$

where $a_\pm(\rho;\lambda_{m,n}) \sim \sum_{j=0}^\infty a_{\pm,j}(\rho)\lambda_{m,n}^{-j}$ are determined by the Dirichlet or Neumann boundary conditions. Moreover, from the L^2-normalization condition $\int_I |\Psi_{m,n}(\rho,\frac{\pi}{2})|^2 d\rho = 1$ it follows that $c_{m,n}(\lambda_{m,n}) \sim \lambda_{m,n}^{1/4}$.

From (2.3) and (2.4), the Gaussian beams are roughly asymptotic to superpositions of $e^{\pm iks}e^{-\lambda_{m,n}y^2}$ (cf. [BB]), where s denotes arc-length along the bouncing ball orbit and y denotes the Fermi normal coordinate. It follows that outside a tube of any given radius $\epsilon > 0$, the Gaussian beam decays on the order $O(e^{-\lambda_{m,n}\epsilon^2})$. Hence on any curve C which is disjoint from the bouncing ball orbit, the restriction of the Gaussian beam to C saturates the description of a 'good' analytic curve.

Remark:

In the case of circles and ellipses, it is elementary to obtain the analytic continuations of the boundary values of Neumann eigenfunctions to the complexification of the boundary. One sees that the growth rate of the analytic continuations are determined by the angular momenta of the eigenfunctions, i.e. the eigenvalue of the boundary Laplacian, and not by the interior eigenvalue. This suggests that the growth rate of boundary values of eigenfunctions and of their analytic continuations should be estimated in terms of a 'boundary frequency function'.

3. Holomorphic extensions of eigenfunctions to Grauert tubes

It is classical that solutions of the Helmholtz equation on a Euclidean domain are real analytic in the interior and hence their restrictions to interior real analytic curves admit holomorphic extensions to the complexification of the curves. A classical presentation can be found in [G] §5.2, and to some extent we try to conform to its notation; see also [MN, S, Mor, V]. Moreover, the Cauchy data along the boundary of eigenfunctions satisfying Dirichlet or Neumann boundary conditions admit analytic continuations into a uniform tube in the complexification of the boundary, independent of the eigenvalue. We now use integral representations for the analytic continuations to obtain upper bounds for the growth rate of the complexified Cauchy data of eigenfunctions in a fixed Grauert tube of interior curves or curves on the boundary as $\lambda_j \to \infty$.

3.1. **Complexification of domains Ω and their boundaries $\partial\Omega$.** When (M,g) is a real analytic Riemannian manifold without boundary, then there exists a complexification $M_{\mathbb{C}}$ of M as an (open) complex manifold, and Laplace eigenfunctions extend to a maximal tube in $M_{\mathbb{C}}$. This has been studied in [Bou, GS1, GS2, GLS, Z] using the analytic continuation of the geodesic flow and wave group. The notion of maximal tube depends on the construction of a special plurisubharmonic exhaustion function ρ adapted to the metric g in the sense that $i\partial\bar\partial\rho$ restricts to the totally real submanifold $M \subset M_{\mathbb{C}}$ as g. The function $\sqrt{\rho}(t)$ is the distance from t to $\bar{t}$.

The analogous results for manifolds with boundary have not apparently been studied before. In this section, we study the complexification of the boundary and analytic continuations of the Cauchy data of eigenfunctions on the boundary (or on interior curves) for domains $\Omega \subset \mathbb{R}^2$ with the Euclidean metric. The complexification of the ambient space is of course $\mathbb{C}^2$ and its Grauert tube function is $|\Im t|$. The novel features concern the influence of the boundary on analytic extensions of eigenfunctions.

We adopt the following notation from Garabedian [G] and Millar [M1, M2]: We denote points in $\mathbb{R}^2$ and also in $\mathbb{C}^2$ by (x, y). We further write $z = x + iy, z^* = x - iy$. Note that z, z^* are independent holomorphic coordinates on $\mathbb{C}^2$ and are characteristic coordinates for the Laplacian, in that the Laplacian analytically extends to $\frac{\partial^2}{\partial z \partial z^*}$. When dealing with kernel functions of two variables, we use (ξ, η) in the same way as (x, y) for the second variable.

When the boundary is real analytic, the complexification $\partial\Omega$ is the image of the analytic continuation of a real analytic parameterization. To simplify notation, unless indicated otherwise, we will assume that the length of $\partial\Omega$ is 2π. We denote a real analytic parametrization by arc-length by $Q : \mathbb{S}^1 \to \partial\Omega \simeq \mathbb{C}$, and also write the parametrization as a periodic function

$$q(t) = Q(e^{it}) : [0, 2\pi] \to \partial\Omega \tag{3.1}$$

on $[0, 2\pi]$. We then put $q(s) = q_1(s) + iq_2(s), \bar{q}(s) = q_1(s) - iq_2(s)$.

We complexify $\partial\Omega$ by holomorphically extending the parametrization to $Q^{\mathbb{C}}$ on the annulus

$$A(\epsilon) := \{t \in \mathbb{C}; e^{-\epsilon} < |t| < e^{\epsilon}\}, \tag{3.2}$$

for $\epsilon > 0$ small enough. Note that the complex conjugate parameterization $\bar{Q}$ extends holomorphically to $A(\epsilon)$ as Q^*_C. Throughout the paper, the subscript $_{\mathbb{C}}$ or superscript $^{\mathbb{C}}$ denotes the holomorphic continuation of a curve or function; when no confusion can arise, we sometimes omit the sub or superscript for notational simplicity. The $q(t)$ parametrization analytically continues to a periodic function $q^{\mathbb{C}}(t)$ on $[0, 2\pi] + i[-\epsilon, \epsilon]$.

Next, we put $r^2((x, y); (\xi, \eta)) = (\xi - x)^2 + (\eta - y)^2$. For $s \in \mathbb{R}$ and $t \in \mathbb{C}$, we have $q(s) = \xi(s) + i\eta(s), q(t) = x(t) + iy(t), q^*(t) = x(t) - iy(t)$ and we write $r^2(s, t) = r^2(q(s); q(t))$. Thus,

$$r^2(s, t) = (q(s) - q(t))(\bar{q}(s) - q^*(t)). \tag{3.3}$$

We denote by $\frac{d}{dn}$ the not-necessarily-unit normal derivative in the direction $iq'(s)$. Thus, in terms of the notation $\frac{\partial}{\partial \nu}$ above, $\frac{d}{dn} = |q'(s)| \frac{\partial}{\partial \nu}$. One has

$$\frac{d}{ds} \log r = \frac{1}{2} \left[\frac{q'(s)}{q(s) - q(t)} + \frac{\overline{q}'(s)}{\overline{q(s)} - q^*(t)} \right], \quad \frac{\partial}{\partial n} \log r = \frac{-i}{2} \left[\frac{q'(s)}{q(s) - q(t)} - \frac{\overline{q}'(s)}{\overline{q(s)} - q^*(t)} \right].$$

When we are using an arc-length parameterization, $\frac{d}{dn} = \frac{\partial}{\partial \nu}$.

To clarify the notation, we consider the case of $S^1 = \partial\Omega$. Then, $q(s) = e^{is}$, $t = \theta + i\xi$, $q(\theta + i\xi) = e^{i(\theta + i\xi)}, q^* = e^{-i(\theta + i\xi)}, \overline{q^*} = e^{i(\theta - i\xi)}$, and

$$r^2(s, \theta + i\xi) = (e^{i(\theta + i\xi)} - e^{is})(e^{-i(\theta + i\xi)} - e^{-is}) \quad = \quad 4 \sin^2 \tfrac{(\theta - s + i\xi)}{2}.$$

Thus, $\log r^2 = \log(4 \sin^2 \frac{(\theta - s + i\xi)}{2})$. Clearly, $\frac{d}{ds} = \frac{d}{d\theta}$, so

$$\frac{d}{ds} \log r^2 = \left[\frac{ie^{is}}{e^{is} - e^{i(\theta + i\xi)}} + \frac{-ie^{-is}}{e^{-is} - e^{-i(\theta + i\xi)}} \right], \quad \frac{\partial}{\partial \nu} \log r = \frac{-i}{2} \left[\frac{ie^{is}}{e^{is} - e^{i(\theta + i\xi)}} - \frac{-ie^{-is}}{e^{-is} - e^{-i(\theta + i\xi)}} \right].$$

3.2. **Complexification of analytic curves $C \subset \overline{\Omega}$.** Theorem 5 concerns complexifications of real analytic curves $C \subset \Omega$, and we now introduce analogous notation for them. As in the case of $\partial\Omega$, we assume that the length of all curves C is 2π. We use the notation $Q_C : \mathbb{S}^1 \to \mathbb{R}^2 \simeq \mathbb{C}$ for a real analytic parameterization of C. We then complexify C by holomorphically extending the parametrization to $Q_C^{\mathbb{C}}$ on the annulus (3.2), and again $\bar{Q}_C$ extends holomorphically to $A(\epsilon)$ as Q_C^*. We also write $q_C(t) = Q_C(e^{it}) : [0, 2\pi] \to C$.

3.3. **Layer potential representations on Ω.** We denote the Cauchy data of the eigenfunction on $\partial\Omega$ by

$$u_{\lambda_j} = \varphi_{\lambda_j}|_{\partial\Omega} \ \ (Neumann); \quad u_{\lambda_j} = \partial_\nu \varphi_{\lambda_j}|_{\partial\Omega} \ \ (Dirichlet), \tag{3.4}$$

and write $u_{\lambda_j}^{\mathbb{C}}$ for its holomorphic extension to $\partial\Omega_{\mathbb{C}}$.

We will represent the analytic continuations of the Cauchy data in terms of layer potentials. Let $G(\lambda_j, x_1, x_2)$ be any 'Green's function' for the Helmholtz equation on Ω, i.e. a solution of $(-\Delta - \lambda_j^2)G(\lambda_j, x_1, x_2) = \delta_{x_1}(x_2)$ with $x_1, x_2 \in \bar{\Omega}$. By Green's formula,

$$\varphi_{\lambda_j}(x, y) = \int_{\partial\Omega} \big(\partial_\nu G(\lambda_j, q, (x, y))\varphi_{\lambda_j}(q) - G(\lambda_j, q, (x, y))\partial_\nu \varphi_{\lambda_j}(q)\big)\, d\sigma(q), \tag{3.5}$$

where $(x, y) \in \mathbb{R}^2$, where $d\sigma$ is arc-length measure on $\partial\Omega$ and where ∂_ν is the normal derivative by the interior unit normal. Our aim is to analytically continue this formula.

In the case of Neumann eigenfunctions φ_λ in Ω,

$$\varphi_{\lambda_j}(x, y) = \int_{\partial\Omega} \frac{\partial}{\partial\nu_q} G(\lambda_j, q, (x, y)) u_{\lambda_j}(q) d\sigma(q), \ \ (x, y) \in \Omega^o \ \ (\text{Neumann}). \tag{3.6}$$

In the Dirichlet case, the corresponding formula is

$$\varphi_{\lambda_j}(x, y) = -\int_{\partial\Omega} G(\lambda_j, q, (x, y)) u_{\lambda_j}(q) d\sigma(q), \ \ (\text{Dirichlet}) \tag{3.7}$$

where u_{λ_j} are as in (3.4).

To obtain concrete representations we need to choose G. We choose the real ambient Euclidean Green's function S (in the notation of [G], §5),

$$S(\lambda_j, \xi, \eta; x, y) = -Y_0(\lambda_j r((x, y); (\xi, \eta))), \tag{3.8}$$

where $r = \sqrt{zz^*}$ is the distance function (the square root of r^2 above) and where Y_0 is the Bessel function of order zero of the second kind. (see equation (9.1) in Appendix 9.) The Euclidean Green's function has the form

$$S(\lambda_j, \xi, \eta; x, y) = A(\lambda_j, \xi, \eta; x, y) \log \frac{1}{r} + B(\lambda_j, \xi, \eta; x, y), \tag{3.9}$$

where A and B are entire functions of r^2 (see (9.1) and (9.2) in Appendix 9 for the formulas). The coefficient $A = J_0(\lambda_j r)$ is known as the Riemann function.

By the 'jumps' formulae, the double layer potential $\frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, (x, y))$ on $\partial\Omega \times \bar{\Omega}$ restricts to $\partial\Omega \times \partial\Omega$ as $\frac{1}{2}\delta_q(\tilde{q}) + \frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, q)$ (see e.g. [T]). Hence in the Neumann case the boundary values u_{λ_j} satisfy,

$$u_{\lambda_j}(q) = 2\int_{\partial\Omega} \frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, q) u_{\lambda_j}(\tilde{q}) d\sigma(\tilde{q}) \ \ (\text{Neumann}). \tag{3.10}$$

In the Dirichlet case, one takes the normal derivative of φ_{λ_j} at the boundary to get a similar formula for $\partial_\nu \varphi_{\lambda_j}|_{\partial\Omega}$, with a sign change on the right side. We have,

$$\frac{\partial}{\partial \nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, q) = -\lambda_j Y_1(\lambda_j r) \cos\angle(q - \tilde{q}, \nu_{\tilde{q}}) \tag{3.11}$$

where the formula for $Y_1(z)$ is given in (9.3) in Appendix 9. As is well-known, the pole of Y_1 is cancelled by the $\cos\angle(q - \tilde{q}, \nu_{\tilde{q}})$ factor.

It is equivalent, and sometimes more convenient, to use the (complex valued) Euclidean outgoing Green's function $\mathrm{Ha}_0^{(1)}(kz)$, where $\mathrm{Ha}_0^{(1)} = J_0 + iY_0$ is the Hankel function of order zero. It has the same form as (3.9) and only differs by the addition of the even entire function J_0 to the B term. If we use the Hankel free outgoing Green's function, then in place of (3.11) we have the kernel

$$\begin{aligned} N(\lambda_j, q(s), q(s')) &= \tfrac{i}{2} \partial_{\nu_y} \mathrm{Ha}_0^{(1)}(\lambda_j |q(s) - y|)|_{y = q(s')} \\ &= -\tfrac{i}{2} \lambda_j \mathrm{Ha}_1^{(1)}(\lambda_j |q(s) - q(s')|) \cos\angle(q(s') - q(s), \nu_{q(s')}), \end{aligned} \tag{3.12}$$

and in place of (3.10) we have the formula

$$u_{\lambda_j}(q(t)) = \int_0^{2\pi} N(\lambda_j, q(s), q(t))\, u_{\lambda_j}(q(s)) ds. \tag{3.13}$$

3.4. **Analytic continuation of layer potential representations.** In this section, we analytically continue the layer potential representations (3.10) and (3.13). The main point is to express the analytic continuations of Cauchy data of Neumann and Dirichlet eigenfunctions in terms of the real Cauchy data. For brevity, we only consider (3.10) but essentially the same arguments apply to the free outgoing representation (3.13).

As mentioned above, both $A(\lambda_j, \xi, \eta, x, y)$ and $B(\lambda_j, \xi, \eta, x, y)$ admit analytic continuations. In the case of A, we use a traditional notation $R(\zeta, \zeta^*, z, z^*)$ for the analytic continuation and for simplicity of notation we omit the dependence on λ_j.

3.4.1. *Interior curves.* Before considering Cauchy data on $\partial\Omega$, we consider the simpler problem of expressing the analytic continuations of the Cauchy data of eigenfunctions along interior real analytic curves C in terms of the real Cauchy data.

The Cauchy data of the eigenfunction on C consists of the pair

$$\varphi_{\lambda_j}|_C, \quad \partial_\nu \varphi_{\lambda_j}|_C.$$

We then restrict the Green's formula (3.5) to C. As above, we let $q_C : [0, 2\pi] \to C$ denote the real-analytic, arclength parametrization of the curve C. To simplify notation put $R(q_C(s), \bar{q}_C(s), z, z^*) =: R(s; z, z^*)$. When the coefficients A, B are restricted to $q_C(t), q_C(s)$ we also abbreviate them by $A(\lambda_j, s, t), B(\lambda_j, s, t)$ or when dealing with the analytic continuation in t by $A(\lambda_j, s, q_C(t), q_C^*(t))$ and $B(\lambda_j, s, q_C(t), q_C^*(t))$; .

For $\epsilon > 0$ sufficiently small, we consider the annulus (3.2) and the corresponding complexification of C given by

$$C_{\mathbb{C}} = \{Q_C^{\mathbb{C}}(e^{i(s+i\tau)}) \in q_C^{\mathbb{C}}(A(\epsilon));\ Q_c^{\mathbb{C}}(e^{i(s+i\tau)}) = q_C^{\mathbb{C}}(s + i\tau);\ 0 \le s \le 2\pi,\ -\epsilon < \tau < \epsilon\},$$

where, we recall that $^{\mathbb{C}}$ denotes holomorphic continuation. Since C is assumed to be an interior curve, it follows by compactness of $\partial\Omega$ that for $|\Im q^{\mathbb{C}}|$ sufficiently small, $r^2|_{C_{\mathbb{C}} \times \partial\Omega} \neq 0$.

As a result, one can choose a globally defined holomorphic branch for $\log r$ and so, the holomorphic continuation formula for Neumann eigenfunctions in this case follows immediately from (3.6):

$$\varphi_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t)) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_{\tilde{q}}} S(\lambda_j, q, q_C^{\mathbb{C}}(t))\, u_{\lambda_j}(q)\, d\sigma(q). \tag{3.14}$$

Note that we are using Green's formula on Ω to study the restriction to C and its complexification.

3.4.2. *When $\partial\Omega$ is real analytic.* As before, $q : [0, 2\pi] \to \partial\Omega$ denotes the real analytic paramaterization of $\partial\Omega$ by arc-length, so that $d\sigma(s) = ds$. Let $u_{\lambda_j}(q(t))$ be the Neumann case of (3.4). From (3.9), we can write (3.10) as

$$\begin{aligned} u_{\lambda_j}(q(t)) &= \tfrac{1}{2\pi}\int_0^{2\pi}(-u_{\lambda_j}(q(s))\tfrac{\partial A}{\partial \nu}(\lambda_j, s, t)) \log r^2 ds \\ &\quad -\tfrac{1}{\pi}\int_0^{2\pi} u_{\lambda_j}(q(s))A(\lambda_j, s, t)\tfrac{1}{r}\tfrac{\partial r}{\partial \nu} ds - \tfrac{1}{\pi}\int_0^{2\pi}(-u_{\lambda_j}(q(s)))\tfrac{\partial B}{\partial \nu}(\lambda_j, s, t) ds. \end{aligned} \tag{3.15}$$

With different choices of B the same formula is valid for the outgoing Green's function as well. We wish to analytically continue this formula to the complex parameter strip $[0, 2\pi] + i[-\epsilon, \epsilon]$. For brevity, we denote by $u_{\lambda_j}^{\mathbb{C}}(q^{\mathbb{C}}(t))$ the analytic continuation of (3.4) to the complex parameter strip.

Since $r^2(s,t) = 0$ when $s = t$, the logarithmic factor in S now gives rise to a multi-valued integrand, and it is not obvious that the representation can be holomorphically extended. The analytic continuation of the representation (3.15) was stated by Millar, and uses the function

$$\Phi(t; z, z^*) = \int_0^t u_{\lambda_j}^{\mathbb{C}}(q^{\mathbb{C}}(s)) \frac{\partial}{\partial n} R(s, z, z^*) ds. \tag{3.16}$$

Proposition 7. [M1, M2] *Let $u_{\lambda_j}^{\mathbb{C}}$ be the analytic continuation to $\partial\Omega_{\mathbb{C}}$ of the boundary trace of the Neumann eigenfunction, φ_{λ_j}. Then, for $\Im t > 0$, resp. < 0,*

$$\begin{aligned} u_{\lambda_j}^{\mathbb{C}}(q^{\mathbb{C}}(t)) &= \pm i\Phi(t, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) \\ &\quad + \int_0^{2\pi}\left[\Phi(s; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) + i u_{\lambda_j}(q(s)) R(s, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t))\right] \tfrac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} ds \\ &\quad + \int_0^{2\pi}\left[\Phi(s; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) - i u_{\lambda_j}(q(s)) R(s, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t))\right] \tfrac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)} ds \\ &\quad -2\int_0^{2\pi} u_{\lambda_j}(q(s)) \tfrac{\partial B}{\partial n}(s; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) ds. \end{aligned} \tag{3.17}$$

Since we rely on this Proposition and since the argument in [M1, M2] is somewhat sketchy, we give a detailed proof in Appendix 8.

3.4.3. *When $\partial\Omega$ is piecewise real analytic.* We now consider the case where $\partial\Omega$ is piecewise real-analytic. This has previously been discussed in [M3].

By a piecewise analytic boundary of length ℓ we mean $\partial\Omega = \bigcup_{k=1}^m C_k$ where

- $C_k \subset \mathbb{R}^2$ are the maximal real analytic components of $\partial\Omega$, enumerated in counterclockwise order so that C_k intersects only C_{k-1} and C_{k+1}.

- The C_k are parameterized by m real analytic functions $q_k(t_k) : [0, \ell_k] \to C_k$ on m parameterizing intervals (where $\ell_k = L(C_k)$ is the length of C_k.) We assume $C_k \cap C_{k-1} = \{q_k(0) = q_{k-1}(\ell_k)\}$ when $m \geq 2$.

We denote the Cauchy data of the eigenfunction φ_{λ_j} on the boundary component C_k by $u^k_{\lambda_j}$. Our aim is to analytically continue $u^k_{\lambda_j}$ to $\bigcup_{k=1}^m C_{k,\mathbb{C}}$ where $C_{k,\mathbb{C}}$ is a complexification of the interior of C_k. Thus, as we define it, $\partial\Omega_{\mathbb{C}}$ is pinched at the corner points and the analytic continuation of the boundary data of φ_{λ_j} is somewhat simpler than in the fully analytic case in that we are analytically continuing to a smaller kind of neighborhood of $\partial\Omega$. We denote the holomorphic extension of $u^k_{\lambda_j} \circ q(t_k)$ to the complex parameter 'strips' $[\delta_k, \ell_k - \delta_k] \times i[-\epsilon, \epsilon]$ by $u^{k\mathbb{C}}_{\lambda_j} \circ q^{\mathbb{C}}(t_k))$. Millar's formula for the analytic extension of u_{λ_j} to $C_{k,\mathbb{C}}$ in the Neumann case is given by:

Proposition 8. [M3] *For $\Re t_k \in [0, \ell_k]$ and $\Im t_k > 0$, resp. < 0,*

$$
\begin{aligned}
u^{k,\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t_k)) \quad = \quad & \pm i\Phi(t_k, q_k^{\mathbb{C}}(t_k), q_k^{\mathbb{C}*}(t_k)) + \tfrac{1}{\pi}\textstyle\sum_{n=1}^m [\int_0^{\ell_n}[\Phi(s_n; q_k^{\mathbb{C}}(t_k), q_k^{\mathbb{C}*}(t_k)) \\
& + iu^n_{\lambda_j}(q(s_n))R(s_n, q_k^{\mathbb{C}}(t_k), q_k^{\mathbb{C}*}(t_k))] \tfrac{q_n'(s_n)}{q_n(s_n) - q_k^{\mathbb{C}}(t_k)} ds_n] \\
& + \tfrac{1}{\pi}\textstyle\sum_{n=1}^m [\int_0^{\ell_n}[\Phi(s_n; q_k^{\mathbb{C}}(t_k), q_k^{\mathbb{C}*}(t_k)) \\
& - iu^n_{\lambda_j}(q(s_n))R(s_n, q_k^{\mathbb{C}}(t_k), q_k^{\mathbb{C}*}(t_k))] \tfrac{\bar{q_n}'(s_n)}{\bar{q_n}(s_n) - q_k^{\mathbb{C}*}(t_k)} ds_n] \\
& - \tfrac{2}{\pi}\textstyle\sum_{n=1}^m \int_0^{\ell_n} u^n_{\lambda_j}(q(s_n)) \tfrac{\partial B}{\partial n}(s_n; q_k^{\mathbb{C}}(t_k), q_k^{\mathbb{C}*}(t_k)) ds_n.
\end{aligned}
\tag{3.18}
$$

A proof is supplied in Appendix 8.

3.4.4. *The case $C \cap \partial\Omega \neq \emptyset$ with $C \neq \partial\Omega$.* This is a somewhat complicated hybrid of the previous cases, but we briefly explain how to deal with it. Unlike the case of interior C, and like the case of $\partial\Omega$, we use Green's formula

$$
\varphi_{\lambda_j}(x, y) = \int_C \big(\partial_\nu G(\lambda_j, q, (x, y))\varphi_{\lambda_j}(q) - G(\lambda_j, q, (x, y))\partial_\nu\varphi_{\lambda_j}(q)\big)\, d\sigma(q) \tag{3.19}
$$

on the domain int (C) bounded by C. Above, $(x, y) \in \mathbb{R}^2, q \in C$, $d\sigma$ is arc-length measure on C and ∂_ν is the normal derivative by the interior unit normal. There are no boundary conditions on C to simplify (3.19) and the jumps formula, so the restriction of φ_λ to C satisfies

$$
\varphi_{\lambda_j}|_C\,(q) = 2\left(\int_C \frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q};, q)\varphi_{\lambda_j}(q) d\sigma(\tilde{q}) - \int_C S(\lambda_j, \tilde{q}, q)\frac{\partial}{\partial\nu_{\tilde{q}}}\varphi_{\lambda_j}(\tilde{q})\, d\sigma(\tilde{q})\right). \tag{3.20}
$$

The first term is handled exactly as in the case $C = \partial\Omega$, while the second term (the single layer potential term) is new.

Proposition 9. *The restrictions $\varphi_{\lambda_j}|_C$ of the eigenfunctions of the Neumann problem to an analytic curve C with $C \cap \partial\Omega \neq \emptyset$ admit the following holomorphic extension to a uniform* *tube around C in its complexification $C_{\mathbb{C}}$: (for $\Im t > 0, < 0$)*

$$\begin{aligned}
\varphi_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t)) \;\; = \;\; & \pm i\Phi(t, q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t)) \\
& + \int_0^{2\pi} \left[\Phi(s; q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t)) + i\varphi_{\lambda_j}(q_C(s)) R(s, q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t))\right] \tfrac{q_C'(s)}{q_C(s) - q_C^{\mathbb{C}}(t)} ds \\
& + \int_0^{2\pi} \left[\Phi(s; q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t)) - i\varphi_{\lambda_j}(q_C(s)) R(s, q_C^{\mathbb{C}*}(t), q_C^{\mathbb{C}}(t))\right] \tfrac{\bar{q_C}'(s)}{\bar{q_C}(s) - q_C^{\mathbb{C}*}(t)} ds \\
& -2\int_0^{2\pi} \varphi_{\lambda_j}(q_C(s)) \tfrac{\partial B}{\partial n}(s; q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t)) ds. \\
& - \int_0^{2\pi} R(s, q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t)) L(s,t) \partial_\nu \varphi_{\lambda_j}(q_C(s))\, ds \\
& \mp 2\pi i \int_0^{t} R(s, q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t)) \partial_\nu \varphi_{\lambda_j}(q_C(s))\, ds \\
& -2\int_0^{2\pi} B(s, q_C^{\mathbb{C}}(t), q_C^{\mathbb{C}*}(t)) \partial_\nu \varphi_{\lambda_j}(q_C(s)) ds.
\end{aligned} \tag{3.21}$$

In (3.21), $L(s,t)$ is a specific branch of the multi-valued analytic continuation of $\log r^2(s,t)$ *defined in Appendix 8.*

4. Growth of zeros and growth of $u_{\lambda_j}^{\mathbb{C}}$

The main purpose of this section is to give an upper bound for the number of complex zeros of $u_{\lambda_j}^{\mathbb{C}}$ in $(\partial\Omega)_{\mathbb{C}}$ in terms of the growth of $\left|u_{\lambda_j}^{\mathbb{C}}(q^{\mathbb{C}}(t))\right|$. The method applies equally well to any closed analytic curve $C \subset \Omega$, so we give an upper bound for the number of zeros of $\varphi_{\lambda_j}|_C^{\mathbb{C}}$ in $C_{\mathbb{C}}$, or more precisely in the annulur region $q_C^{\mathbb{C}}(A(\epsilon))$ where $A(\epsilon) = \{z \in \mathbb{C}; e^{-\epsilon} < |z| < e^{\epsilon}\}$.

To simplify notation, in this section we write $u_{\lambda_j} = \varphi_{\lambda_j}|_C$ for any real analytic curve $C \subset \Omega$ regardless of whether or not $C = \partial\Omega$. For $\lambda_j \in Sp(\sqrt{\Delta})$ and for a region $D \subset C_{\mathbb{C}}$ we denote by

$$n(\lambda_j, D) = \#\{q_C^{\mathbb{C}}(t) \in D : u_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t)) = 0\}. \tag{4.1}$$

To orient the reader, we recall that the classical distribution theory of holomorphic functions is concerned with the relation between the growth of the number of zeros of a holomorphic function f and the growth of $\max_{|z|=r} \log|f(z)|$ on discs of increasing radius. In contrast, our problem concerns the family of functions $\{\varphi_{\lambda_j}\}$ on a fixed domain. The following estimate, suggested by Lemma 6.1 of Donnelly-Fefferman [DF], gives an upper bound on the number of zeros in terms of the growth of the family:

Proposition **10.** *Suppose that C is a good real analytic curve in the sense of (1.4). Normalize u_{λ_j} so that $||u_{\lambda_j}||_{L^2(C)} = 1$. Then, there exists a constant $C(\epsilon) > 0$ such that for any $\epsilon > 0$,*

$$n(\lambda_j, Q_C^{\mathbb{C}}(A(\epsilon/2))) \leq C(\epsilon) \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon))} \left|\log|u_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))|\right|.$$

Proof. Let G_ϵ denote the Dirichlet Green's function of the 'annulus' $Q_C^{\mathbb{C}}(A(\epsilon))$. Also, let $\{a_k\}_{k=1}^{n(\lambda_j, Q_C^{\mathbb{C}}(A(\epsilon/2)))}$ denote the zeros of $u_{\lambda_j}^{\mathbb{C}}$ in the sub-annulus $Q_C^{\mathbb{C}}(A(\epsilon/2))$. Let $U_{\lambda_j} =$

$\frac{u^{\mathbb{C}}_{\lambda_j}}{||u^{\mathbb{C}}_{\lambda_j}||_{Q^{\mathbb{C}}_C(A(\epsilon))}}$ where $||u||_{Q^{\mathbb{C}}_C(A(\epsilon))} = \max_{\zeta \in Q^{\mathbb{C}}_C(A(\epsilon))} |u(\zeta)|$. Then,

$$\begin{aligned}\log |U_{\lambda_j}(q^{\mathbb{C}}_C(t))| &= \int_{Q^{\mathbb{C}}_C((A(\epsilon/2)))} G_\epsilon(q^{\mathbb{C}}_C(t), w)\partial\bar{\partial} \log |u^{\mathbb{C}}_{\lambda_j}(w)| + H_{\lambda_j}(q^{\mathbb{C}}_C(t)) \\ &= \sum_{a_k \in Q^{\mathbb{C}}_C(A(\epsilon/2)): u^{\mathbb{C}}_{\lambda_j}(a_k)=0} G_\epsilon(q^{\mathbb{C}}_C(t), a_k) + H_{\lambda_j}(q^{\mathbb{C}}_C(t)),\end{aligned}$$

since $\partial\bar{\partial} \log |u^{\mathbb{C}}_{\lambda_j}(w)| = \sum_{a_k \in C_{\mathbb{C}}: u^{\mathbb{C}}_{\lambda_j}(a_k)=0} \delta_{a_k}$. Moreover, the function H_{λ_j} is sub-harmonic on $Q^{\mathbb{C}}_C(A(\epsilon))$ since

$$\partial\bar{\partial} H_{\lambda_j} = \partial\bar{\partial} \log |U_{\lambda_j}(q^{\mathbb{C}}_C(t))| - \sum_{a_k \in Q^{\mathbb{C}}_C(A(\epsilon/2)): u^{\mathbb{C}}_{\lambda_j}(a_k)=0} \partial\bar{\partial} G_\epsilon(q^{\mathbb{C}}_C(t), a_k) = \sum_{a_k \in Q^{\mathbb{C}}_C(A(\epsilon)) \setminus Q^{\mathbb{C}}_C(A(\epsilon/2))} \delta_{a_k} > 0.$$

So, by the maximum principle for subharmonic functions,

$$\max_{Q^{\mathbb{C}}_C(A(\epsilon))} H_{\lambda_j}(q^{\mathbb{C}}_C(t)) \leq \max_{\partial Q^{\mathbb{C}}_C(A(\epsilon))} H_{\lambda_j}(q^{\mathbb{C}}_C(t)) = \max_{\partial Q^{\mathbb{C}}_C(A(\epsilon))} \log |U_{\lambda_j}(q^{\mathbb{C}}_C(t))| = 0.$$

It follows that

$$\log |U_{\lambda_j}(q^{\mathbb{C}}_C(t))| \leq \sum_{a_k \in Q^{\mathbb{C}}_C(A(\epsilon/2)): u^{\mathbb{C}}_{\lambda_j}(a_k)=0} G_\epsilon(q^{\mathbb{C}}_C(t), a_k), \tag{4.2}$$

hence that

$$\max_{q^{\mathbb{C}}_C(t) \in Q^{\mathbb{C}}_C(A(\epsilon/2))} \log |U_{\lambda_j}(q^{\mathbb{C}}_C(t))| \leq \left(\max_{z,w \in Q^{\mathbb{C}}_C(A(\epsilon/2))} G_\epsilon(z, w) \right) \; n(\lambda_j, Q^{\mathbb{C}}_C(A(\epsilon/2))). \tag{4.3}$$

Now $G_\epsilon(z, w) \leq \max_{w \in Q^{\mathbb{C}}_C(\partial A(\epsilon))} G_\epsilon(z, w) = 0$ and $G_\epsilon(z, w) < 0$ for $z, w \in Q^{\mathbb{C}}_C(A(\epsilon/2))$. It follows that there exists a constant $\nu(\epsilon) < 0$ so that $\max_{z,w \in Q^{\mathbb{C}}_C(A(\epsilon/2))} G_\epsilon(z, w) \leq \nu(\epsilon)$. Hence,

$$\max_{q^{\mathbb{C}}_C(t) \in Q^{\mathbb{C}}_C(A(\epsilon/2))} \log |U_{\lambda_j}(Q^{\mathbb{C}}_C(t))| \leq \nu(\epsilon) \;\; n(\lambda_j, Q^{\mathbb{C}}_C(A(\epsilon/2))). \tag{4.4}$$

Since both sides are negative, we obtain

$$\begin{aligned} n(\lambda_j, Q^{\mathbb{C}}_C(A(\epsilon/2))) &\leq \tfrac{1}{|\nu(\epsilon)|} \left| \max_{q^{\mathbb{C}}_C(t) \in Q^{\mathbb{C}}_C(A(\epsilon/2))} \log |U_{\lambda_j}(q^{\mathbb{C}}_C(t))| \right| \\ &\leq \tfrac{1}{|\nu(\epsilon)|} \left(\max_{q^{\mathbb{C}}_C(t) \in Q^{\mathbb{C}}_C(A(\epsilon))} \log |u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}_C(t))| - \max_{q^{\mathbb{C}}_C(t) \in Q^{\mathbb{C}}_C(A(\epsilon/2))} \log |u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}_C(t))| \right) \\ &\leq \tfrac{1}{|\nu(\epsilon)|} \;\; \max_{q^{\mathbb{C}}_C(t) \in Q^{\mathbb{C}}_C(A(\epsilon))} \log |u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}_C(t))|, \end{aligned} \tag{4.5}$$

where in the last step we use that $\max_{q^{\mathbb{C}}_C(t) \in Q^{\mathbb{C}}_C(A(\epsilon/2))} \log |u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}_C(t))| \geq 0$, which holds since $|u^{\mathbb{C}}_{\lambda_j}| \geq 1$ at some point in $Q^{\mathbb{C}}_C(A(\epsilon/2))$. Indeed, by our normalization, $\|u_{\lambda_j}\|_{L^2(C)} = 1$, and so there must already exist points on the real curve C with $|u_{\lambda_j}| \geq 1$. Putting $C(\epsilon) = \frac{1}{|\nu(\epsilon)|}$ finishes the proof. $\square$

5. Zeros on interior curves: Proof of Theorem 6

In this section we prove Theorem 6. We prove it before Theorem 5 since it is simpler and provides a model for the more difficult boundary case. It is simpler because the analytic continuation of the layer potential formula (3.13) is straightforward and does not use Proposition 7. We use the analytic continuation to bound $\max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon))} \Big| \log |\varphi_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))| \Big|$ from above, and then an application of Proposition 10 concludes the proof.

Proof. As above, the arc-length parametrization of C is denoed by by $q_C : [0, 2\pi] \to C$ and the corresponding arc-length parametrization of the boundary, $\partial\Omega$, by $q : [0, 2\pi] \to \partial\Omega$. Since the boundary and C do not intersect, the logarithm $\log r^2(q(s); q_C^C(t))$ is well defined and the holomorphic continuation of equation (3.13) is given by:

$$\varphi_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t)) = \int_0^{2\pi} N(\lambda_j, q(s), q_C^{\mathbb{C}}(t))\, u_{\lambda_j}(q(s)) d\sigma(s), \tag{5.1}$$

From the basic formula (3.12) for $N(\lambda_j, q, q_C)$ and the standard integral formula for $\mathrm{Ha}_1^{(1)}(z)$ (see (9.5) in Appendix 9), one easily gets an asymptotic expansion in λ_j of the form:

$$N(\lambda_j, q(s), q_C^{\mathbb{C}}(t)) = e^{i\lambda_j r(q(s); q_C^{\mathbb{C}}(t))} \sum_{m=0}^{k} a_m(q(s), q_C^{\mathbb{C}}(t))\, \lambda_j^{1/2-m} + O(e^{i\lambda_j r(q(s); q_C^{\mathbb{C}}(t))}\, \lambda_j^{1/2-k-1}). \tag{5.2}$$

Note that the expansion in (5.2) is valid since for interior curves,

$$C_0 := \min_{(q_C(t), q(s)) \in C \times \partial\Omega} |q_C(t) - q(s)|^2 > 0.$$

Then, $\Re r^2(q(s); q_C^C(t)) > 0$ as long as

$$|\Im q_C^{\mathbb{C}}(t)|^2 < C_0. \tag{5.3}$$

So, the principal square root of r^2 has a well-defined holomorphic extension to the tube (5.3) containing C. We have denoted this square root by r in (5.2).

Substituting (5.2) in the analytically continued single layer potential integral formula (5.1) proves that for $t \in A(\epsilon)$ and $\lambda_j > 0$ sufficiently large,

$$\varphi_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t)) = 2\pi \lambda_j^{1/2} \int_0^{2\pi} e^{i\lambda_j r(q(s): q_C^{\mathbb{C}}(t))} a_0(q(s), q_C^{\mathbb{C}}(t))(1 + O(\lambda_j^{-1}))\, u_{\lambda_j}(q(s)) d\sigma(s). \tag{5.4}$$

Taking absolute values of the integral on the RHS in (5.4) and applying the Cauchy-Schwartz inequality proves

Lemma 11. *For $t \in [0, 2\pi] + i[-\epsilon, \epsilon]$ and $\lambda_j > 0$ sufficiently large*

$$|\varphi_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))| \leq C_1 \lambda_j^{1/2} \exp \lambda_j \left(\max_{q(s) \in \partial\Omega} \Re\, i r(q(s); q_C^{\mathbb{C}}(t)) \right) \cdot \|u_{\lambda_j}\|_{L^2(\partial\Omega)}.$$

From the pointwise upper bounds in Lemma 11, it is immediate that

$$\log \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon))} |\varphi_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))| \leq C_{\max} \lambda_j + C_2 \log \lambda_j + \log \|u_{\lambda_j}\|_{L^2(\partial\Omega)}, \tag{5.5}$$

where,

$$C_{\max} = \max_{(q(s), q_C^{\mathbb{C}}(t)) \in \partial\Omega \times Q_C^{\mathbb{C}}(A(\epsilon))} \Re\, ir(q(s); q_C^{\mathbb{C}}(t)).$$

Finally, we use that $\log \|u_{\lambda_j}\|_{L^2(\partial\Omega)} = O(\lambda_j)$ by the assumption that C is a good curve and apply Proposition 10 to get that $n(\lambda_j, C) = O(\lambda_j)$.

This completes the proof of Theorem 6.

□

5.1. **A remark on goodness and unique continuation.** Before leaving Theorem 6, we make a few remarks on the goodness requirement (1.4): On any interior curve $C \subset \Omega$, goodness is implied by an exponential growth estimate involving only the Cauchy data $(\varphi_{\lambda_j}|_C, \partial_\nu \varphi_{\lambda_j}|_C)$ along C. This is a consequence of the following unique continuation argument.

Assume that C is a closed curve in the interior of Ω. Let U_C be the domain with boundary $C \cup \partial\Omega$. It follows from the Sobolev restriction theorem that

$$\|\varphi_{\lambda_j}\|^2_{L^2(\partial\Omega)} \leq C \|\varphi_{\lambda_j}\|^2_{H^{1/2}(U_C)}. \tag{5.6}$$

Let $int(C)$ be the interior of the domain bounded by the curve C and take $x \in int(C)$. From the potential layer formula (see 3.5) $\varphi_{\lambda_j}(x) = \int_C (\partial_{\nu(q)} G(x, q; \lambda) \varphi_{\lambda_j}(q) - G(x, q; \lambda_j) \partial_{\nu_q} \varphi_{\lambda_j}(q)) d\sigma(q)$ and so, by squaring both sides, using the bounds $|G(x, q, \lambda_j)| = O(1)$ and $|\partial_{\nu_q} G(x, q; \lambda_j)| = O(\lambda_j^{1/2})$ and applying Cauchy Schwartz, one gets

$$\|\varphi_{\lambda_j}\|^2_{L^2(int(C))} \leq C\lambda_j (\|\varphi_{\lambda_j}\|^2_{L^2(C)} + \|\partial_\nu \varphi_{\lambda_j}\|^2_{L^2(C)}). \tag{5.7}$$

By a standard Carleman estimate/unique continuation argument [EZ, Ta]:

$$\|\varphi_{\lambda_j}\|^2_{H^{1/2}(U_C)} \leq e^{C\lambda_j} \|\varphi_{\lambda_j}\|^2_{L^2(int(C))} \leq \lambda_j e^{C\lambda_j} (\|\varphi_{\lambda_j}\|^2_{L^2(C)} + \|\partial_\nu \varphi_{\lambda_j}\|^2_{L^2(C)}),$$

where the last inequality follows from (5.7). Inserting the last bound on the RHS in (5.6) yields the comparison estimate relating Cauchy data along C and $\partial\Omega$:

$$\|\varphi_{\lambda_j}\|_{L^2(\partial\Omega)} \leq e^{C\lambda_j} (\|\varphi_{\lambda_j}\|_{L^2(C)} + \|\partial_\nu \varphi_{\lambda_j}\|_{L^2(C)}). \tag{5.8}$$

As an immediate consequence of (5.8) we note that (1.4) follows from the exponential bound

$$\|\partial_\nu \varphi_{\lambda_j}\|_{L^2(C)} \leq e^{C\lambda_j} \|\varphi_{\lambda_j}\|_{L^2(C)} \tag{5.9}$$

involving only Cauchy data along C.

A natural question is whether (5.9) is automatically satisfied when φ_{λ_j} does not vanish identically on C?

6. Boundary zeros: proof of Theorem 5

The proof of Theorem 5 is more complicated than that for interior curves because we need to invert the Volterra operator of Proposition 7.

We recall that the analytic continuation of u_{λ_j} is the solution of a Volterra equation,

$$(I + K_{\lambda_j}) u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t)) = U_{\lambda_j}(q^{\mathbb{C}}(t)), \tag{6.1}$$

where $U_{\lambda_j}(q^{\mathbb{C}}(t))$ has an explicit analytic continuation, and where

$$K_{\lambda_j} u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t)) = \int_0^t \frac{\partial R}{\partial \nu}(\lambda_j, s, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(s)) ds. \tag{6.2}$$

Here, $R = A$ is the Riemann function, so explicitly

$$K_{\lambda_j} u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t)) = \int_0^t \frac{\partial J_0(\lambda_j r)}{\partial \nu}(\lambda_j, s, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(s)) ds.$$

Therefore,

$$K_{\lambda_j}(t,s) = \mathbf{1}_{[0,t]}(s) J_1(\lambda_j r) r \frac{\partial \log r}{\partial \nu}(s,t) = \mathbf{1}_{[0,t]}(s) r J_1(\lambda_j r) \left(\frac{q(s)}{q(s) - q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)} \right),$$

where $\mathbf{1}_{[0,t]}$ is the indicator function of the interval $[0,t]$. We note that the pole of $\frac{q(s)}{q(\Re t + is) - q^{\mathbb{C}*}(t)}$ at the upper limit of integration $s = t$ is cancelled because the Taylor expansion of $rJ_1(r)$ begins with r^2. So the integrand is regular and holomorphic along the path of integration.

On the right side of the Volterra equation,

$$\begin{aligned} u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t)) \mp i\Phi(t, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) &= \int_0^{2\pi} \left[\Phi(s; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) + i u_{\lambda_j}(q(s)) R(s, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) \right] \tfrac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} ds \\ &+ \int_0^{2\pi} \left[\Phi(s; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) - i u_{\lambda_j}(q(s)) R(s, q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) \right] \tfrac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)} ds \\ &- 2 \int_0^{2\pi} u_{\lambda_j}(q(s)) \tfrac{\partial B}{\partial \nu}(s; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) ds \end{aligned} \tag{6.3}$$

the Cauchy data u_{λ_j} is only integrated over the real domain where by a standard Sobolev estimate it has polynomial growth in λ_j. And further, the Riemann function and other special functions occurring there have exponential growth with exponent bounded by the ambient complexified distance function. The main problem is thus to invert the Volterra operator $I + K_{\lambda_j}$ and to obtain a similar growth estimate for $(I + K_{\lambda_j})^{-1}(RHS)$.

We first simplify the operator, K_{λ_j}. Given $t = \Re t + i \Im t$ we may choose the contour to go along the real interval $[0, \Re t]$ and then to go along the vertical line segment $\Re t + is$ for $s \in [0, \Im t]$. This decomposes $K_{\lambda_j} = K^{(1)}_{\lambda_j} + K^{(2)}_{\lambda_j}$, where

$$K^{(1)}_{\lambda_j} u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t)) = \int_0^{\Re t} u_{\lambda_j}(q(s)) \frac{\partial}{\partial \nu} R(\lambda_j; s; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) ds \tag{6.4}$$

and where

$$K^{(2)}_{\lambda_j} u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t)) = \int_0^{\Im t} u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(\Re t + is)) \frac{\partial}{\partial \nu} R(\lambda_j; \Re t + is; q^{\mathbb{C}}(t), q^{\mathbb{C}*}(t)) ds.$$

We move the $K^{(1)}_{\lambda_j}$ term again to the right side since it only involves the Cauchy data on the real domain.

We now write $t = \Re t + i \Im t$ and treat $\Re t$ as a parameter. We need to study the mapping properties of $K^{(2)}_{\lambda_j}$ and $(I + K^{(2)}_{\lambda_j})^{-1}$ on the weighted Hilbert space $L^2([-\epsilon, \epsilon], e^{-\lambda_j |\Im t|} d\Im t)$.

6.0.1. *Model example.* As a model example, we consider the operator $K_{\lambda_j} u(y) = \int_0^y e^{\lambda_j (y-s)} u(s) ds$. A simple calculation shows that for $n \geq 0$,

$$K_{\lambda_j}^{n+1}(y,s) = e^{\lambda_j (y-s)} \frac{(y-s)^n}{n!},$$

and

$$(I - K_{\lambda_j})^{-1}(y,s) = e^{(\lambda_j + 1)(y-s)}.$$

Hence, in the model example, the exponential growth of the kernel $(I - K_{\lambda_j})^{-1}(s, \Im t)$ is the same as for $K_{\lambda_j}(s, \Im t)$.

6.1. **Upper bounds.** In view of the growth estimate for complex zeros in Proposition 10, one needs to determine asymptotic pointwise upper bounds for the $|u_{\lambda_j}^{\mathbb{C}}(q^{\mathbb{C}}(t))|$ as $\lambda_j \to \infty$. In this section, we prove:

LEMMA **12.** *Given $t \in [0,l] + i[-\epsilon, \epsilon]$ and $\lambda_j > 0$ sufficiently large, there exists a constant $C > 0$ such that*

$$\left| u_{\lambda_j}^{\mathbb{C}}(q^{\mathbb{C}}(t)) \right| \leq \exp C\lambda_j |\Im t| \cdot \|u_{\lambda_j}\|_{L^2(\partial\Omega)}.$$

Proof. Let $C_0 > 0$ be a constant. To bound the kernel $K_{\lambda_j}^{(2)}(\Im t, s)$ we split the analysis into two cases: (i) $|r(\Re t + is, t)| \leq \frac{1}{C_0}$ and (ii) $|r(\Re t + is, t)| \geq \frac{1}{C_0}$.

6.1.1. *The range $|r| \geq \frac{1}{C_0}$.* In this range, J_1 has an asymptotic expansion given by

$$J_1(\lambda_j r) = \sum_{m=0}^{k} \lambda_j^{-1/2-m} a_m(r) e^{i\lambda_j r} + O(\lambda_j^{-1/2-k-1} e^{\lambda_j \Im r}).$$

From the identity

$$\begin{aligned} K_{\lambda_j}^{(2)} &= \partial_\nu J_0(\lambda_j r) \\ &= \mathbf{1}_{[0,\Im t]}(s)\, J_1(\lambda_j r)\, r\, \tfrac{\partial \log r}{\partial \nu} \\ &= \mathbf{1}_{[0,\Im t]}(s)\, r\, J_1(\lambda_j r)\, [\tfrac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} - \tfrac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)}], \end{aligned}$$

it follows that there exists a symbol S_{λ_j} of order $-\frac{1}{2}$ such that

$$|K_{\lambda_j}^{(2)}(\Im t, s)| \leq S_{\lambda_j}(\Im t, s)\, \mathbf{1}_{[0,\Im t]}(s)\, e^{\lambda_j |\Im r(\Re t + is, t)|}. \tag{6.5}$$

The estimate (6.5) is locally uniform in $\Re t$ and the dependence on the parameter $\Re t$ is implicit.

6.1.2. *The range $|r| \leq \frac{1}{C_0}$.* In this range, the asymptotic expansion breaks down when $|r| \ll \frac{1}{\lambda}$. Instead, we use the standard integral representation for J_1 (see (9.4) in Appendix 9) to get the bound

$$|J_1(\lambda_j r)| \leq C e^{\lambda_j |\Im r|}, \tag{6.6}$$

and so, when $|r(\Re t + is, t)| \leq \frac{1}{C_0}$,

$$|K_{\lambda_j}^{(2)}(\Im t, s)| \leq C \mathbf{1}_{[0,\Im t]}(s)\, e^{\lambda_j |\Im r(\Re t + is, t)|}. \tag{6.7}$$

Combining the estimates (6.5) and (6.7), it follows that

$$|K^{(2)}_{\lambda_j}(\Im t, s)| \leq C\mathbf{1}_{[0,\Im t]}(s)\, e^{\lambda_j |\Im r(\Re t+is,t)|}, \tag{6.8}$$

locally uniformly in $|s|+|\Im t|$ and in $\Re t$. Again, the dependence of K_{λ_j} on the parameter $\Re t$ has been suppressed.

6.1.3. *Pointwise estimates for r.* By definition,

$$r(\Re t+is,t) = \langle q^{\mathbb{C}}(\Re t+i\Im t) - q^{\mathbb{C}}(\Re t+is), q^{\mathbb{C}}(\Re t+i\Im t) - q^{\mathbb{C}}(\Re t+is)\rangle^{\frac{1}{2}} \tag{6.9}$$

Taylor expansion around $s=\Im t$ in (6.9) gives

$$|r(\Re t+is,t)| \leq C|\Im t - s|, \tag{6.10}$$

since,

$$q^{\mathbb{C}}(t) - q^{\mathbb{C}}(\Re t+is) = \int_0^1 \frac{d}{dt} q^{\mathbb{C}}(\Re t+i(t\Im t+(1-t)s))dt = (\Im t - s)\int_0^1 (q^{\mathbb{C}})'(\Re t+i(t\Im t+(1-t)s))dt.$$

From (6.10) and the bound (6.8) it follows that there are constants $C_j>0; j=1,2$, such that

$$|K^{(2)}_{\lambda_j}(\Im t, s)| \leq C_1\mathbf{1}_{[0,\Im t]}(s) e^{C_2|\Im t - s|}. \tag{6.11}$$

Next, we expand $(I-K^{(2)}_{\lambda_j})^{-1}$ in a norm convergent geometric series,

$$\sum_{n=0}^{\infty}[K^{(2)}_{\lambda_j}]^n(\Im t, s), \quad ; [K^{(2)}_{\lambda_j}]^n(\Im t, s) := \int_0^{\Im t}\int_0^{s_n}\cdots\int_0^{s_1} K^{(2)}_{\lambda_j}(\Im t, s_n)\cdots K^{(2)}_{\lambda_j}(s_1,s)\, ds_1\cdots ds_n.$$

We recall that the n-simplex Δ_n is defined by

$$\{(s_1,\dots,s_n) : 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1\}.$$

Let $\Im t \cdot \Delta_n$ be the scaled simplex. Applying the estimate (6.11) to each factor in the above formula for $[K^{(2)}_{\lambda_j}]^n$ gives the following pointwise bound:

$$|[K^{(2)}_{\lambda_j}]^n(\Im t, s)| \leq \int_{\Im t\cdot\Delta_n} e^{C\lambda_j(\Im t - s_n)}\cdot e^{C\lambda_j(s_n - s_{n-1})}\cdots e^{C\lambda_j(s_2-s_1)}\cdot e^{C\lambda(s_1-s)}\, ds_1\cdots ds_n.$$

So, by the model example,

$$|(I-K^{(2)}_{\lambda_j})^{-1}(\Im t, s)| \leq e^{(C+1)\lambda_j|\Im t - s|}\cdot \mathbf{1}_{[0,\Im t]}(s).$$

To complete the proof of Lemma 12, we note that for $q^{\mathbb{C}}(t) \in Q^{\mathbb{C}}(A(\epsilon))$, the right side of the analytic continuation formula (6.3) together with the $K^{(1)}_{\lambda}$ term satisfies the estimate

$$(**) \leq C_1 \exp\left(\lambda_j \max_{q(s)\in\partial\Omega} \Re\, ir(t,s)\right)\cdot \|u_{\lambda_j}\|_{L^2(\partial\Omega)} \leq C_1 e^{C_2\lambda_j|\Im t|}, \tag{6.12}$$

since by our normalization, $\|u_{\lambda_j}\|_{L^2(\partial\Omega)}=1$. It follows that

$$(I-K^{(2)}_{\lambda_j})^{-1}(**) \leq C\int_0^{\Im t} e^{(C+1)\lambda_j(\Im t - s)} e^{C_2\lambda_j s} ds \leq C\exp(\lambda_j \max\{C+1, C_2\}|\Im t|). \tag{6.13}$$

This finishes the proof of Lemma 12. $\square$

By Lemma 12, $\left|\log \max_{q^{\mathbb{C}}(t)\in Q^{\mathbb{C}}(A(\epsilon))} |u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t))|\right| \leq C\lambda_j$ and Proposition 10 then implies that $n(\lambda_j;\partial\Omega) =\leq C_\Omega \lambda_j$. □

7. Critical points: Proof of Theorems 3 and 5 (2)

We now prove part (2) of Theorem 5 concerning the growth of critical points. It immediately implies Theorem 3. The argument is similar to that for counting zeros, the only change being that we now take the derivative and restrict to the boundary in (3.7). For the sake of brevity we only sketch the proof.

In the Dirichlet case, the jumps formula for the double layer potential gives (3.15) except that now u_{λ_j} denotes the restriction to the boundary of $\frac{\partial \varphi_{\lambda_j}}{\partial \nu}$. We refer to [T, HZ] for background. We then define $n(\lambda_j, D)$ as in (4.1) but for the new u_{λ_j}. The layer potential representation implies the analogue of Lemma 12 and by Proposition 10 we conclude that the number of complex zeros (hence real zeros) is $O(\lambda_j)$.

In the Neumann case, we must take the tangential derivative $\frac{\partial}{\partial t}(u_{\lambda_j}\circ q)(t)$. Since the normal derivative is zero, the critical points of the tangential derivative are critical points of the eigenfunction along the boundary. The tangential derivative now has the representation,

$$\begin{aligned}\frac{\partial}{\partial t}(u_{\lambda_j}\circ q)(t) \qquad &= \frac{1}{2\pi}\int_0^{2\pi}(-u_{\lambda_j}(q(s))\frac{\partial^2 A}{\partial t\partial \nu}(s,t))\log r^2 ds\\ &+\int_0^{2\pi}(-u_{\lambda_j}(q(s))\frac{\partial A}{\partial \nu}(s,t))\frac{\partial}{\partial t}\log r^2 ds \\ &-\frac{1}{\pi}\int_0^{2\pi} u_{\lambda_j}(q(s))(\frac{\partial}{\partial t}(A(s,t)\frac{1}{r}\frac{\partial r}{\partial \nu}))ds - \frac{1}{\pi}\int_0^{2\pi}(-u_{\lambda_j}(q(s))\frac{\partial^2 B}{\partial\nu\partial t}(s,t)ds.\end{aligned} \tag{7.1}$$

The analytic continuation of Proposition 7 applies equally to the equation (7.1) and the analytic continuation of $\frac{\partial}{\partial t}(u_{\lambda_j}\circ q)(t)$ has the form (6.1) with a slight change in K_{λ_j}. However, the phase function is the same, so the proof of Lemma 12 applies with only small modifications to the new Volterra operator, giving the upper bound

$$\left|\partial_t u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t))\right| \leq \lambda_j \exp C\lambda_j|\Im t| \cdot \|u_{\lambda_j}\|_{L^2(\partial\Omega)}. \tag{7.2}$$

Only the exponential growth rate is significant here. We then apply Proposition 10 to bound the number of zeros by the maximum of

$$\log\frac{|\partial_t u^{\mathbb{C}}_{\lambda_j}(q^{\mathbb{C}}(t))|}{||\partial_t u_{\lambda_j}||_{L^2(\partial\Omega)}} \leq C_1\lambda_j + C_2\log\lambda_j + C_3\log\frac{||u_{\lambda_j}||_{L^2(\partial\Omega)}}{||\partial_t u_{\lambda_j}||_{L^2(\partial\Omega)}}. \tag{7.3}$$

Of course this estimate assumes $\partial_t u_{\lambda_j}(q(t)) \neq 0$ identically, as can happen with radial eigenfunctions on the disc. Assuming (1.2), the third term of (7.3) is $\leq C_4\lambda_j$, completing the proof.

8. Appendix: Proofs of Propositions 7, 8 and 9

This Appendix is devoted to Propositions 7, 8 and 9. Since they do not appear to be well-known, or to be proved in detail in [M1, M2, V], we supply the details of the proof. To simplify notation, when the context is clear, we often denote the holomorphic functions $q^{\mathbb{C}}(t)$ (resp. $q^{\mathbb{C}*}(t)$) simply by $q(t)$ (resp. $q^*(t)$).

8.1. **Proof of Proposition 7.**

Proof. We will analytically continue the formula (3.15). Although u_{λ_j} is real analytic on $\partial\Omega$ and hence admits an analytic continuation to a small complex 'tube' $(\partial\Omega)_{\mathbb{C}}$, it is not clear that the representation (3.15) can be extended analytically due to singularities of the integrand. Moreover, it is not clear that the right side of (3.17) is in fact complex analytic. The main task in the proof is to clarify these points.

We begin by showing that the last two terms of (3.15) analytically continue in a straightforward way.

LEMMA **13.** *The integrals (i)* $\frac{1}{\pi}\int_0^{2\pi} u_{\lambda_j}(q(s))A(s,t)\frac{1}{r}\frac{\partial r}{\partial\nu}(s,t)ds$, *resp. (ii)* $\frac{1}{\pi}\int_0^{2\pi} u_{\lambda_j}(q(s))\frac{\partial B}{\partial\nu}(s,t)ds$, *are real analytic on the parameter interval* S^1 *parametrizing* $\partial\Omega$ *and are holomorphically extended to an annulus by the formulae*

$$(i)\quad \int_0^{2\pi} iu_{\lambda_j}(q(s))R(s,q(t),q^*(t))\left(\frac{q'(s)}{q(s)-q(t)}-\frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}\right)ds,$$

resp.

$$(ii)\quad -2\int_0^{2\pi} u_{\lambda_j}(q(s))\frac{\partial B}{\partial\nu}(s;q(t),q^*(t))ds.$$

Proof. Any derivative of $\log r^2$ is unambiguously defined and we have

$$\frac{1}{r}\frac{\partial r}{\partial n}=\frac{\partial\log r}{\partial n}=\frac{1}{2i}[\frac{q'(s)}{q(s)-q(t)}-\frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}].$$

In the real domain, $q^*(t)=\bar{q}(t)$, so

$$\frac{1}{r}\frac{\partial r}{\partial n}=\Im\frac{q'(s)}{q(s)-q(t)}.$$

We recall that $\frac{\partial r}{\partial\nu}=|q'(s)|^{-1}\frac{\partial r}{\partial n}$. In terms of the real parametrization $q(s)$,

$$\frac{\partial r}{\partial\nu}=\cos\angle(q(t)-q(s),\nu_{q(s)})$$

vanishes to order one on the diagonal in the real domain so that $\frac{1}{r}\frac{\partial r}{\partial\nu}$ is real and continuous. In complex notation, the same statement follows from the fact that

$$\lim_{t\to s}\frac{q(s)-q(t)}{s-t}=q'(s)\implies\frac{q'(s)}{q(s)-q(t)}=\frac{1}{s-t}+O(1),\quad(s\to t),$$

where $\frac{1}{s-t}$ is real when $s,t\in\mathbb{R}$. Hence, $\Im\frac{q'(s)}{q(s)-q(t)}$ is continuous for $s,t\in[0,2\pi]$ and since $q(s),q(t)$ are real analytic, the map $s\to[\frac{q'(s)}{q(s)-q(t)}-\frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}]$ is a continuous map from $s\in[0,2\pi]$ to the space of holomorphic functions of t.

Since $A=J_0(kr)$ is an analytic function of r^2, $A\frac{\partial\log r}{\partial\nu}$ has the form

$$F(r^2)\;\frac{1}{2i}[\frac{q'(s)}{q(s)-q(t)}-\frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}],$$

for an analytic function F. Clearly, $F(r^2(s,t))$ is also a continuous map from $s\in[0,2\pi]$ to the space of holomorphic functions of t. Hence, so is the product and therefore so is the integral over $s\in[0,2\pi]$ of the product.

Similarly for case (ii). In this case, B is an entire function $H(r^2)$ of r^2 which is of the form $r^2 h(r^2)$ for another entire h. Hence,

$$\frac{\partial B}{\partial \nu} = \quad r^2 H'(r^2) \times \frac{1}{2i}[\frac{q'(s)}{q(s)-q(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}].$$

So the integral (ii) also admits an analytic continuation.

□

Thus, the difficulty in analytic continuation of the representation is entirely with the integral $\int_0^{2\pi}(-u_{\lambda_j}(q(s))\frac{\partial A}{\partial \nu}(s,t))\log r^2(s,t)ds$. Due to the logarithm, the analytic continuation of the integrand is multi-valued in any neighborhood of $\partial\Omega$. Nevertheless, the integral admits a single-valued analytic continuation there.

LEMMA **14.** *The integral $\int_0^{2\pi}(-u_{\lambda_j}(q(s))\frac{\partial A}{\partial \nu}(s,t))\log r^2(s,t)ds$ extends to a holomorphic function of t in a neighborhood of $[0,2\pi]$ in $[0,2\pi]\times[-\epsilon,\epsilon]$ given by*

$$\pm i\Phi(t,q(t),q^*(t)) + \int_0^{2\pi}[\Phi(s;q(t),q^*(t))]\frac{q'(s)}{q(s)-q(t)}ds + \int_0^{2\pi}[\Phi(s;q(t),q^*(t))]\frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}ds,$$

where, $\pm$ corresponds to $\Im t>0$, resp. <0.

Proof. We first observe that

$$\frac{\partial A}{\partial \nu} = J_0'(r)\frac{\partial r}{\partial \nu} = J_1(r)\frac{\partial r}{\partial \nu}. \tag{8.1}$$

Now J_1 is odd in r so (8.1) has the form

$$F(r^2)\ r\frac{\partial r}{\partial \nu} = F(r^2)\ r^2\,\frac{\partial \log r}{\partial \nu} = F(r^2)\ r^2\ \left(\frac{1}{2i}[\frac{q'(s)}{q(s)-q(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}]\right), \tag{8.2}$$

where F is a holomorphic function. From (8.2) it follows that $\frac{\partial A}{\partial \nu}(t,s)$ is a smoothly varying family of holomorphic functions of t in a sufficiently small annulus.

Thus, our problem is a special case of the general problem of analytically continuing the integral $\int_0^{2\pi} f(s)\log r^2(s,t)ds$ where f is real analytic and where $r^2(s,t)$ is given by (3.3). In our case, $f(s)$ also depends holomorphically on t but this does not affect the analytic continuation issue.

We have slit the complex parameter annulus through the vertical segment through 0 to obtain the complex t parameter strip $I=[0,2\pi]+i(-\epsilon,\epsilon)$. To define the analytic continuation, we specify a branch $L(s,t)$ of the multi-valued analytic continuation of $\log r^2(s,t)$ on $[0,2\pi]\times I$. Our integrals only involve pairs $(s,t)\in[0,2\pi]\times I$. For each t, we remove the set $\{s : 0\le s<\Re t\}$. For fixed s, these cuts disconnect the t-strip into four 'quadrants', defined by the inequalities $\Im t>0$ (resp. $\Im t<0$) and $0\le \Re t\le s$ (resp. $s\le \Re t\le 2\pi$). In the right 'half-plane' where $s>\Re t$, we define $\Im L(s,t)$ so that it is continuous in the right 'half-plane' and tends 0 as $\Im t\to 0$ from either top or bottom. In the slit left 'half-plane', $\{s<\Re t\}\backslash[0,\Re t]$ we define $L(s,t)$ by continuation from the right half plane. It then tends to $\mp 2\pi$ as $\Im t\to 0$ from above, resp. below the cut along the 'negative' real axis $s<\Re t$.

To illustrate, we consider the basic case of the circle, where $q(t)=e^{it}$ and where we are defining $\arg((e^{is}-e^{it})(e^{-is}-e^{-it}))$. We fix $\Re t=t_0$ and consider the map $(s,\tau)\to(e^{is}-e^{it})(e^{-is}-e^{-it})$ where $t=t_0+i\tau$. In the 'first quadrant' $s>t_0, \Im t>0$, this map is anti-holmorophic and takes the real axis $\Im t=0$ to the positive real axis and the 'imaginary axis', $s = t_0$ and $\Im t > 0$, to the negative real axis. Since the map is anti-holomorphic, the image of a counter-clockwise path in the first quadrant from the real to imaginary axis is a clock-wise path from the positive real axis to the negative real axis, so the arg equals $-\pi$ on the imaginary axis. As the path in the domain moves counter-clockwise in the second quadrant to $s < t_0, \Im t = 0$ the image path moves to argument -2π. Similarly, the continuation in the fourth and third quadrants leads to a value of 2π on the axis $s < t_0$.

We now make:

Claim: (cf. Millar [M2]) If f admits an analytic continuation to a neighborhood of $\partial\Omega$, then the integral $\int_0^{2\pi} f(s) \log r^2(s,t) ds$ admits an analytic continuation to a neighborhood of $\partial\Omega$ in $(\partial\Omega)_{\mathbb{C}}$ by

$$\int_0^{2\pi} f(s) \log r^2(s,t) ds \to \int_0^{2\pi} f(s) L(s,t) ds \pm 2\pi i \int_0^t f(s) ds, \tag{8.3}$$

where the path from 0 to t is defined in the integral is the same as the path used to analytically continue $\log r^2(s,t)$, and where the $+$ sign is taken for $\Im t > 0$ and the $-$ sign when $\Im t < 0$. Furthermore, the right side of (8.3) is periodic.

We check the last statement first. It follows from the fact that

$$\begin{aligned}\int_0^{2\pi} f(s) L(s, 2\pi + i\Im t) ds \pm 2\pi i \int_0^{2\pi + i\Im t} f(s) ds &= \int_0^{2\pi} L(s, i\Im t) f(s) ds \\ &\pm 2\pi i \int_0^{2\pi + i\Im t} f(s) ds \mp 2\pi i \int_0^{2\pi} f(s) ds \\ &= \int_0^{2\pi} L(s, i\Im t) f(s) ds \pm 2\pi i \int_0^{i\Im t} f(s) ds,\end{aligned}$$

where the last line follows from the Cauchy formula and the fact that $f(t + 2\pi) = f(t)$.

To prove the first statement of the claim it suffices to show that the right side in (8.3) is

- (i) Holomorphic in the upper annulus $\Im t > 0$;
- (ii) Holomorphic in the lower annulus $\Im t < 0$;
- (iii) Continuous in the whole annulus, and restricts to $\int_0^{2\pi} f(s) \log r^2(s,t) ds$ for real t.

Let us prove (iii) first, since it explains the second term on the right side of (8.3). With no loss of generality, suppose that $\Im t \to 0^+$ with $t \to t_0$. Then

$$\int_0^{2\pi} f(s) L(s,t) ds + 2\pi i \int_0^t f(s) ds \to \int_0^{2\pi} f(s) L(s,t_0) ds + 2\pi i \int_0^{t_0} f(s) ds,$$

and we must show that

$$\int_0^{2\pi} f(s) L(s,t_0) ds + 2\pi i \int_0^{t_0} f(s) ds = \int_0^{2\pi} f(s) \log r^2(s,t_0) ds.$$

Here, $\arg r^2(t,s) = 0$ while $\Im L(s,t)$ equals zero for $s \geq t$ and equals -2π for $s \leq t$. Hence, the imaginary part of the left side cancels and we obtain the right side.

Now let us prove (i)-(ii). Since the proofs are essentially the same we only prove (i).

We first note that all branches of analytic continuation of $\log r^2(s,t)$ differ by constants in $2\pi i\mathbb{Z}$. Hence, if the period $\langle f \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(s) ds$ of f vanishes, then all choices of branch of $\log r^2$ give the same value of the integral $\int_0^{2\pi} f(s)\log r^2(s,t)ds$. Similarly, the integral $\int_0^t f(s)ds$ is only multi-valued due to the period of f. Hence, when the $\langle f\rangle = 0$, both terms on the right side of the claim are well-defined independently of any choice of integration path or branch of $\log r^2$. Since we can write $f = (f - \langle f\rangle) + \langle f\rangle$, we only need to show:

(1) $\int_0^{2\pi} f(s)L(s,t)ds \pm 2\pi i \int_0^t f(s)ds$ is holomorphic for $\Im t > 0$ when $\langle f\rangle = 0$;
(2) $\int_0^{2\pi} L(s,t)ds \pm 2\pi i \int_0^t ds$ is single-valued holmorphic function for $\Im t > 0$.

To prove (1), we assume $\langle f\rangle = 0$ and let $F(t) = \int_0^t f(s)ds$ be the (well-defined) primitive of F in the annulus. We then integrate by parts in the first integral to obtain

$$\begin{aligned}
&\int_0^{2\pi} F'(s)L(s,t)ds \\
&= F(2\pi)L(2\pi,t) - F(0)L(0,t) - \int_0^{2\pi} F(s)\tfrac{q'(s)}{q(s)-q(t)}ds - \int_0^{2\pi} F(s)\tfrac{\overline{q}'(s)}{\overline{q}(s)-q^*(t)}ds \\
&= -\int_0^{2\pi} F(s)\tfrac{q'(s)}{q(s)-q(t)}ds - \int_0^{2\pi} F(s)\tfrac{\overline{q}'(s)}{\overline{q}(s)-q^*(t)}ds
\end{aligned} \tag{8.4}$$

Here, we use that $F(2\pi) = F(0)$ since it is a single-valued holomorphic function on the annulus and that $F(0) = 0$ by definition. It is clear that the right side of (8.4) is holomorphic in $\Im t > 0$ since poles occur only when $\Im t = 0$. Since $F(t)$ is single valued and holomorphic, this proves (1).

To prove (2) we write

$$\begin{aligned}
\int_0^{2\pi} L(s,t)ds \pm 2\pi i \int_0^t ds \quad = \quad & \int_0^{2\pi} \log \tfrac{Q(e^{is})-Q(e^{it})}{e^{is}-e^{it}}\tfrac{Q^*(e^{is})-Q^*(e^{it})}{e^{-is}-e^{-it}}ds \\
& + \int_0^{2\pi} \log\left((e^{is}-e^{it})(e^{-is}-e^{-it})\right)ds \pm 2\pi i \int_0^t ds,
\end{aligned} \tag{8.5}$$

We observe that the first term is holomorphic for $\Im t > 0$ since the arg of both numerator and denominator are continued so that each arg tends to -2π as $\Im t \to 0$ for $s \in [0, \Re t]$ and so that each arg tends to zero for $s \in [\Re t, 2\pi]$. Hence, the arg of the ratio tends to zero as $\Im t \to 0$ in both integrals. Since the *arg* is only ambiguous up to a constant in $2\pi i\mathbb{Z}$, it follows that the arg of the ratio is well defined and single valued and the integrand is well-defined as a single-valued holomorphic function for $\Im t > 0$. Therefore, to prove (2) it suffices to show that for $\Im t \geq 0$

$$g(t) := \int_0^{2\pi} L(s,t)ds + 2\pi i \int_0^t ds = 0, \tag{8.6}$$

where $L(s,t) = \log\left((e^{is}-e^{it})(e^{-is}-e^{-it})\right)$ with our choice above of the branch cut at $s = \Re t$. Note that g is an analytic continuation of the integral $\int_0^{2\pi} \log|e^{is}-e^{it}|^2 ds = 0$ for real t, so analyticity of g is equivalent to $g = 0$.

This reduces the analysis to the integral

$$\int_0^{2\pi} \log\left((e^{is}-e^{it})(e^{-is}-e^{-it})\right)ds = \int_0^{2\pi} \log\left(2 - 2\cos(s-t)\right)ds,$$

where as above the logarithm is defined by breaking up the integral into $\int_0^{\Re t} + \int_{\Re t}^{2\pi}$ and defining the arg by the method above. Note that formally the integral is constant in t by a change of variables but that such a change of variables is not consistent with the definition of the logarithm. However, the integrand is a function of $s-t$ and so, $\frac{d}{dt}\log\left(2-2\cos(s-t)\right) =$

$\frac{d}{d(t-s)} \log(2 - 2\cos(t-s))$ is well-defined independent of the branch of log used (since these differ by integer multiples of $2\pi i$). Hence,

$$\begin{aligned} \frac{d}{dt}\int_0^{2\pi} \log(2-2\cos(s-t))\,ds &= -\int_0^{2\pi} \frac{d}{ds}\log(2-2\cos(s-t))\,ds \\ &= -\log(2-2\cos(s-t))\,|_0^{2\pi} \\ &= -2\pi i, \end{aligned}$$

by definition of the logarithm. It follows from (8.6) that

$$\frac{d}{dt}g(t) = \frac{d}{dt}\int_0^{2\pi} \log(2-2\cos(s-t))\,ds + \frac{d}{dt}(2\pi i t) = -2\pi i + 2\pi i = 0.$$

Hence g is constant and as noted above it equals 0 for real t.

□

This completes the proof of the Claim and hence of the Proposition.

□

Remark: By integrating by parts directly in the integral $Lf(t) := \int_0^{2\pi} \log r^2(s,t) f(s) ds$ for t *real* and using that $\int_0^{2\pi} \log|e^{is} - e^{it}|^2 ds = 0$, one gets the formula

$$Lf(t) = -\int_0^{2\pi}\left(\frac{q'(s)}{q(s)-q(t)} + \frac{\bar{q}'(s)}{\bar{q}(s)-\bar{q}(t)}\right)\cdot(F(s)-F(t))\,ds + \langle f\rangle \int_0^{2\pi} \log\frac{|Q(e^{is})-Q(e^{it})|^2}{|e^{is}-e^{it}|^2}ds, \tag{8.7}$$

where, $F(t) := \int_0^t (f - \langle f\rangle)ds$. It follows from (8.7) that for $t \in [0,2\pi] + i[-\epsilon,\epsilon]$ there is an alternative formula for the analytic continuation of Lf which is given by

$$\begin{aligned} (Lf)^{\mathbb{C}}(t) = -\int_0^{2\pi}\left(\frac{q'(s)}{q(s)-q(t)} + \frac{\bar{q}'(s)}{\bar{q}(s)-q^*(t)}\right)\cdot(F(s)-F(t))\,ds \\ +\langle f\rangle\int_0^{2\pi}\log\frac{[Q(e^{is})-Q(e^{it})][Q^*(e^{is})-Q^*(e^{it})]}{[e^{is}-e^{it}][e^{-is}-e^{-it}]}ds. \end{aligned} \tag{8.8}$$

We note that the log in the second term on the RHS of (8.8) is defined unambiguously (independent of branch) since as $\Im t \to 0$ and $s \to \Re t$ from either side, we have that $\arg[Q(e^{is}) - Q(e^{it})] - \arg[e^{is} - e^{it}] \to 0$ and so the arguments cancel. The same thing is true for the ratio involving Q^*.

8.2. **Proof of Propositions 8 and 9.**

Proof. We only sketch the proofs, because they only involve a small modification of Proposition 7.

First, consider Proposition 8. Green's formulae (3.5)-(3.6) remain correct in the piecewise analytic case, with the definition that on the kth component, the normal derivative is calculated by taking the limit from within the kth component.

The verification of the Millar formula is then similar to the fully analytic case. The main difference is that we now have pairs (s_n, t_k) of parameter points which may come from different intervals. When $n = k$ there is no difference in the argument except that $\partial\Omega_k^{\mathbb{C}}$ is not an annulus but rather two regions meeting along a common interval. But the same choice of branch of the logarithm extends u_{λ_j} holomorphically above and below the interval, and the first term on the right side ensures that the two holomorphic extensions agree on the common interval. When $n \neq k$, one defines $\arg r^2(s_n, t_k) = 0$ for all real s_n, t_k. Since $q_n(s_n) \neq q_k(t_k)$ when $n \neq k$ the logarithm extends to a holomorphic function in t_k with this choice of branch.

In the case of Proposition 9, the additional step is to analytically continue the second term of (3.20). To do so, we use the arc-length parameterization $q_C(s)$ of C to write

$$-\frac{1}{2}\int_0^{2\pi} A(\lambda_j, s, t)\, \log r^2(s,t)\, \partial_\nu \varphi_{\lambda_j}(q_C(s))ds - \int_0^{2\pi} B(\lambda_j, s, t)\, \partial_\nu \varphi_{\lambda_j}(q_C(s))\, ds.$$

Since $B = F(r^2)$ where F is entire, the second term above has a straightforward analytic continuation. The first term is another case of the integrals discussed in the previous section, and by the proof of Proposition 7, its holomorphic continuation is the one stated in the Proposition.

□

9. Appendix on Hankel and Bessel functions

For the reader's convenience, we summarize the facts we use about these special functions. We recall that Bessel's function of order ν is a solution of the equation $x^2y''+xy'+(x^2-\nu^2)y = 0$.

A fundamental set of solutions for Bessel's equation of order 0 is given by J_0, Y_0 where

$$\begin{cases} J_0(z) := \sum_{k=0}^{\infty}(-1)^k \frac{(z)^{2k}}{2^{2k}(k!)^2} \\ \\ Y_0(z) = J_0(z)\log(z) - \sum_{m=1}^{\infty} \frac{(-1)^m(1+2\cdots+\frac{1}{m})(z)^{2m}}{4^m(m!)^2}. \end{cases} \tag{9.1}$$

The coefficients A, B in (3.9) then have the form

$$A = J_0(\lambda_j r), \quad B = -\sum_{m=1}^{\infty} \frac{(-1)^m(1+2\cdots+\frac{1}{m})(\lambda r)^{2m}}{4^m(m!)^2} + J_0(\lambda r)\log\lambda. \tag{9.2}$$

A fundamental set of solutions for Bessel's equation of order one is given by J_1, Y_1 where:

$$\begin{cases} J_1(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{2^{2k+1}\, k!(k+1)!} \\ \\ \pi Y_1(z) = \frac{-2}{z} + 2\; J_1(z)\; (\log(z/2)+\gamma) - \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k!(k-1)!}\; (z/2)^{2k-1}[\frac{1}{k} + 2\sum_{m=1}^{k}\frac{1}{m}]. \end{cases} \tag{9.3}$$

Here, γ is Euler's constant. We also have the integral formula

$$J_1(z) = -\pi i \int_0^{\pi} e^{iz\,\cos\theta}\, \cos\theta\, d\theta. \tag{9.4}$$

Finally, we also use the Hankel function $\mathrm{Ha}_1^{(1)}(z) = J_1 + iY_1$, which is given by the integral formula,

$$\mathrm{Ha}_1^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{i(z-3\pi/4)}}{\Gamma(3/2)} \int_0^{\infty} e^{-s}s^{-\frac{1}{2}}\,(1-\frac{s}{2iz})^{\frac{1}{2}}\, ds. \tag{9.5}$$

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