NUMBER VARIANCE OF RANDOM ZEROS

BERNARD SHIFFMAN AND STEVE ZELDITCH

Abstract. The main results of this article are asymptotic formulas for the variance of the number of zeros of a Gaussian random polynomial of degree N in an open set $U \subset \mathbb{C}$ as the degree $N \to \infty$, and more generally for the zeros of random holomorphic sections of high powers of any positive line bundle over any Riemann surface. The formulas were conjectured in special cases by Forrester and Honner. In higher dimensions, we give similar formulas for the variance of the volume inside a domain U of the zero hypersurface of a random holomorphic section of a high power of a positive line bundle over any compact Kähler manifold. These results generalize the variance asymptotics of Sodin and Tsirelson for special model ensembles of chaotic analytic functions in one variable to any ample line bundle and Riemann surface. We also combine our methods with those of Sodin-Tsirelson to generalize their asymptotic normality results for smoothed number statistics.

Contents

1. Introduction

This article is concerned with number and volume statistics for Gaussian random holomorphic functions (and sections). To introduce our subject, let us start with the simplest case of holomorphic polynomials p_N of degree N of one complex variable. By homogenizing, we may identify the space $\mathcal{P}_N$ of polynomials of degree N with the space $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ of holomorphic sections of the N-th power of the hyperplane section bundle over $\mathbb{CP}^1$. This space carries a natural $SU(2)$-invariant inner product and associated Gaussian measure γ_N. To each polynomial p_N we associate its zero set $Z_{p_N} \subset \mathbb{CP}^1$ and thus obtain a random

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point process on $\mathbb{CP}^1$. Given an open subset $U \subset \mathbb{CP}^1$, we define the integer-valued random variable

$$\mathcal{N}_N^U(p_N) = \#\{z \in U : p_N(z) = 0\} \tag{1}$$

on $\mathcal{P}_N$ counting the number of zeros of p_N which lie in U. Clearly, $\mathcal{N}_N^U$ is discontinuous along the set of polynomials having a zero on the boundary ∂U. It is easy to see from the SU(2) invariance that the expected value of this random variable is given by

$$\mathbf{E}(\mathcal{N}_N^U) = N \int_U \frac{i}{2\pi}\,\Theta_h\,,$$

where $\mathbf{E}(X)$ denotes the expectation of a random variable X and where Θ_h is the curvature form of the Fubini-Study metric; i.e., the expected zero distribution is uniform on $\mathbb{CP}^1$ with respect to its $SU(2)$ invariant area form. The variance

$$\mathrm{Var}(\mathcal{N}_N^U) = \mathbf{E}\big(\mathcal{N}_N^U - \mathbf{E}(\mathcal{N}_N^U)\big)^2$$

of $\mathcal{N}_N^U$ measures the fluctuations of $\mathcal{N}_N^U$, i.e. the extent to which the number of zeros of individual polynomials conforms to or deviates from the expected number. More generally, we can study the same problem for Gaussian random holomorphic sections $s_N \in H^0(M, L^N)$ of powers of any positive holomorphic line bundle $L \to M$ over any compact Riemann surface M. In [SZ1], we showed that in this case, the expected value of the random variable $\mathcal{N}_N^U$ has the asymptotics

$$\frac{1}{N}\,\mathbf{E}(\mathcal{N}_N^U) = \frac{i}{2\pi}\int_U \Theta_h \;+\; O\left(\frac{1}{N}\right)\,. \tag{2}$$

Our first result gives an estimate for the variance of the number of zeros on a domain in a compact complex curve, extending and sharpening a result of Forrester and Honner [FH] (see also Sodin-Tsirelson [ST]):

Theorem 1.1. *Let (L, h) be a positive Hermitian holomorphic line bundle over a compact complex curve M. We give $H^0(M, L^N)$ the Hermitian Gaussian measure induced by h and the area form $\omega = \frac{i}{2}\Theta_h$. Let U be a domain in M with piecewise $\mathcal{C}^2$ boundary and no cusps. Then for random sections $s_N \in H^0(M, L^N)$, we have*

$$\mathrm{Var}\big(\#\{z \in U : s_N(z) = 0\}\big) = \sqrt{N}\left[\frac{\zeta(3/2)}{8\pi^{3/2}}\,\mathrm{Length}(\partial U) + O(N^{-\frac{1}{2}+\varepsilon})\right]\,.$$

This theorem proves a strong form of self-averaging for the number of zeros in U. Here, a sequence X_N of random variables is called *self-averaging* if the fluctuations of X_N are of smaller order than its typical values, or in other words if $\frac{\mathrm{Var}(X)}{(\mathbf{E}X)^2} \to 0$.

In higher dimensions, the analogous point process is defined by the simultaneous zeros of m polynomials in m variables, or more generally, m sections on an m-dimensional complex manifold. In this article we consider instead the simpler 'volume analogue' of number statistics for one random polynomial or section s in m dimensions. We let Z_s denote the zero set of the random holomorphic section s. Recall that the volume of Z_s in a domain U is given by

$$\mathrm{Vol}_{2m-2}[Z_s \cap U] = \int_{Z_s \cap U} \frac{1}{(m-1)!}\omega^{m-1} = \left([Z_s],\ \chi_U\,\frac{1}{(m-1)!}\omega^{m-1}\right)\,,$$

where $[Z_s]$ denotes the current of integration over Z_s. Our higher dimensional generalization of Theorem 1.1 is the following asymptotic formula for the variance of the volume of the zero divisor in a domain with nice boundary.

Theorem **1.2.** *Let (L, h) be a positive Hermitian holomorphic line bundle over a compact Kähler manifold (M, ω), where $\omega = \frac{i}{2}\Theta_h$. We give $H^0(M, L^N)$ the Hermitian Gaussian measure induced by h, ω (see Definition 2.2). Let U be a domain in M with piecewise $\mathcal{C}^2$ boundary and no cusps. Then for random sections $s_N \in H^0(M, L^N)$, we have*

$$\operatorname{Var}\big(\operatorname{Vol}_{2m-2}[Z_{s_N} \cap U]\big) = N^{-m+3/2}\left[\nu_m \operatorname{Vol}_{2m-1}(\partial U) + O(N^{-\frac{1}{2}+\varepsilon})\right] ,$$

where

$$\nu_m = \frac{\pi^{m-5/2}}{8}\,\zeta(m + \tfrac{1}{2}) .$$

Here, we say that U has piecewise $\mathcal{C}^k$ boundary without cusps if for each boundary point $z_0 \in \partial U$, there exists a (not necessarily convex) closed polyhedral cone $C \subset \mathbb{R}^{2m}$ and a $\mathcal{C}^k$ diffeomorphism $\rho : V \to \rho(V) \subset \mathbb{R}^{2m}$, where V is a neighborhood of z_0, such that $\rho(V \cap \overline{U}) = \rho(V) \cap C$. If M is a complex curve, this condition means that ∂U is a piecewise $\mathcal{C}^k$ curve with distinct tangents at corners and self-intersection points.

A model case of Theorem 1.2 (as well as of the results stated below) is where $M = \mathbb{CP}^m$ and $L = \mathcal{O}(1)$ is the hyperplane section bundle with the $\mathrm{SU}(m+1)$-invariant Hermitian metric h. Then sections in $H^0(M, L^N)$ are homogeneous degree N holomorphic polynomials on $\mathbb{C}^{m+1}$, and volumes are computed with respect to the Fubini-Study metric $\omega_{\mathrm{FS}} = \frac{i}{2}\Theta_h = \frac{i}{2}\partial\bar\partial \log |z|^2$ on $\mathbb{CP}^m$.

In addition to the number variance problem raised by Forrester-Honner [FH], the main motivation for this article came from the variance and asymptotic normality theorems of Sodin-Tsirelson [ST] for certain model random analytic functions. They consider the smooth analogue of number statistics, sometimes called 'linear statistics', defined by the random variables

$$N_\varphi(s) = (Z_s, \varphi) = \sum_{\{z \in M : s(z) = 0\}} \varphi(z), \tag{3}$$

where $\varphi \in \mathcal{C}^3_c(M)$ is a test function. In our early paper [SZ1], we showed that

$$\mathbf{E}(Z_{s_N}, \varphi) = N \int_M \omega \wedge \varphi + O(1) , \tag{4}$$

and we gave a crude bound (see [SZ1, Lemma 3.3])

$$\frac{\operatorname{Var}(Z_N, \varphi)}{\big[\,\mathbf{E}(Z_N, \varphi)\,\big]^2} = O\left(\frac{1}{N^2}\right) \tag{5}$$

on the variance, which was sufficient to prove a strong law of large numbers for the distribution of zeros. In certain model ensembles, Sodin-Tsirelson [ST] improved this result to a sharp estimate as an ingredient in their asymptotic normality result for zeros. Our next result generalizes their variance asymptotics for the zeros of random polynomials f_N of degree N (and their counterparts for model chaotic analytic functions in $\mathcal{O}(D)$ and $\mathcal{O}(\mathbb{C})$) to any compact Kähler manifold:

THEOREM **1.3.** *Let (L, h) be a positive Hermitian holomorphic line bundle over a compact Kähler manifold (M, ω), where $\omega = \frac{i}{2}\Theta_h$. Let φ be a real $(2m-2)$-form on M with $\mathcal{C}^3$ coefficients. We give $H^0(M, L^N)$ the Hermitian Gaussian measure induced by h and the area form ω (see Definition 2.2). Then for random sections $s_N \in H^0(M, L^N)$, we have*

$$\mathrm{Var}\big(Z_{s_N}, \varphi\big) = N^{-m}\left[\kappa_m \|\partial\bar\partial\varphi\|_2^2 + O(N^{-\frac12+\varepsilon})\right] ,$$

where

$$\kappa_m = \frac{\pi^{m-2}}{4}\,\zeta(m+2) .$$

Here, $\|\partial\bar\partial\varphi\|_2$ denotes the $\mathcal{L}^2$ norm of $\partial\bar\partial\varphi$, i.e. writing $i\partial\bar\partial\varphi = \psi\frac{1}{m!}\omega^m$, we have $\|\partial\bar\partial\varphi\|_2^2 = \int \psi^2\frac{1}{m!}\omega^m = \int i\psi\partial\bar\partial\varphi$. (Of course, we may assume that φ is of bidegree $(m-1, m-1)$, since $(Z_{s_N}, \varphi) = 0$ for forms φ of other bidegrees.)

In particular, for the case $\dim M = 1$, we note that $|\partial\bar\partial\varphi| = \frac12|\Delta\varphi|$, and thus

$$\mathrm{Var}\big(Z_{s_N}, \varphi\big) = N^{-1}\left[\frac{\zeta(3)}{16\pi}\|\Delta\varphi\|_2^2 + O(N^{-\frac12+\varepsilon})\right] . \tag{6}$$

The leading term in (6) was obtained by Sodin and Tsirelson [ST] for the case of random polynomials $s_N \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ and random holomorphic functions on $\mathbb{C}$ and on the disk. (The constant $\frac{\zeta(3)}{16\pi}$ was given in a private communication from M. Sodin.)

Our final result is an asymptotic normality result of the type proved in [ST]. It follows very easily from the analysis underlying Theorem 1.3 together with a general asymptotic normality result of Sodin-Tsirelson.

THEOREM **1.4.** *With the same notation and hypotheses as in Theorem 1.3, the distributions of the random variables*

$$\frac{(Z_{s_N}, \varphi) - \mathbf{E}(Z_{s_N}, \varphi)}{\sqrt{\mathrm{Var}(Z_{s_N}, \varphi)}}$$

converge weakly to the standard Gaussian distribution $\mathcal{N}(0, 1)$ as $N \to \infty$.

We let $\mathcal{N}(0, \sigma)$ denote the (real) Gaussian distribution of mean zero and variance σ^2. Substituting the values of the expectation and variance of (Z_{s_N}, φ) from (4) and Theorem 1.3, respectively, we have

COROLLARY **1.5.** *With the same hypotheses as in Theorem 1.3, the distributions of the random variables $N^{m/2}(Z_{s_N} - N\omega, \varphi)$ converge weakly to $\mathcal{N}(0, \sqrt{\kappa_m}\,\|\partial\bar\partial\varphi\|_2)$ as $N \to \infty$.*

Let us briefly mention some key ideas in the proofs and the relation of the Sodin-Tsirelson methods to ours. The Sodin-Tsirelson estimate was based on their construction of a 'bi-potential' for the pair correlation measures, i.e. functions $G_N(z, w)$ such that

$$\Delta_z\Delta_w G_N(z, w) = K_2^N(z, w). \tag{7}$$

Here, K_2^N is the 'pair correlation function' for the zeros of sections of $\mathcal{S}_N$, that is, the probability density that a section in $\mathcal{S}_N$ has zeros at two points z and w of $\mathbb{CP}^1$. The bi-potential is given in [ST] as a power series in the Szegö kernel for $\mathcal{O}(N) \to \mathbb{CP}^1$. (Here, the notation in (7) is taken from [BSZ1] and is not used in [ST].) In fact, the same bipotential already arose in [BSZ1] in the setting of line bundles over a compact Kähler manifold as the bi-potential for the 'pair correlation current,' i.e.

$$-\partial_z\bar{\partial}_z\partial_w\bar{\partial}_w G_N(z,w) = \mathbf{E}\left(Z_{s_N}(z)\otimes Z_{s_N}(w)\right) . \tag{8}$$

We build on our analysis of this bi-potential in [BSZ1] to prove Theorem 1.3.

The other main ingredient in the proofs are estimates derived from the off-diagonal asymptotics of the Szegö kernel in [SZ2]. For the sake of completeness, we review the derivation of these off-diagonal asymptotics in §4. Off-diagonal estimates of the Szegö kernel with sharper (exponentially small) remainder estimates are given in [DLM, MM], but the estimates of [SZ2] already suffice for our applications.

Although we are emphasizing positive line bundles over compact Kähler manifolds, our results (and their proofs) extend with no essential change to positive line bundles over noncompact Kähler manifolds for which the orthogonal projector Π_N onto $\mathcal{L}^2H^0(M,L^N)$, the L^2 holomorphic sections of a positive line bundle with respect to a Hermitian metric and the Kähler volume form, has analytic properties similar to those in the compact case. A model for a positive line bundle over a noncompact Kähler manifold is provided by the Heisenberg line bundle $L_{\mathbf{H}}\to\mathbb{C}^m$ associated to the reduced Heisenberg group by the identity character, as described in detail in [BSZ2, BSZ3]. In this case, the analogue of Theorem 1.2 is an asymptotic formula (Corollary 2.5) for the volume variance of the zeros of random Gaussian entire functions on the dilates $\sqrt{N}\,U$ of a domain $U\subset\mathbb{C}^m$. Other model examples are given by homogeneous Hermitian line bundles over bounded symmetric domains with curvature equal to the Bergman Kähler metric. We briefly discuss the extension to random holomorphic sections in the noncompact case in §2.2.

It will readily be recognized that the variance and normality problems in higher dimensions make sense for the simultaneous zeros of k independent sections $s_1,\dots,s_k$ of a line bundle over an m-dimensional complex manifold and are perhaps most interesting for the full codimension case $k=m$. The same problem may be posed for the critical points of a single Gaussian random section. However, new technical ideas seem to be necessary to obtain limit formula for the intersections of the random zero currents Z_{s_j}. We hope to return to this problem elsewhere.

In conclusion, we thank M. Sodin for discussions of his work with B. Tsirelson on number variance and asymptotic normality for random analytic functions of one variable.

2. Expected distribution of zeros and Szegö kernels

In this section, we review the basic formula from [BSZ1, BSZ2, SZ1] for the expected distribution of zeros of Gaussian random sections of holomorphic line bundles. We state it here in a general framework which we shall use in our forthcoming paper on zeros of random fewnomials [SZ3].

We let (L,h) be a Hermitian holomorphic line bundle over a complex manifold M (not necessarily compact), and let $\mathcal{S}$ be a finite-dimensional subspace of $H^0(M,L)$. We suppose that $\dim\mathcal{S}\geq 2$ and we give $\mathcal{S}$ a Hermitian inner product. The inner product induces the complex Gaussian probability measure

$$d\gamma(s)=\frac{1}{\pi^m}e^{-|c|^2}dc\,,\qquad s=\sum_{j=1}^n c_jS_j\,, \tag{9}$$

on $\mathcal{S}$, where $\{S_j\}$ is an orthonormal basis for $\mathcal{S}$ and dc is $2n$-dimensional Lebesgue measure. This Gaussian is characterized by the property that the $2n$ real variables $\operatorname{Re} c_j, \operatorname{Im} c_j$ ($j = 1, \dots, n$) are independent complex Gaussian random variables with mean 0 and variance 1; i.e.,

$$\mathbf{E} c_j = 0, \quad \mathbf{E} c_j c_k = 0, \quad \mathbf{E} c_j \bar{c}_k = \delta_{jk}\,.$$

We let

$$\Pi_{\mathcal{S}}(z,z) = \sum_{j=1}^n \|S_j(z)\|_h^2\,, \qquad z \in M\,, \tag{10}$$

denote the *Szegö kernel* for $\mathcal{S}$ on the diagonal. (See §2.1 for a discussion of the Szegö kernel.) We now consider a local holomorphic frame e_L over a trivializing chart U, and we write $S_j = f_j e_L$ over U. Any section $s \in \mathcal{S}$ may then be written as

$$s = \langle c, F\rangle e_L^{\otimes N}\,, \quad \text{where} \quad F = (f_1, \dots, f_k)\,, \quad \langle c, F\rangle = \sum_{j=1}^n c_j f_j\,.$$

If $s = f e_L$, its Hermitian norm is given by $\|s(z)\|_h = a(z)^{-\frac{1}{2}}|f(z)|$ where

$$a(z) = \|e_L(z)\|_h^{-2}\,. \tag{11}$$

Recall that the curvature form of (L,h) is given locally by

$$\Theta_h = \partial\bar\partial \log a\,,$$

and the *Chern form* $c_1(L,h)$ is given by

$$c_1(L,h) = \frac{\sqrt{-1}}{2\pi}\Theta_h = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log a\,. \tag{12}$$

The current of integration Z_s over the zeros of $s = \langle c, F\rangle e_L$ is then given locally by the *Poincaré-Lelong formula*:

$$Z_s = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log|\langle c, F\rangle|\,. \tag{13}$$

It is of course independent of the choice of local frame e_L and basis $\{S_j\}$.

We now state our formula for the expected zero divisor for the linear system $\mathcal{S}$:

PROPOSITION **2.1.** *Let (L,h) be a Hermitian line bundle on a complex manifold M, and let $\mathcal{S}$ be a finite dimensional subspace of $H^0(M,L)$. We give $\mathcal{S}$ a Hermitian inner product and we let γ be the induced Gaussian probability measure on $\mathcal{S}$. Then the expected zero current of a random section $s \in \mathcal{S}$ is given by*

$$\mathbf{E}_\gamma(Z_s) \;=\; \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \Pi_{\mathcal{S}}(z,z) + c_1(L,h)\,.$$

Proof. Let $\{S_j\}$ be an orthonormal basis of $\mathcal{S}$. As above, we choose a local nonvanishing section e_L of L over $U \subset M$, and we write

$$s = \sum_{j=1}^n c_j S_j = \langle c, F\rangle e_L\,,$$

where $S_j = f_j e_L,\ F = (f_1, \dots, f_k)$. As in the proof of [SZ1], Proposition 3.1, we then write $F(x) = |F(x)| u(x)$ so that $|u| \equiv 1$ and

$$\log |\langle c, F\rangle| = \log |F| + \log |\langle c, u\rangle| \ . \tag{14}$$

A key point is that $\mathbf{E}\big(\log|\langle c,u\rangle|\big)$ is independent of z (and in fact, is a universal constant depending only on n), and hence $\mathbf{E}\big(d\log|\langle c,u\rangle|\big) = 0$.

Thus by (13), we have

$$\begin{aligned}\big(\mathbf{E}_\gamma(Z_s),\varphi\big) &= \frac{\sqrt{-1}}{\pi}\int_{\mathbb{C}^n}\big(\log|\langle c,F\rangle|,\partial\bar\partial\varphi\big)\,d\gamma(c)\\ &= \frac{\sqrt{-1}}{\pi}\int_{\mathbb{C}^n}\big(\log|F|,\partial\bar\partial\varphi\big)\,d\gamma(c) + \frac{\sqrt{-1}}{\pi}\int_{\mathbb{C}^n}\big(\log|\langle c,u\rangle|,\partial\bar\partial\varphi\big)\,d\gamma(c)\ ,\end{aligned}$$

for all test forms $\varphi \in \mathcal{D}^{m-1,m-1}(U)$. The first term is independent of c so we may remove the Gaussian integral. The vanishing of the second term follows by noting that

$$\begin{aligned}\int_{\mathbb{C}^n}\big(\log|\langle c,u\rangle|,\partial\bar\partial\varphi\big)\,d\gamma(c) &= \int_{\mathbb{C}^n} d\gamma(c)\int_M \log|\langle c,u\rangle|\,\partial\bar\partial\varphi\\ &= \int_M\left[\int_{\mathbb{C}^n}\log|\langle c,u\rangle| d\gamma(c)\right]\partial\bar\partial\varphi \ = \ 0\ ,\end{aligned}$$

since $\int \log|\langle c,u\rangle|\,d\gamma(c) = \frac{1}{\pi}\int_{\mathbb{C}}\log|c_1|\,e^{-|c_1|^2}dc_1$ is constant, by the $\mathrm{U}(n)$-invariance of $d\gamma$. Fubini's Theorem can be applied above since

$$\int_{M\times\mathbb{C}^n}\big|\log|\langle c,u\rangle|\,\partial\bar\partial\varphi\big|\ d\gamma(c) = \left(\frac{1}{\pi}\int_{\mathbb{C}}\big|\log|c_1|\big|\,e^{-|c_1|^2}dc_1\right)\left(\int_M|\partial\bar\partial\varphi|\right) < +\infty\ .$$

Thus

$$\mathbf{E}_\gamma(Z_s) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|F|^2 = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\left(\log\sum\|S_j\|_h^2 + \log a\right)\ . \tag{15}$$

Recalling that $\Pi_{\mathcal{S}}(z,z) = \sum\|S_j(z)\|_h^2$ and that $c_1(L,h) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log a$, the formula of the proposition follows. ☐

Remark: The complex manifold M, the line bundle L and space $\mathcal{S}$ as well as its inner product in Proposition 2.1 are all completely arbitrary. We do not assume that M is compact or that (L,h) has positive curvature. We do not even assume that $\mathcal{S}$ is base point free. If $\mathcal{S}$ has no base points (points where all sections in $\mathcal{S}$ vanish), then we have the alternate formula (see [SZ1])

$$\mathbf{E}_\gamma(Z_s) = \Phi_{\mathcal{S}}^*\omega_{\mathrm{FS}}\ ,$$

where $\Phi_{\mathcal{S}} : M \to \mathbb{P}\mathcal{S}^*$ is the Kodaira map and ω_{FS} is the Fubini-Study form on $\mathbb{P}\mathcal{S}^*$. In the general case where there are base points, we have

$$\mathbf{E}_\gamma(Z_s) = \Phi_{\mathcal{S}}^*\omega_{\mathrm{FS}} + D\ ,$$

where D is the fixed component of the linear system $\mathbb{P}\mathcal{S}$.

2.1. **Powers of an ample line bundle.** We now let $L \to M$ be an ample line bundle on a compact complex manifold M. We consider tensor powers $L^N = L^{\otimes N}$ of the line bundle, and we let $\mathcal{S} = H^0(M, L^N)$. We further choose a Hermitian metric h on L with strictly positive curvature and we give M the Kähler form $\omega = \frac{i}{2}\Theta_h = \pi c_1(L, h)$.

We now describe the natural Gaussian probability measures on $H^0(M, L^N)$ used in [SZ1, SZ2, BSZ1, BSZ2]. For the case of polynomials in one variable, these Gaussian ensembles are equivalent to the SU(2) ensembles studied in [BBL, Han, NV, SZ1] and elsewhere.

Definition **2.2.** *Let $(L, h) \to (M, \omega)$ be as above, and let h^N denote the Hermitian metric on L^N induced by h. We give $H^0(M, L^N)$ the inner product induced by the Kähler form ω and the Hermitian metric h^N:*

$$\langle s_1, \bar{s}_2 \rangle = \int_M h^N(s_1, s_2) \frac{1}{m!}\omega^m , \qquad s_1, s_2 \in H^0(M, L^N) \ . \tag{16}$$

The Hermitian Gaussian measure *on $H^0(M, L^N)$ is the complex Gaussian probability measure γ_N induced by the inner product* (16)*:*

$$d\gamma_N(s) = \frac{1}{\pi^m} e^{-|c|^2} dc , \qquad s = \sum_{j=1}^{d_N} c_j S_j^N ,$$

where $\{S_1^N, \dots, S_{d_N}^N\}$ is an orthonormal basis for $H^0(M, L^N)$.

As in [SZ1, BSZ1] and elsewhere, we analyze the Szegö kernel for $H^0(M, L^N)$ by lifting it to the circle bundle $X \xrightarrow{\pi} M$ of unit vectors in the dual bundle $L^{-1} \to M$ with respect to h. In the standard way (loc. cit.), sections of L^N lift to equivariant functions on X. Then $s \in H^0(M, L^N)$ lifts to a CR holomorphic functions on X satisfying $\hat{s}(e^{i\theta}x) = e^{iN\theta}\hat{s}(x)$. We denote the space of such functions by $\mathcal{H}_N^2(X)$. The *Szegö projector* is the orthogonal projector $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}_N^2(X)$, which is given by the *Szegö kernel*

$$\Pi_N(x, y) = \sum_{j=1}^{d_N} \widehat{S}_j^N(x) \overline{\widehat{S}_j^N(y)} \qquad (x, y \in X) \ .$$

(Here, the functions $\widehat{S}_j^N$ are the lifts to $\mathcal{H}_N^2(X)$ of the orthonormal sections S_j^N; they provide an orthonormal basis for $\mathcal{H}_N^2(X)$.)

Further, the covariant derivative ∇s of a section s lifts to the horizontal derivative $\nabla_h \hat{s}$ of its equivariant lift $\hat{s}$ to X; the horizontal derivative is of the form

$$\nabla_h \hat{s} = \sum_{j=1}^{m} \left(\frac{\partial \hat{s}}{\partial z_j} - A_j \frac{\partial \hat{s}}{\partial \theta} \right) dz_j. \tag{17}$$

For further discussion and details on lifting sections, we refer to [SZ1].

We shall write

$$|\Pi_N(z, w)| := |\Pi_N(x, y)| , \quad z = \pi(x),\, w = \pi(y) \in M \ .$$

In particular, on the diagonal we have $\Pi_N(z, z) = \Pi_N(x, x)$, where $\pi(x) = z$. Note that $\Pi_N(z, z) = \Pi_{\mathcal{S}}(z, z)$ as defined by (10) with $\mathcal{S} = H^0(M, L^N)$.

2.2. **Random functions and Szegö kernels on noncompact domains.** As mentioned in the introduction, Theorems 1.1–1.4 extend with no essential change to positive line bundles over noncompact complete Kähler manifolds as long as the orthogonal projection onto the space $\mathcal{L}^2 H^0(M, L^N)$ of L^2 holomorphic sections with respect to the inner product (16) possesses the analytical properties stated in Theorem 4.1 (and mostly proved in [SZ2]) for Szegö kernels in the compact case. It would take us too far afield to discuss in detail the properties of Szegö kernels and random holomorphic sections in the noncompact setting, but we can illustrate the ideas with homogeneous models.

Before discussing our specific noncompact models, we first note that Proposition 2.1 holds for infinite-dimensional spaces of Gaussian random holomorphic sections. There are several equivalent ways to describe Gaussian random analytic sections or functions in an infinite dimensional space (e.g., [Ja, GJ, ST]). To take a simple approach, we suppose that $\{S_1, S_2, \dots, S_n, \dots\}$ is an infinite sequence of holomorphic sections of a Hermitian line bundle (L, h) on a (noncompact) complex manifold M such that

$$\sup_{z\in K} \sum_{j=1}^{\infty} \|S_j(z)\|_h^2 < +\infty \quad \text{for all compact } \ K \subset M \, . \tag{18}$$

We then consider the ensemble $(\mathcal{S}, d\gamma)$ of sections of L of the form

$$\mathcal{S} = \left\{ s = \sum_{j=1}^{\infty} c_j S_j : c_j \in \mathbb{C} \right\} , \qquad d\gamma = \prod_{j=1}^{\infty} \left(\frac{1}{\pi} e^{-|c_j|^2} dc_j \right) , \tag{19}$$

i.e. we consider random sections $s = \sum_{j=1}^{\infty} c_j S_j$, where the c_j are i.i.d. standard complex Gaussian random variables. It is well known that (18) implies that the series in (19) almost surely converges uniformly on compact sets (see e.g. [Ja, Kah]), and hence with probability one, $s \in H^0(M, L)$. We then have:

Proposition **2.3.** *The expected zero current of the random section $s \in \mathcal{S}$ in* (19) *is given by*

$$\mathbf{E}(Z_s) = \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log \Pi(z, z) + c_1(L, h) \, ,$$

where

$$\Pi(z, z) = \sum_{j=1}^{\infty} \|S_j(z)\|_h^2 \, .$$

Proof. The proof is exactly the same as the proof of Proposition 2.1 using the ensemble (19) with the infinite product measure, except we cannot use unitary invariance to show that

$$\int \log |\langle c, u\rangle| \, d\gamma(c) = \tfrac{1}{\pi} \int_{\mathbb{C}} \log |c_1| \, e^{-|c_1|^2} dc_1, \ \int \big| \log |\langle c, u\rangle| \big| d\gamma(c) = \tfrac{1}{\pi} \int_{\mathbb{C}} \big| \log |c_1| \big| \, e^{-|c_1|^2} dc_1. \tag{20}$$

To verify (20) in this case, we note that $\langle c, u(z)\rangle$ is a complex Gaussian random variable of mean 0 and variance 1 (see [Ja, Kah]), and hence

$$\int f(\langle c, u\rangle) \, d\gamma(c) = \frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \, e^{-|\zeta|^2} \, d\zeta \ , \quad \text{for all } \ f \in \mathcal{L}^1(\mathbb{C}, e^{-|\zeta|^2} d\zeta) \, .$$

The identities of (20) then follow by letting $f(\zeta) = \log |\zeta|$, resp. $f(\zeta) = \big| \log |\zeta| \big|$. $\square$

We are interested in the case where (L,h) has positive curvature, M is complete with respect to the Kähler metric $\omega = \frac{i}{2}\Theta_h$, and $\{S_j\}$ is an orthonormal basis of $\mathcal{L}^2H^0(M,L)$ with respect to the inner product (16). Note that with probability one, a random section s is not an $\mathcal{L}^2$ section (since $\|s\|_2 = \|c\|_2 = +\infty$ *a.s.*), but is a holomorphic section of L. (Equivalently, $\mathcal{L}^2H^0(M,L)$ carries a Gaussian measure in the sense of Bochner-Minlos; see [GJ].)

The first model noncompact case is known as the Bargmann-Fock space

$$\mathcal{F} := \mathcal{H}^2(\mathbb{C}^m, e^{-|z|^2}) = \left\{ f \in \mathcal{O}(\mathbb{C}^m) : \int_{\mathbb{C}^m} |f|^2 e^{-|z|^2}\,dz < +\infty \right\}.$$

We can regard elements of $\mathcal{F}$ as $\mathcal{L}^2$ sections of the trivial bundle $L_{\mathbf{H}}$ over $\mathbb{C}^m$ with metric $h = e^{-|z|^2}$. The associated circle bundle X can be identified with the reduced Heisenberg group; see [BSZ3, §2.3] or [BSZ2, §1.3.2]. Then $\mathcal{F} = \mathcal{L}^2H^0(\mathbb{C}^m, L_{\mathbf{H}}) = \mathcal{H}^2_1(X)$, and more generally,

$$\mathcal{L}^2H^0(\mathbb{C}^m, L^N_{\mathbf{H}}) = \mathcal{H}^2_N(X) = \left\{ f \in \mathcal{O}(\mathbb{C}^m) : \int_{\mathbb{C}^m} |f|^2 e^{-N|z|^2}\,dz < +\infty \right\}. \qquad (21)$$

In dimension one, this example is referred to as the 'flat model' in [ST].

An orthonormal basis for the Hilbert space $\mathcal{L}^2H^0(\mathbb{C}^m, L^N_{\mathbf{H}})$ with inner product $\langle f_1, \bar{f}_2\rangle = \int_{\mathbb{C}^m} f_1 \bar{f}_2^2 e^{-|z|^2}\,dz$ is

$$\left\{ S^N_k(z) = \frac{N^{m/2}}{\pi^{m/2}} \frac{(N^{1/2}z)^k}{\sqrt{k!}} \right\}_{k\in\mathbb{N}^m},$$

where we use the usual conventions $z^k = z_1^{k_1}\cdots z_m^{k_m}$, $k! = k_1!\cdots k_m!$, $|k| = k_1 + \cdots + k_m$. A Gaussian random section is defined by

$$f_N(z) = \sum_{k\in\mathbb{N}^m} c_k S^N_k(z) = \frac{N^{m/2}}{\pi^{m/2}} \sum_{k\in\mathbb{N}^m} \frac{c_k}{\sqrt{k!}} (N^{1/2}z)^k , \qquad (22)$$

where the coefficients c_k are independent standard complex Gaussian random variables as in (9). As mentioned above, the random sections f_N are almost surely not in $\mathcal{L}^2H^0(\mathbb{C}^m, L^N_{\mathbf{H}})$. However, they are almost surely entire functions of finite order 2 in the sense of Nevanlinna theory. Indeed, we easily see from (19) that

$$\gamma\left(\left\{c \in \mathbb{C}^\infty : |c_k|^2 \le 2\textstyle\sum_j \log k_j \quad \text{for} \quad k_j \ge 2\right\}\right) > 0 ,$$

and hence it follows from the zero-one law that $|c_k|^2 = O\left(\sum_j \log k_j\right)$ *a.s.* Therefore, by Cauchy-Schwartz,

$$|f(z)| = O\left(\sum (1-\varepsilon)^{|k|}\right)^{1/2} \left(\sum \frac{(1+2\varepsilon)^{|k|}}{k!} N^{|k|} |z^k|^2\right)^{1/2} = O\left(e^{(\frac12+\varepsilon)N|z|^2}\right) \quad a.s.$$

for all $\varepsilon > 0$. Thus we have an upper bound for the Nevanlinna growth function,

$$T(f_N, r) := \mathrm{Ave}_{\{|z|=r\}} \log^+|f(z)| \le \sup_{|z|\le r} \log^+|f(z)| \le \left[\frac{N}{2} + o(1)\right] r^2 \quad a.s.$$

(where $o(1)$ denotes a term that goes to 0 as $r \to \infty$, for each fixed $N \geq 1$). On the other hand, if $T(f_N, r) = O(e^{(\frac{1}{2}-\varepsilon)N|z|^2})$, then $f_N \in \mathcal{L}^2 H^0(\mathbb{C}^m, L^N_{\mathbf{H}})$, which has probability zero. Thus,

$$\limsup_{r\to\infty} \frac{T(f_N, r)}{r^2} = \frac{N}{2} \qquad a.s.$$

To use the proofs in §§5–7 to show that Theorems 1.1–1.4 hold for the line bundle $L_{\mathbf{H}}$, we need only to verify that the Szegö kernel $\Pi^{\mathbf{H}}_N$, i.e. the kernel of the orthogonal projection to $\mathcal{L}^2 H^0(\mathbb{C}^m, L_{\mathbf{H}})$, satisfies the diagonal and off-diagonal asymptotics in Theorem 4.1. In the model Heisenberg case, the Szegö kernel is given by

$$\Pi^{\mathbf{H}}_N(z,\theta; w,\varphi) = e^{iN(\theta-\varphi)} \sum_{k\in\mathbb{N}^m} S_k(z)\overline{S_k(w)} = \frac{N^m}{\pi^m} e^{iN(\theta-\varphi)+Nz\cdot\bar{w}-\frac{N}{2}(|z|^2+|w|^2)} \,, \tag{23}$$

(see [BSZ2]) and visibly has these properties.

Another class of homogeneous examples are the bounded symmetric domains $\Omega \subset \mathbb{C}^m$, equipped with their Bergman metrics $\omega = \frac{i}{2}\partial\bar{\partial}\log K(z,\bar{z})$ where $K(z,\bar{z})$ denotes the Bergman kernel function of Ω. Let $(L,h) \to \Omega$ be the holomorphic homogeneous Hermitian line bundle over Ω with curvature $(1,1)$ form ω. It was observed by Berezin [Ber] that the Szegö kernels Π_N for $\mathcal{L}^2 H^0(\Omega, L^N)$ also have the form $C_N e^{N\psi(z,\bar{w})}$ where C_N is a normalizing constant and $\psi = \log K(z,\bar{w})$. In the case of the unit disc $D \subset \mathbb{C}$ with its Bergman (hyperbolic metric) $\frac{-i}{2}\partial\bar{\partial}\log(1-|z|^2)$, the space $\mathcal{L}^2 H^0(D, L^N)$ may be identified with the holomorphic discrete series irreducible representation $\mathcal{D}^+_N$ of $SU(1,1)$ (cf. [Kn, p. 40]), that is with the space of holomorphic functions on D with inner product

$$\|f\|^2_N = \int_D |f(z)|^2 (1-|z|^2)^{N-2} dz.$$

The factor $e^{N\log(1-|z|^2)}$ comes from the Hermitian metric. An orthonormal basis for the holomorphic sections of L^N is then given by the monomials $\binom{N+n-1}{n}^{1/2} z^n$ $(n = 0,1,2,\dots)$. The Szegö kernels are given by $\Pi_N(z,w) = (1-z\bar{w})^N$. The Szegö kernels also visibly have the properties stated in Theorem 4.1. These ensembles are called the hyperbolic model in [ST]. Random SU(1,1) polynomials are studied in [BR], where further details can be found.

Thus our proofs also yield the following result:

Theorem 2.4. *Theorems 1.1–1.4 hold for the zeros of sections in the following ensembles:*

- *Gaussian random sections* $f_N \in H^0(\mathbb{C}^m, L^N_{\mathbf{H}})$ *given by (22);*
- *Gaussian random sections of the holomorphic homogeneous Hermitian line bundle* (L,h) *over a bounded symmetric domain* $\Omega \subset \mathbb{C}^m$*, as described above.*

We note that taking the N-th power of the line bundle $L_{\mathbf{H}} \to \mathbb{C}^m$ (i.e., taking the N-th power of the metric $e^{-|z|^2}$) corresponds to dilating $\mathbb{C}^m$ by $\sqrt{N}$. Precisely, the map

$$\tau_N : \mathcal{L}^2 H^0(\mathbb{C}^m, L_{\mathbf{H}}) \to \mathcal{L}^2 H^0(\mathbb{C}^m, L^N_{\mathbf{H}}) \,, \qquad (\tau_N f)(z) := N^{m/2} f(N^{1/2} z) \,,$$

is unitary. Thus we can restate our result on the volume (or number, in dimension 1) variance for the Bargmann-Fock ensemble as follows:

Corollary 2.5. *Let*

$$f(z) = \sum_{k\in\mathbb{N}^m} \frac{c_k}{\sqrt{k!}} z^k \,,$$

where the coefficients c_k are independent complex Gaussian random variables with mean 0 and variance 1. Let U be a domain in $\mathbb{C}^m$ with piecewise $\mathcal{C}^2$ boundary and no cusps, and consider its dilates $U_N := \sqrt{N}\,U$. Then,

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2}[Z_f \cap U_N]\big) = \nu_m \, \mathrm{Vol}_{2m-1}(\partial U_N) + O(N^{-\frac{1}{2}+\varepsilon}) \ ,$$

where $\nu_m = \frac{\pi^{m-5/2}}{8}\,\zeta(m+\frac{1}{2})$.

Note that $\partial U_N = N^{m-1/2}\partial U$, so we have $\mathrm{Var}\big(\mathrm{Vol}_{2m-2}[Z_f \cap \sqrt{N}\,U]\big) \sim N^{m-1/2}$.

Off-diagonal estimates for general Bergman kernels of positive line bundles over complete Kähler manifolds are proved in [MM] using heat kernel methods. The relevant issue for this article is the approximation of the L^2 Szegö kernel by its Boutet de Monvel -Sjöstrand parametrix in the noncompact case. The analysis of Szegö kernels on noncompact spaces lies outside the scope of this article, so we do not state the general results here. But it appears that the general results of [MM] give sufficient control over Szegö kernels in the noncompact case to allow Theorems 1.1–1.4 to be extended to all positive line bundles over complete Kähler manifolds.

3. A bipotential for the variance

Our proofs of Theorems 1.1– 1.3 are based on a bipotential implicitly given in [SZ1]. To describe our bipotential $Q_N(z,w)$, we define the function

$$\widetilde{G}(t) := -\frac{1}{4\pi^2}\int_0^{t^2} \frac{\log(1-s)}{s}\, ds \ , \qquad 0 \le t \le 1. \tag{24}$$

Alternately,

$$\widetilde{G}(e^{-\lambda}) = -\frac{1}{2\pi^2}\int_\lambda^{\infty} \log(1-e^{-2s})\, ds \ , \qquad \lambda > 0 \ . \tag{25}$$

(The function $\widetilde{G}$ is a modification of the function G defined in [SZ1]; see (41).) We also introduce the *normalized Szegö kernel*

$$P_N(z,w) := \frac{|\Pi_N(z,w)|}{\Pi_N(z,z)^{\frac{1}{2}}\,\Pi_N(w,w)^{\frac{1}{2}}} \ . \tag{26}$$

Definition **3.1.** *Let $(L,h) \to (M,\omega)$ be as in Theorems 1.2–1.3. The* variance bipotential *is the function $Q_N : M \times M \to [0,+\infty)$ given by*

$$Q_N(z,w) = \widetilde{G}(P_N(z,w)) = -\frac{1}{4\pi^2}\int_0^{P_N(z,w)^2} \frac{\log(1-s)}{s}\, ds \ . \tag{27}$$

We remark here that Q_N is $\mathcal{C}^\infty$ off the diagonal, but is only $\mathcal{C}^1$ and not $\mathcal{C}^2$ at all points on the diagonal in $M \times M$, as the computations in §6 show.

The variance in Theorems 1.1–1.3 can be given by a double integral of the bipotential, as stated in Propositions 3.2 and 3.5 below:

Proposition **3.2.** *Let $(L,h) \to (M,\omega)$ and φ be a $(2m-2)$-form on M with $\mathcal{C}^2$ coefficients. Then*

$$\mathrm{Var}(Z_{s_N},\varphi) = \int_M\int_M Q_N(z,w)\,(i\partial\bar\partial\varphi(z))\,(i\partial\bar\partial\varphi(w)) \ . \tag{28}$$

To begin the proof of the proposition, we write

$$\Psi_N = (S_1^N, \dots, S_{d_N}^N) \in H^0(M, L^N)^{d_N} \ , \tag{29}$$

where $\{S_j^N\}$ is an orthonormal basis of $H^0(M, L^N)$. As in the proof of Proposition 2.1, we write

$$\Psi_N(z) = |\Psi_N(z)|\, u_N(z) \ ,$$

where $|\Psi_N| := (\sum_j \|S_j^N\|_{h^N}^2)^{1/2}$, so that $|u_N| \equiv 1$.

We first establish a less explicit variance formula:

Lemma 3.3.

$$\mathrm{Var}(Z_{s_N}, \varphi) = \frac{1}{\pi^2} \int_M \int_M (i\partial\bar\partial\varphi(z))(i\partial\bar\partial\varphi(w)) \int_{\mathbb{C}^{d_N}} \log|\langle u_N(z), c\rangle| \, \log|\langle u_N(w), c\rangle| \, d\gamma_N(c) \ .$$

Proof. We write sections $s_N \in H^0(M, L^N)$ as

$$s_N = \sum_{j=1}^{d_N} c_j S_j^N = \langle c, \Psi_N\rangle \ , \qquad c = (c_1, \dots, c_{d_N}) \ . \tag{30}$$

Writing $\Psi_N = F e_L^{\otimes N}$, where e_L is a local nonvanishing section of L, and recalling that

$$\omega = \frac{i}{2}\Theta_h = -i\partial\bar\partial \log \|e_L\|_h \ ,$$

we have by (13),

$$\begin{aligned} Z_{s_N} &= \frac{i}{\pi}\partial\bar\partial \log|\langle c, F\rangle| = \frac{i}{\pi}\partial\bar\partial \log|\langle c, \Psi_N\rangle| - \frac{i}{\pi}\partial\bar\partial \log\|e_L^{\otimes N}\|_h \\ &= \frac{i}{\pi}\partial\bar\partial \log|\langle c, \Psi_N\rangle| + \frac{N}{\pi}\,\omega \ . \end{aligned} \tag{31}$$

Let φ be a test form, and consider the random variable

$$Y_N := \left(\frac{i}{\pi}\partial\bar\partial \log|\langle c, \Psi_N\rangle| \, , \, \varphi\right) = \left(\log|\langle c, \Psi_N\rangle| \, , \, \frac{i}{\pi}\partial\bar\partial\varphi\right) \ , \tag{32}$$

so that $(Z_{s_N}, \varphi) = Y_N + NC$, where $C = \frac{1}{\pi}\int_M \omega \wedge \varphi$; hence

$$\mathrm{Var}(Z_{s_N}, \varphi) = \mathrm{Var}(Y_N) \ .$$

By (15), we have

$$\mathbf{E}(Y_N) = \left(\frac{i}{\pi}\partial\bar\partial \log|\Psi_N| \, , \, \varphi\right) = \frac{i}{\pi}\int_M \log|\Psi_N| \, \partial\bar\partial\varphi \ , \tag{33}$$

whereas by (32), we have

$$\mathbf{E}(Y_N^2) = \frac{1}{\pi^2} \int_M \int_M (i\partial\bar\partial\varphi(z)) \, (i\partial\bar\partial\varphi(w)) \int_{\mathbb{C}^{d_N}} \log|\langle c, \Psi_N(z)\rangle| \, \log|\langle c, \Psi_N(w)\rangle| \, d\gamma_N(c) \ . \tag{34}$$

Recalling that $\Psi_N = |\Psi_N| u_N$ with $|u_N| \equiv 1$, we have

$$\begin{aligned} \log|\langle \Psi_N(z), c\rangle| \, \log|\langle \Psi_N(w), c\rangle| &= \log|\Psi_N(z)| \, \log|\Psi_N(w)| + \log|\Psi_N(z)| \, \log|\langle u_N(w), c\rangle| \\ &\quad + \log|\Psi_N(w)| \, \log|\langle u_N(z), c\rangle| \\ &\quad + \log|\langle u_N(w), c\rangle| \, \log|\langle u_N(z), c\rangle| \ , \end{aligned} \tag{35}$$

which decomposes (34) into four terms. By (33), the first term contributes

$$\frac{1}{\pi^2}\int_M\int_M (i\partial\bar\partial\varphi(z))(i\partial\bar\partial\varphi(w))\log|\Psi_N(z)|\,\log|\Psi_N(w)| = (\mathbf{E}Y_N)^2\,. \tag{36}$$

The c-integral in the second term is independent of w and hence the second term vanishes. The third term likewise vanishes. Therefore, the fourth term gives the variance $\mathrm{Var}(Z_{s_N},\varphi)$. □

We now complete the proof of Proposition 3.2 by evaluating the c-integral of Lemma 3.3:

LEMMA **3.4.** *We have:*

$$\frac{1}{\pi^2}\int_{\mathbb{C}^{d_N}}\log|\langle u_N(z),c\rangle|\,\log|\langle u_N(w),c\rangle|\,d\gamma_N(c) = Q_N(z,w)+K\,,$$

where K is a universal constant.

Proof. We showed in [SZ1, p. 779] by an elementary computation that

$$\int_{\mathbb{C}^{d_N}}\log|\langle u_N(z),c\rangle|\,\log|\langle u_N(w),c\rangle|\,d\gamma_N(c) = G(|\langle u_N(z),\overline{u_N(w)}\rangle|)\,, \tag{37}$$

where

$$G(t) = \frac{1}{\pi^2}\int_{\mathbb{C}^2} e^{-(|c_1|^2+|c_2|^2)}\log|c_1|\,\log\left|c_1t+c_2\sqrt{1-t^2}\right|\,dc_1dc_2\,. \tag{38}$$

The computation of the integral (38) was begun in [SZ1, §4]. Let us finish it. By (47)–(50) in [SZ1] (with $\lambda=\frac{1}{2}r^2$), we have

$$G(e^{-\lambda}) = k_1 + k_2\lambda + \frac{1}{2}\int_0^\lambda \log(1-e^{-2s})\,ds\,.$$

Since $G(0)$ is finite, $k_2=0$ and hence

$$G(e^{-\lambda}) = k_0 - \frac{1}{2}\int_\lambda^\infty \log(1-e^{-2s})\,ds\,, \tag{39}$$

or equivalently,

$$G(t) = k_0 - \frac{1}{4}\int_0^{t^2}\frac{\log(1-s)}{s}\,ds \qquad (0\le t\le 1)\,. \tag{40}$$

Hence,

$$\widetilde{G}(t) = \frac{1}{\pi^2}[G(t)-k_0]\,. \tag{41}$$

The lemma follows from (37) and (41) with

$$t = P_N(z,w) = \frac{|\langle\Psi_N(z),\overline{\Psi_N(w)}\rangle|}{|\Psi_N(z)|\,|\Psi_N(w)|} = |\langle u_N(z),\overline{u_N(w)}\rangle|\,. \tag{42}$$

□

Proposition 3.2 is an immediate consequence of Lemmas 3.3–3.4. □

To state the bipotential formula for the volume variance of Theorem 1.2 (and number variance of Theorem 1.1), we let

$$\Phi := \frac{1}{(m-1)!}\omega^{m-1}\,, \tag{43}$$

so that $\mathrm{Vol}_{2m-2}[Z_{s_N}\cap U] = \big(Z_{s_N}, \chi_U\,\Phi\big)$. (If $m=1$, we set $\Phi = 1$.)

Proposition **3.5.** *Let* $(L,h)\to(M,\omega)$, $U\subset M$ *be as in Theorem 1.3. Then*

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2}[Z_{s_N}\cap U]\big) = -\int_{\partial U\times\partial U}\bar\partial_z\bar\partial_w Q_N(z,w)\wedge\Phi(z)\wedge\Phi(w)\,,$$

where Q_N *is given by* (27). *In particular, for the case where* $\dim M = 1$, *we have*

$$\mathrm{Var}\big(\#[Z_{s_N}\cap U]\big) = -\int_{\partial U\times\partial U}\bar\partial_z\bar\partial_w Q_N(z,w)\,.$$

Proof. As in the proof of Lemma 3.3, we let

$$Y_N^U := \left(\frac{i}{\pi}\partial\bar\partial\log|\langle c,\Psi_N\rangle|\,,\,\chi_U\,\Phi\right) = \int_U\frac{i}{\pi}\partial\bar\partial\log|\langle c,\Psi_N\rangle|\wedge\Phi = \frac{i}{\pi}\int_{\partial U}\bar\partial\log|\langle c,\Psi_N\rangle|\wedge\Phi\,, \tag{44}$$

where Ψ_N is given by (29). As before, we have $(Z_{s_N},\chi_U\,\Phi) = Y_N^U + NC$, where $C = \frac{1}{\pi}\int_U\omega\wedge\Phi = \frac{m}{\pi}\mathrm{Vol}(U)$. Hence,

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2}[Z_{s_N}\cap U]\big) = \mathrm{Var}(Z_{s_N},\chi_U\,\Phi) = \mathrm{Var}(Y_N^U)\,. \tag{45}$$

By (44) (see also the proof of Proposition 2.1), we then have

$$\mathbf{E}\big(Y_N^U\big) = \frac{i}{\pi}\int_{\partial U}\bar\partial\log|\Psi_N|\wedge\Phi\,. \tag{46}$$

Also by (44), we have

$$\mathbf{E}\big((Y_N^U)^2\big) = \frac{-1}{\pi^2}\int_{\mathbb{C}^{d_N}}\int_{\partial U}\int_{\partial U}\big[\bar\partial\log|\langle c,\Psi_N(z)\rangle|\wedge\Phi(z)\big]\,\big[\bar\partial\log|\langle c,\Psi_N(w)\rangle|\wedge\Phi(w)\big]\,d\gamma_N(c)\,. \tag{47}$$

Again following the proof of Lemma 3.3, we write $\Psi_N = |\Psi_N|u_N$ with $|u_N|\equiv 1$, and use (35) to decompose (47) into four terms. The first term contributes

$$\frac{-1}{\pi^2}\int_{\partial U}\int_{\partial U}\big[\bar\partial\log|\Psi_N(z)|\wedge\Phi(z)\big]\,\big[\bar\partial\log|\Psi_N(w)|\wedge\Phi(w)\big] = \big(\mathbf{E}Y_N^U\big)^2\,. \tag{48}$$

The second term vanishes since $\int\log|\langle c,u_N(w)\rangle|\,d\gamma_N(c)$ is independent of w and hence $\bar\partial$ kills it. Similarly, the third term vanishes, since it contains $\int\log|\langle c,u_N(z)\rangle|\,d\gamma_N(c)$, which is independent of z. Therefore,

$$\begin{aligned}\mathrm{Var}(Y_N^U) &= \frac{-1}{\pi^2}\int_{\mathbb{C}^{d_N}}\int_{\partial U}\int_{\partial U}\big[\bar\partial\log|\langle u_N(z),c\rangle|\wedge\Phi(z)\big]\,\big[\bar\partial\log|\langle u_N(w),c\rangle|\wedge\Phi(w)\big]\,d\gamma_N(c)\\ &= \frac{-1}{\pi^2}\int_{\mathbb{C}^{d_N}}\bar\partial_z\bar\partial_w\left(\int_{\partial U\times\partial U}\log|\langle u_N(z),c\rangle|\,\log|\langle u_N(w),c\rangle|\,\Phi(z)\wedge\Phi(w)\right)d\gamma_N(c)\,.\end{aligned}$$

The formula of the proposition then follows from Lemma 3.4. □

4. Off-diagonal asymptotics and estimates for the Szegö kernel

In this section, we use the off-diagonal asymptotics for $\Pi_N(z,w)$ from [SZ2] to provide the off-diagonal estimates for the normalized Szegö kernel $P_N(z,w)$ that we need for our variance formulas. Our estimates are of two types: (1) 'near-diagonal' asymptotics (Propositions 4.3–4.4) for $P_N(z,w)$ where the distance $d(z,w)$ between z and w satisfies an upper bound $d(z,w) \le b\left(\frac{\log N}{N}\right)^{1/2}$ ($b \in \mathbb{R}^+$); (2) 'far-off-diagonal' asymptotics (Proposition 4.2) where $d(z,w) \ge b\left(\frac{\log N}{N}\right)^{1/2}$.

As discussed in §2.1 (cf. [Ze, SZ1, SZ2]), we obtain the asymptotics by identifying the line bundle Szegö kernel Π_N with a scalar Szegö kernel $\Pi_N(x,y)$ on the unit circle bundle $X \subset L^{-1} \to M$ associated to the Hermitian metric h. Given $z_0 \in M$, we choose a neighborhood U of z_0, a local normal coordinate chart $\rho : U, z_0 \to \mathbb{C}^m, 0$ centered at z_0, and a *preferred* local frame at z_0, which we defined in [SZ2] to be a local frame e_L such that

$$\|e_L(z)\|_h = 1 - \frac{1}{2}\|\rho(z)\|^2 + \cdots \ . \tag{49}$$

For $u = (u_1, \dots, u_m) \in \rho(U)$, $\theta \in (-\pi, \pi)$, we let

$$\tilde{\rho}(u_1, \dots, u_m, \theta) = \frac{e^{i\theta}}{|e_L^*(\rho^{-1}(u))|_h} e_L^*(\rho^{-1}(u)) \in X \,, \tag{50}$$

so that $(u_1, \dots, u_m, \theta) \in \mathbb{C}^m \times \mathbb{R}$ give local coordinates on X. As in [SZ2], we write

$$\Pi_N^{z_0}(u, \theta; v, \varphi) = \Pi_N(\tilde{\rho}(u,\theta), \tilde{\rho}(v,\varphi)) \ .$$

Note that $\Pi_N^{z_0}$ depends on the choice of coordinates and frame; we shall assume that we are given normal coordinates and local frames for each point $z_0 \in M$ and that these normal coordinates and local frames are smooth functions of z_0.

The scaling asymptotics of $\Pi_N^{z_0}(u,\theta; v,\varphi)$ lead to the model Heisenberg Szegö kernel (23) discussed in §2.2 for the Bargmann-Fock space of functions on $\mathbb{C}^m$. We shall use the following (near and far) off-diagonal asymptotics from [SZ2]:

Theorem **4.1.** *Let $(L,h) \to (M,\omega)$ be as in Theorem 1.3, and let $z_0 \in M$. Then using the above notation,*

i) $N^{-m}\Pi_N^{z_0}(\frac{u}{\sqrt{N}}, \frac{\theta}{N}; \frac{v}{\sqrt{N}}, \frac{\varphi}{N})$

$$= \Pi_1^{\mathbf{H}}(u,\theta; v,\varphi)\left[1 + \textstyle\sum_{r=1}^k N^{-r/2} p_r(u,v) + N^{-(k+1)/2} R_{Nk}(u,v)\right] \ ,$$

where the p_r are polynomials in (u,v) of degree $\le 5r$ (of the same parity as r), and

$$|\nabla^j R_{Nk}(u,v)| \le C_{jk\varepsilon b} N^{\varepsilon} \quad \textit{for} \ \ |u| + |v| < b\sqrt{\log N}\,,$$

for $\varepsilon, b \in \mathbb{R}^+$, $j,k \ge 0$. Furthermore, the constant $C_{jk\varepsilon b}$ can be chosen independently of z_0.

ii) *For $b > \sqrt{j + 2k + 2m}$, $j,k \ge 0$, we have*

$$\left|\nabla_h^j \Pi_N(z,w)\right| = O(N^{-k}) \qquad \textit{uniformly for} \ \ d(z,w) \ge b\sqrt{\frac{\log N}{N}} \ .$$

Here $\nabla_h^j = (\nabla_h)^j$ is the j-th iterated horizontal covariant derivative; see (17). Theorem 4.1 is equivalent to equations (95)–(96) in [SZ2], where the result was shown to hold for almost-complex symplectic manifolds. (The remainder in (i) was given for $v = 0$, but the proof holds without any change for $v \neq 0$. Also the statement of the result was divided into the two cases where the scaled distance is less or more, respectively, than $N^{1/6}$ instead of $\sqrt{\log N}$ in the above formulation, which is more useful for our purposes.) A description of the polynomials p_r in part (i) is given in [SZ2], but we only need the $k = 0$ case in this paper. For the benefit of the reader, we give a complete proof of Theorem 4.1 in §4.1 below.

Remark: The Szegö kernel actually satisfies the sharper 'Agmon decay estimate' away from the diagonal:

$$\nabla^j \Pi_N(z,\theta;w,\varphi) = O\left(e^{-A_j\sqrt{N}\,d(z,w)}\right)\,, \qquad j \geq 0\,. \tag{51}$$

In particular,

$$|\Pi_N(z,w)| = O\left(e^{-A\sqrt{N}\,d(z,w)}\right)\,. \tag{52}$$

A short proof of (52) is given in [Be, Th. 2.5]; similar estimates were established by M. Christ [Ch], H. Delin [De], and N. Lindholm[Li]. (See also [DLM, MM] for off-diagonal exponential estimates in a more general setting.) We do not need Agmon estimates for this paper; instead Theorem 4.1 suffices.

We now state our far-off-diagonal decay estimate for $P_N(z,w)$, which follows immediately from Theorem 4.1(ii) and the fact that $\Pi_N(z,z) = \frac{1}{\pi^m}N^m(1+O(N^{-1}))$ (by [Ze] or Theorem 4.1(i)).

Proposition **4.2.** *Let $(L,h) \to (M,\omega)$ be as in Theorem 1.3, and let $P_N(z,w)$ be the normalized Szegö kernel for $H^0(M,L^N)$ given by* (26)*. For $b > \sqrt{j+2k}$, $j,k \geq 0$, we have*

$$\nabla^j P_N(z,w) = O(N^{-k}) \qquad \text{uniformly for } \; d(z,w) \geq b\sqrt{\frac{\log N}{N}}\,.$$

The normalized Szegö kernel P_N also satisfies Gaussian decay estimates valid very close to the diagonal. To give the estimate, we write by abuse of notation,

$$P_N(z_0+u, z_0+v) := P_N(\rho^{-1}(u),\rho^{-1}(v)) = \frac{|\Pi_N^{z_0}(u,0;v,0)|}{\Pi_N^{z_0}(u,0;u,0)^{1/2}\Pi_N^{z_0}(v,0;v,0)^{1/2}}\,.$$

As an immediate consequence of Theorem 4.1(i), we have:

Proposition **4.3.** *Let $P_N(z,w)$ be as in Proposition 4.2, and let $z_0 \in M$. For $b,\varepsilon > 0$, $j \geq 0$, there is a constant $C_j = C_j(\varepsilon,b)$, independent of the point z_0, such that*

$$\begin{aligned} P_N\left(z_0+\tfrac{u}{\sqrt{N}}, z_0+\tfrac{v}{\sqrt{N}}\right) &= e^{-\frac12|u-v|^2}[1+R_N(u,v)] \\ &\qquad |\nabla^j R_N(u,v)| \leq C_j\, N^{-1/2+\varepsilon} \quad \text{for } \; |u|+|v| < b\sqrt{\log N}\,. \end{aligned}$$

As a corollary we have:

Proposition **4.4.** *The remainder R_N in Proposition 4.3 satisfies*

$$|R_N(u,v)| \leq \frac{C_2}{2}\,|u-v|^2 N^{-1/2+\varepsilon}, \quad |\nabla R_N(u)| \leq C_2\,|u-v|\,N^{-1/2+\varepsilon}, \quad \text{for } \; |u|+|v| < b\sqrt{\log N}.$$

Proof. Since $P_N(z_0+u, z_0+v) \le 1 = P_N(z_0+u, z_0+u)$, we conclude that $R_N(u,u) = 0$, $dR_N|_{(u,u)} = 0$, and thus by Proposition 4.3,

$$|\nabla R_N(u,v)| \le \sup_{0\le t\le 1} |\nabla^2 R_N(u,(1-t)u+tv)|\,|u-v| \le C_2\,|u-v|\,N^{-1/2+\varepsilon}\ .$$

Similarly,

$$|R_N(u,v)| \le \frac{1}{2}\sup_{0\le t\le 1} |\nabla^2 R_N(u,(1-t)u+tv)|\,|u-v|^2 \le \frac{C_2}{2}\,|u-v|^2\,N^{-1/2+\varepsilon}\ .$$

□

4.1. **Proof of Theorem 4.1.** In this section, we sketch the proof of Theorem 4.1. The argument is essentially contained in [SZ2], but we add some details relevant to the estimates in Theorem 4.1.

The Szegö kernels $\Pi_N(x,y)$ are the Fourier coefficients of the total Szegö projector $\Pi(x,y) : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$; i.e. $\Pi_N(x,y) = \frac{1}{2\pi}\int e^{-iN\theta}\Pi(e^{i\theta}x,y)\,d\theta$. The estimates for $\Pi_N(z,w)$ are then based on the Boutet de Monvel-Sjöstrand construction of an oscillatory integral parametrix for the Szegö kernel:

$$\begin{gathered}\Pi(x,y) = S(x,y) + E(x,y)\ ,\\ \text{with}\ \ S(x,y) = \int_0^\infty e^{it\psi(x,y)} s(x,y,t)dt\,, \qquad E(x,y) \in \mathcal{C}^\infty(X\times X)\,.\end{gathered} \tag{53}$$

The amplitude has the form $s \sim \sum_{k=0}^\infty t^{m-k} s_k(x,y) \in S^m(X\times X\times\mathbb{R}^+)$. The phase function ψ is of positive type, and as described in [BSZ2], is given by:

$$\psi(z,\theta,w,\varphi) = i\left[1 - \frac{a(z,\bar w)}{\sqrt{a(z)}\sqrt{a(w)}}\, e^{i(\theta-\varphi)}\right]\ , \tag{54}$$

where $a \in \mathcal{C}^\infty(M\times M)$ is an almost holomorphic extension of the function $a(z,\bar z) := a(z)$ on the anti-diagonal $A = \{(z,\bar z) : z \in M\}$, i.e., $\bar\partial a$ vanishes to infinite order along A. We recall from (11) that $a(z)$ describes the Hermitian metric on L in our preferred holomorphic frame at z_0, so by (49), we have $a(u) = 1 + |u|^2 + O(|u|^3)$, and hence

$$a(u,\bar v) = 1 + u\cdot\bar v + O(|u|^3 + |v|^3)\ . \tag{55}$$

For further background and notation on complex Fourier integral operators we refer to [BSZ2] and to the original paper of Boutet de Monvel and Sjöstrand [BoSj].

As above, denote the N-th Fourier coefficient of these operators relative to the S^1 action by $\Pi_N = S_N + E_N$. Since E is smooth, we have $E_N(x,y) = O(N^{-\infty})$, where $O(N^{-\infty})$ denotes a quantity which is uniformly $O(N^{-k})$ on $X\times X$ for all positive k. Then, $E_N(z,w)$ trivially satisfies the remainder estimates in Theorem 4.1.

Hence it is only necessary to verify that the oscillatory integral

$$S_N(x,y) = \int_0^{2\pi} e^{-iN\theta} S(e^{i\theta}x,y)d\theta = \int_0^\infty\int_0^{2\pi} e^{-iN\theta + it\psi(e^{i\theta}x,y)} s(e^{i\theta}x,y,t)d\theta dt \tag{56}$$

satisfies Theorem 4.1. This follows from an analysis of the stationary phase method and remainder estimate for the rescaled parametrix

$$S_N^{z_0}\left(\frac{u}{\sqrt{N}},0;\frac{v}{\sqrt{N}},0\right) = N\int_0^\infty\int_0^{2\pi} e^{iN\left(-\theta+t\psi\left(\frac{u}{\sqrt{N}},\theta;\frac{v}{\sqrt{N}},0\right)\right)} s\left(\frac{u}{\sqrt{N}},\theta;\frac{v}{\sqrt{N}},0,Nt\right) d\theta dt\,, \tag{57}$$

where we changed variables $t\mapsto Nt$. For background on the stationary phase method when the phase is complex we refer to [Hö]. We are particularly interested in the dependence of the stationary phase expansion and remainder estimate on the parameters (u,v) satisfying the constraints in (i)-(ii) of Theorem 4.1.

To clarify the constraints, we recall from [SZ2] (95) that the Szegö kernel satisfies the following far from diagonal estimates:

$$\left|\nabla_h^j \Pi_N(z,w)\right| = O(N^{-K}) \qquad \text{for all } \ j,K \text{ when } \ d(z,w)\geq \frac{N^{1/6}}{\sqrt{N}}\,. \tag{58}$$

Hence we may assume from now on that $z=z_0+\frac{u}{\sqrt{N}}, w=z_0+\frac{v}{\sqrt{N}}$ with

$$|u|+|v|\leq \delta N^{1/6} \tag{59}$$

for a sufficiently small constant $\delta>0$ (see 65).

By (54)–(55), the rescaled phase in (57) has the form:

$$\widetilde{\Psi} := t\psi\left(\frac{u}{\sqrt{N}},\theta;\frac{v}{\sqrt{N}},0\right)-\theta = it\left[1-\frac{a\left(\frac{u}{\sqrt{N}},\frac{\bar v}{\sqrt{N}}\right)}{a\left(\frac{u}{\sqrt{N}},\frac{\bar u}{\sqrt{N}}\right)^{\frac12} a\left(\frac{v}{\sqrt{N}},\frac{\bar v}{\sqrt{N}}\right)^{\frac12}}\,e^{i\theta}\right]-\theta \tag{60}$$

and the N-expansion

$$\widetilde{\Psi} = it[1-e^{i\theta}]-\theta-\frac{it}{N}\psi_2(u,v)e^{i\theta}+tR_3^\psi\left(\frac{u}{\sqrt{N}},\frac{v}{\sqrt{N}}\right)e^{i\theta}\,, \tag{61}$$

where

$$\psi_2(u,v) = u\cdot\bar v-\frac12(|u|^2+|v|^2) = -\frac12|u-v|^2+i\,\mathrm{Im}\,(u\cdot\bar v)$$

is the phase function of (23). After multiplying by iN, we move the last two terms of (61) into the amplitude. Indeed, we absorb all of $\exp\{(\psi_2+iNR_3^\psi)te^{i\theta}\}$ into the amplitude so that (57) is an oscillatory integral

$$N\int_0^\infty\int_0^{2\pi} e^{iN\Psi(t,\theta)}A(t,\theta;z_0,u,v)d\theta dt + O(N^{-\infty}) \tag{62}$$

with phase

$$\Psi(t,\theta) := it(1-e^{i\theta})-\theta \tag{63}$$

and with amplitude

$$A(t,\theta;z_0,u,v) := e^{te^{i\theta}\psi_2(u,v)+ite^{i\theta}NR_3^\psi\left(\frac{u}{\sqrt{N}},\frac{v}{\sqrt{N}}\right)}\, s\big(\frac{u}{\sqrt{N}},\theta;\frac{v}{\sqrt{N}},0,Nt\big). \tag{64}$$

The phase Ψ is independent of the parameters (u,v), satisfies $\mathrm{Re}\,(i\Psi) = -t(1-\cos\theta)\leq 0$ and has a unique critical point at $\{t=1,\theta=0\}$ where it vanishes.

The factor $e^{te^{i\theta}\psi_2(u,v)}$ is of exponential growth in some regions. However, since it is a rescaling of a complex phase of positive type, the complex phase $iN\Psi + te^{i\theta}\psi_2(u,v)$ is of positive type,

$$\mathrm{Re}\,(iN\Psi + te^{i\theta}\psi_2(u,v)) < 0 \tag{65}$$

once the cubic remainder $Nte^{i\theta}R_3^{\psi}(\frac{u}{\sqrt{N}},\frac{v}{\sqrt{N}})$ is smaller than $iN\Psi + te^{i\theta}\psi_2(u,v)$, which occurs for all (t,θ,u,v) when (u,v) satisfy (59) with δ sufficiently small.

To estimate the joint rate of decay in (N,u,v), we follow the stationary phase expansion and remainder estimate in Theorem 7.7.5 of [Hö], with extra attention to the unbounded parameter u.

The first step is to use a smooth partition of unity $\{\rho_1(t,\theta),\rho_2(t,\theta)\}$ to decompose the integral (57) into a region $(1-\varepsilon,1+\varepsilon)_t \times (-\varepsilon,\varepsilon)_\theta$ containing the critical point and one over the complementary set containing no critical point. We claim that the ρ_2 integral is of order $N^{-\infty}$ and can be neglected. This follows by repeated partial integration as in the standard proof together with the fact that the exponential factors in (65) decay, so that the estimates are integrable and uniform in u.

We then apply [Hö] Theorem 7.7.5 to the ρ_1 integral. The first term of the stationary phase expansion equals $N^m e^{te^{i\theta}\psi_2(u,v)}$ and the remainder satisfies

$$|\widehat{R}_J(P_0,u,v,N)| \le CN^{-m+J}\sum_{|\alpha|\le 2J+2}\sup_{t,\theta}|D^\alpha_{t,\theta}\rho_1 A(t,\theta;P_0,u,v)|. \tag{66}$$

From the formula in (64) and the fact that s is a symbol, A has a polyhomogeneous expansion of the form

$$\begin{aligned} A(t,\theta;P_0,u,v) &= \rho_1(t,\theta)e^{te^{i\theta}\psi_2(u,v)}N^m\left[\sum_{n=0}^{K}N^{-n/2}f_n(u,v;t,\theta,P_0)+R_K(u,v,t,\theta)\right], \\ &\qquad |\nabla^j R_{Nk}(u,v)| \le C_{jk\varepsilon b}e^{\varepsilon(|u|^2+|v|^2)}N^{-\frac{K+1}{2}}\big). \end{aligned} \tag{67}$$

The exponential remainder factor $e^{\varepsilon(|u|^2+|v|^2)}$ comes from the fact $\mathrm{Re}\,e^{i\theta}\psi_2 = \cos\theta\mathrm{Re}\,\psi - \sin\theta\mathrm{Im}\,\psi$ with $\mathrm{Re}\,\psi\le 0$ and $|\sin\theta|<\varepsilon$ on the support of ρ_1. Hence, the supremum of the amplitude in a neighborhood of the stationary phase set (in the support of ρ_1) is bounded by $e^{\varepsilon|\mathrm{Im}\,\psi_2|}$. The remainder term is smaller than the main term asymptotically as $N\to\infty$ as long as (u,v) satisfies (59). Part(i) of Theorem 4.1 is an immediate consequence of (67) since $e^{\varepsilon(|u|^2+|v|^2)}\le N^\varepsilon$ for $|u|+|v|\le\sqrt{\log N}$.

To prove part (ii), we may assume from (58)–(59) that $\sqrt{\log N}\le |u|+|v|\le \delta\, N^{1/6}$. In this range the asymptotics (67) are valid. We first rewrite the horizontal z-derivatives $\frac{\partial^h}{\partial z_j}$ as u_j derivatives, which for L^N have the form $\sqrt{N}\frac{\partial}{\partial u_j} - NA_j(\frac{u}{\sqrt{N}})$ and thus ∇_h contributes a factor of $\sqrt{N}$. We thus obtain an asymptotic expansion and remainder for $\nabla_h^j\Pi_N(z,w)$ by applying ∇_h^j to the expansion (i) with $k=0$:

$$\Pi_1^{\mathbf{H}}(u,\theta;v,\varphi)\left[1+N^{-1/2}R_{N0}(u,v)\right].$$

The operator ∇_h^j contributes a factor of $N^{j/2}$ to each term, and thus

$$\begin{aligned}\left|\nabla_h^j \Pi_N(z,w)\right| &= O\left(N^{m+j/2}\, e^{-(1-\varepsilon)\frac{|u|^2+|v|^2}{2}}\right)\\ &= O(N^{-k}) \qquad \text{uniformly for} \quad |u|^2+|v|^2 \geq (j+2k+2m+\varepsilon')\log N\ ,\end{aligned}$$

where $\varepsilon' = (j+2k+2m+1)\varepsilon$. ☐

5. The sharp variance estimate: Proof of Theorem 1.3

We first give the proof of Theorem 1.3, which uses the same method as the proof of Theorem 1.2, but has simpler computations.

We begin with some off-diagonal asymptotics for the function $Q_N = \widetilde{G} \circ P_N$ defined in (27). By Proposition 4.2, we see that

$$Q_N(z,w) \leq \frac{1}{4\pi^2}\int_0^{C/N^{2m}} \frac{-\log(1-s)}{s}\,ds = O\left(\frac{1}{N^{2m}}\right)\ , \quad \text{for} \;\; d(z,w) > \frac{b\sqrt{\log N}}{\sqrt{N}}, \tag{68}$$

with $b > \sqrt{2m}$.

Next we show the near-diagonal estimate

$$Q_N\Big(z_0, z_0 + \frac{v}{\sqrt{N}}\Big) = \widetilde{G}(e^{-\frac12|v|^2}) + O(N^{-1/2+\varepsilon})\ , \qquad \text{for} \;\; |v| \leq b\sqrt{\log N}. \tag{69}$$

To verify (69), we apply Proposition 4.4. Since $P_N(z_0,z_0) = 1$ and $\widetilde{G}'(t) \to \infty$ as $t \to 1$, we need a short argument: let

$$\Lambda_N = -\log P_N\ . \tag{70}$$

Recalling (25), we write,

$$F(\lambda) := \widetilde{G}(e^{-\lambda}) = -\frac{1}{2\pi^2}\int_\lambda^\infty \log(1-e^{-2s})\,ds \qquad (\lambda > 0)\ , \tag{71}$$

so that

$$Q_N = F \circ \Lambda_N\ .$$

By Proposition 4.4,

$$\Lambda_N\left(z_0, z_0 + \frac{v}{\sqrt{N}}\right) = \frac12|v|^2 + \widetilde{R}_N(v)\ ,$$

where

$$\widetilde{R}_N = -\log(1+R_N) = O(|v|^2 N^{-1/2+\varepsilon}) \tag{72}$$

By (71) and Proposition 4.4,

$$|F'(\lambda)| = -\frac{1}{2\pi^2}\log(1-e^{-2\lambda}) \leq \frac{1}{2\pi^2}\max\left(\log\frac{1}{\lambda}, 1\right) = O\left(1 + \log^+\frac{1}{|v|}\right)\ . \tag{73}$$

Since $\frac{1}{2}|v|^2+\widetilde{R}_N(v)=|v|^2\left(\frac{1}{2}+o(N)\right)$, it follows from (72)–(73) that

$$\begin{aligned} Q_N\Big(z_0, z_0+\frac{v}{\sqrt{N}}\Big) &= F\left(\frac{1}{2}|v|^2+\widetilde{R}_N(v)\right) \\ &= F\left(\frac{1}{2}|v|^2\right)+O\left(\left[1+\log^+\frac{1}{|v|}\right]\widetilde{R}_N(v)\right) \\ &= \widetilde{G}(e^{-\frac{1}{2}|v|^2})+O(N^{-1/2+\varepsilon})\,, \qquad \text{for } \ |v|\le b\sqrt{\log N}, \end{aligned}$$

which gives (69). (This computation also shows that Q_N is $\mathcal{C}^1$ and has vanishing first derivatives on the diagonal in $M\times M$.)

Next, we note that $\widetilde{G}(t)=\frac{1}{4\pi^2}\left(t^2+\frac{t^4}{2^2}+\frac{t^6}{3^2}+\cdots+\frac{t^{2n}}{n^2}+\cdots\right)$, and hence

$$\begin{aligned} \int_{\mathbb{C}^m}\widetilde{G}(e^{-\frac{1}{2}|v|^2})\,dv &= \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\int_{\mathbb{C}^m}\frac{e^{-k|v|^2}}{k^2}\,dv \\ &= \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{\pi^m}{k^{m+2}} = \frac{\pi^{m-2}}{4}\,\zeta(m+2)\,. \end{aligned} \tag{74}$$

By Proposition 3.2, we have

$$\operatorname{Var}\big(Z_{s_N},\varphi\big)=\int_M \mathcal{I}(z)\,i\partial\bar\partial\varphi(z)\,, \tag{75}$$

where

$$\mathcal{I}(z)=\int_{\{z\}\times M}Q_N(z,w)\,i\partial\bar\partial\varphi(w)\,. \tag{76}$$

We let

$$\Omega=\frac{1}{m!}\omega^m$$

denote the volume form of M, and we write

$$i\partial\bar\partial\varphi=\psi\,\Omega\,, \qquad \psi\in\mathcal{C}^1. \tag{77}$$

To evaluate $\mathcal{I}(z_0)$ at a fixed point $z_0\in M$, we choose a normal coordinate chart centered at z_0 as in §4. By (68) and (76)–(77),

$$\mathcal{I}(z_0)=\int_{|v|\le b\sqrt{\log N}}Q_N\left(z_0,z_0+\frac{v}{\sqrt{N}}\right)\,\psi\left(z_0+\frac{v}{\sqrt{N}}\right)\,\Omega\left(z_0+\frac{v}{\sqrt{N}}\right)+O\left(\frac{1}{N^{2m}}\right)\,. \tag{78}$$

We recall that $\omega=\frac{i}{2}\partial\bar\partial\log a=\frac{i}{2}\partial\bar\partial\,[|z|^2+O(|z|^3)]$ in normal coordinates. Hence

$$\Omega\left(z_0+\frac{v}{\sqrt{N}}\right)=\frac{1}{m!}\left[\frac{i}{2N}\partial\bar\partial|v|^2+O\left(\frac{|v|}{N^{3/2}}\right)\right]^m=\frac{1}{N^m}\left[\Omega_E(v)+O\left(\sqrt{\frac{\log N}{N}}\right)\right]\,, \tag{79}$$

for $|v|\le b\sqrt{\log N}$, where

$$\Omega_E(v)=\frac{1}{m!}\left(\frac{i}{2}\partial\bar\partial|v|^2\right)^m=\prod_{j=1}^m\frac{i}{2}dv_j\wedge d\bar v_j$$

denotes the Euclidean volume form. Since $\varphi \in \mathcal{C}^3$ and hence $\psi(z+\frac{v}{\sqrt{N}}) = \psi(z)+O(|v|/\sqrt{N})$, we then have by (69) and (78)–(79),

$$\begin{aligned}\mathcal{I}(z_0) &= \frac{1}{N^m}\left[\int_{|v|\le b\sqrt{\log N}} \left\{\widetilde{G}(e^{-\frac12|v|^2}) + O(N^{-1/2+\varepsilon})\right\}\left\{\psi(z_0)+O(N^{-1/2+\varepsilon})\right\}\right.\\ &\qquad\left.\times\left\{\Omega_E(v)+O(N^{-1/2+\varepsilon})\right\}\right] + O\left(\frac{1}{N^{2m}}\right) . \end{aligned} \tag{80}$$

Since $\widetilde{G}(e^{-\frac12|v|^2}) \in \mathcal{L}^1$ by (74), we have

$$\mathcal{I}(z_0) = \frac{\psi(z_0)}{N^m}\left[\int_{|v|\le b\sqrt{\log N}} \widetilde{G}(e^{-\frac12|v|^2})\Omega_E(v) + O\left(N^{-1/2+\varepsilon}(\log N)^m\right)\right] . \tag{81}$$

Since $\widetilde{G}(e^{-\lambda}) = O(e^{-2\lambda})$ and hence

$$\int_{|v|\ge b\sqrt{\log N}} \widetilde{G}(e^{-\frac12|v|^2})\,\Omega_E(v) = O(N^{-2m}) , \tag{82}$$

we can replace the integral over the $(b\sqrt{\log N})$-ball with one over all of $\mathbb{C}^m$, and therefore

$$\mathcal{I}(z_0) = \frac{\psi(z_0)}{N^m}\left[\int_{C^m} \widetilde{G}(e^{-\frac12|v|^2})\Omega_E(v) + O(N^{-1/2+\varepsilon'})\right] = \frac{\psi(z_0)}{N^m}\left[\kappa_m + O(N^{-1/2+\varepsilon'})\right] , \tag{83}$$

for all $\varepsilon' > \varepsilon$, where $\kappa_m = \frac{\pi^{m-2}}{4}\,\zeta(m+2)$ by (74). Therefore, by (75) and (83),

$$\mathrm{Var}\big(Z_{s_N},\varphi\big) = \frac{1}{N^m}\int_M \left[\kappa_m + O(N^{-1/2+\varepsilon'})\right]\psi(z)^2\Omega(z) . \tag{84}$$

Since

$$\int_M \psi(z)^2\Omega(z) = \int |\partial\bar\partial\varphi|^2\,\Omega = \|\partial\bar\partial\varphi\|_2^2 ,$$

(84) yields the variance formula of Theorem 1.3 □

6. Variance of zeros in a domain: Proof of Theorems 1.1–1.2

Following the approach of §5, we now prove Theorem 1.2 and, as a consequence, we also obtain Theorem 1.1, which is the one-dimensional case of Theorem 1.2.

By Proposition 3.5, we have

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2}[Z_{s_N}\cap U]\big) = \int_{\partial U} \Upsilon \wedge \Phi , \tag{85}$$

where $\Phi = \frac{1}{(m-1)!}\omega^{m-1}$ and

$$\Upsilon(z) = -\bar\partial_z \int_{\{z\}\times\partial U} \bar\partial_w Q_N(z,w)\wedge\Phi(w) . \tag{86}$$

From (27) and Proposition 4.2, we conclude that

$$\bar\partial_z\bar\partial_w Q_N(z,w) = O(N^{-m}) , \quad \text{for}\ \ d(z,w) > b\sqrt{\frac{\log N}{N}} , \tag{87}$$

where we choose $b=\sqrt{2m+3}$. Thus, we only need to integrate (86) over a small ball about z in ∂U when using (85)–(86) to compute the variance.

To evaluate $\Upsilon(z_0)$ at a fixed point $z_0 \in \partial U$, we choose normal holomorphic coordinates $\{z_j\}$ at z_0, defined in a neighborhood V 0f z_0. By (86)–(87), we have

$$\begin{aligned}\Upsilon(z_0) &= -\sum_{j,k}\left(\int_{\partial U}\frac{\partial^2}{\partial\bar z_j\partial\bar w_k}Q_N(z,w)\Big|_{z=z_0} d\bar w_k\wedge\Phi(w)\right)d\bar z_j\\ &= -\sum_{j,k}\left(\int_{\left\{z_0+w\in\partial U:|w|<b\sqrt{\frac{\log N}{N}}\right\}}\frac{\partial^2}{\partial\bar z_j\partial\bar w_k}Q_N(z_0+z,z_0+w)\Big|_{z=0} d\bar w_k\wedge\Phi(w)\right)d\bar z_j\\ &\qquad + O\left(\frac{1}{N^m}\right) \qquad (88)\end{aligned}$$

As in the proof of Proposition 1.3, we write $Q_N = F\circ\Lambda_N$, where $\Lambda_N = -\log P_N$ and F is given by (71). By Propositions 4.3–4.4,

$$\Lambda_N(z_0+z,z_0+w) = \frac{N}{2}|z-w|^2 + \widetilde R_N(z,w)\ , \qquad (89)$$

where $\widetilde R_N(z,w) = -\log\left[1+R_N\left(\sqrt{N}\,z,\sqrt{N}\,w\right)\right]$ satisfies the estimates

$$\begin{aligned}&|\widetilde R_N(z,w)| = O(|z-w|^2N^{1/2+\varepsilon}),\quad |\nabla\widetilde R_N(z,w)| = O(|z-w|N^{1/2+\varepsilon}),\\ &|\nabla^2\widetilde R_N(z,w)| = O(N^{1/2+\varepsilon}),\qquad \text{for } |z|+|w|<b\sqrt{\tfrac{\log N}{N}}.\end{aligned} \qquad (90)$$

By (71),

$$F'(\lambda) = \frac{1}{2\pi^2}\log(1-e^{-2\lambda}),\qquad F''(\lambda) = \frac{1}{\pi^2(e^{2\lambda}-1)}\ .$$

We now let $\lambda = \Lambda_N(z_0+z,z_0+w)$. By (89)–(90),

$$\begin{aligned}|F'(\lambda)| &\le \frac{1}{2\pi^2}\max\left(\log\frac{1}{\lambda},1\right) = O\left(1+\log^+\frac{1}{|z-w|}\right)\ ,\\ F''(\lambda) &\le \frac{1}{2\pi^2\lambda} = O\left(\frac{1}{N|z-w|^2}\right)\ ,\qquad \text{for } |z|+|w|<b\sqrt{\frac{\log N}{N}}\ . \end{aligned}\qquad (91)$$

Hence, for $|w|<b\sqrt{\frac{\log N}{N}}$, we have

$$\begin{aligned}\frac{\partial^2}{\partial\bar z_j\partial\bar w_k}Q_N(z_0+z,z_0+w)\Big|_{z=0} &= \left[F''(\lambda)\frac{\partial\lambda}{\partial\bar z_j}\frac{\partial\lambda}{\partial\bar w_k}+F'(\lambda)\frac{\partial^2\lambda}{\partial\bar z_j\partial\bar w_k}\right]\Big|_{z=0}\\ &= F''(\lambda)\left[-\frac12 Nw_j+O(|w|N^{1/2+\varepsilon})\right]\left[\frac12 Nw_k+O(|w|N^{1/2+\varepsilon})\right]\\ &\quad +F'(\lambda)\cdot O(N^{1/2+\varepsilon})\\ &= -\frac14 N^2F''(\lambda)w_jw_k+O(N^{1/2+\varepsilon})\left(1+\log^+\frac{1}{|w|}\right)\ .\end{aligned}$$

Furthermore, since

$$-F^{(3)}(t) = \frac12\operatorname{csch}^2 t\le t^{-2}\,, \qquad (92)$$

we have

$$F''(\lambda) = \frac{1}{\pi^2(e^{N|z-w|^2}-1)} + O(|z-w|^{-2}N^{-3/2+\varepsilon}) \,,$$

and hence

$$\begin{aligned}\frac{\partial^2}{\partial\bar z_j\partial\bar w_k}Q_N(z_0+z,z_0+w)\Big|_{z=0}\\ = \frac{-N^2}{4\pi^2(e^{N|w|^2}-1)}w_jw_k + O(N^{1/2+\varepsilon})\left(1+\log^+\frac{1}{|w|}\right) \,. \qquad (93)\end{aligned}$$

We note that under our hypothesis that ∂U is piecewise $\mathcal{C}^2$ without cusps, we have the estimate

$$\int_{\{z_0+w\in\partial U:|w|<\delta\}}\left(1+\log^+\frac{1}{|w|}\right)d\mathrm{Vol}(w) = O\left(\delta^{2m-1}|\log\delta|\right) \,, \quad \text{for } \delta\le\frac{1}{2}\,. \qquad (94)$$

Substituting (93)–(94) into (88), we obtain

$$\Upsilon(z_0) = \sum_{j,k}\left(\frac{N^2}{4\pi^2}\int_{\left\{z_0+w\in\partial U:|w|<b\sqrt{\frac{\log N}{N}}\right\}}\frac{w_jw_kd\bar w_k}{e^{N|w|^2}-1}\wedge\Phi(w) + O\left(N^{-m+1+2\varepsilon}\right)\right)d\bar z_j. \qquad (95)$$

We first consider the case where ∂U is $\mathcal{C}^2$ smooth (without corners). We can choose our normal coordinates $\{z_j\}$ about z_0 so that the real hyperplane $\{\mathrm{Im}\, z_1 = 0\}$ is tangent to ∂U at $z_0 = 0$. We can then write (after shrinking V if necessary),

$$U\cap V = \{z\in V : \mathrm{Im}\, z_1 + \varphi(z) > 0\} \,,$$

where $\varphi : V \to \mathbb{R}$ is a $\mathcal{C}^2$ function of $(\mathrm{Re} z_1,\, z_2,\dots,z_m)$ such that $\varphi(0)=0,\ d\varphi(0)=0$.

We let

$$\tau(v) = (v_1 + i\varphi(v), v_2,\dots,v_m) \,,$$

so that $\partial U = \{\mathrm{Im}\, v_1 = 0\}$ in terms of the (non-holomorphic) v coordinates. We make the change of variables

$$w = \tau_N(v) := \tau\left(\frac{v}{\sqrt N}\right)$$

in the integral (95):

$$\Upsilon(z_0) = \sum_{j,k}\left(\frac{N^2}{4\pi^2}\int_{B_N^{2m-1}}\tau_N^*\left[\frac{w_jw_kd\bar w_k}{e^{N|w|^2}-1}\wedge\Phi(w)\right] + O\left(N^{-m+1+2\varepsilon}\right)\right)d\bar z_j, \qquad (96)$$

where

$$\left\{v\in\mathbb{R}\times\mathbb{C}^{m-1} : |v| < (b-1)\sqrt{\log N}\right\}\subset B_N^{2m-1}\subset\left\{v\in\mathbb{R}\times\mathbb{C}^{m-1} : |v| < (b+1)\sqrt{\log N}\right\}.$$

To evaluate the integrand in (96), we first note that

$$w_1 = \frac{v_1}{\sqrt N} + O\left(\frac{|v|^2}{N}\right) \,, \quad d\bar w_1 = \frac{1}{\sqrt N}d\bar v_1 + O\left(\frac{|v|}{N}\right), \quad w_2 = \frac{v_2}{\sqrt N},\ \dots,\ w_m = \frac{v_m}{\sqrt N}\,.$$

Thus $N|w|^2 = |v|^2 + O\left(\frac{|v|^4}{N}\right)$, and hence by (92),

$$\frac{1}{e^{N|w|^2}-1} = \frac{1}{e^{|v|^2}-1} + O\left(\frac{|v|^4}{N}\right)O(|v|^{-2}) = \frac{1}{e^{|v|^2}-1} + O(N^{-1+\varepsilon}) \,,$$

for $|v| < 2b\sqrt{\log N}$. Finally, we have

$$\tau_N^*\Phi = \frac{1}{(m-1)!}\left[\frac{i}{2N}\partial\bar\partial|v|^2 + O\left(\frac{|v|}{N^{3/2}}\right)\right]^{m-1} = \frac{1}{N^{m-1}}\left[\Phi_E(v) + O\left(\sqrt{\frac{\log N}{N}}\right)\right] \tag{97}$$

on B_N^{2m-1}, where

$$\Phi_E(v) = \frac{1}{(m-1)!}\left(\frac{i}{2}\partial\bar\partial|v|^2\right)^{m-1} .$$

Therefore, (96) becomes

$$\Upsilon(z_0) = N^{-m+3/2}\sum_{j=1}^m\left[\frac{1}{4\pi^2}\int_{B_N^{2m-1}}\frac{v_j}{e^{|v|^2}-1}\,\bar\partial|v|^2\wedge\Phi_E(v) + O(N^{-1/2+2\varepsilon})\right]d\bar z_j\ . \tag{98}$$

We note that

$$\int_{B_N^{2m-1}}\frac{v_j}{e^{|v|^2}-1}\,\bar\partial|v|^2\wedge\Phi_E(v) = \int_{B_N^{2m-1}}\frac{v_jv_1}{e^{|v|^2}-1}\,d\mathrm{Vol}_{\mathbb{R}\times\mathbb{C}^{m-1}}(v)\ . \tag{99}$$

Since

$$\int_{x\in\mathbb{R}^n:|x|>b\sqrt{\log N}}\frac{|x|^2}{e^{|x|^2}-1}\,dx = O(N^{-b^2+1})\ ,$$

we can replace the integral in (99) with an affine integral, and hence

$$\begin{aligned}\Upsilon(z_0) &= N^{-m+3/2}\sum_{j=1}^m\left[\frac{1}{4\pi^2}\int_{\mathbb{R}\times\mathbb{C}^{m-1}}\frac{v_jv_1}{e^{|v|^2}-1}\,d\mathrm{Vol}_{\mathbb{R}\times\mathbb{C}^{m-1}}(v) + O(N^{-1/2+2\varepsilon})\right]d\bar z_j\\ &= N^{-m+3/2}\,\nu_m\,d\bar z_1 + O(N^{-m+1+2\varepsilon})\ ,\end{aligned} \tag{100}$$

where

$$\begin{aligned}\nu_m &= \frac{1}{4\pi^2}\int_{\mathbb{R}^{2m-1}}\frac{x_1^2}{e^{|x|^2}-1}\,dx \;=\; \frac{1}{4\pi^2(2m-1)}\int_{\mathbb{R}^{2m-1}}\frac{|x|^2}{e^{|x|^2}-1}\,dx\\ &= \frac{1}{4\pi^2(2m-1)}\,\frac{2\pi^{m-1/2}}{\Gamma(m-1/2)}\int_0^\infty\frac{r^{2m}}{e^{r^2}-1}\,dr\\ &= \frac{\pi^{m-5/2}}{4\,\Gamma(m+1/2)}\sum_{k=1}^\infty\int_0^\infty e^{-kr^2}\,r^{2m}\,dr\\ &= \frac{\pi^{m-5/2}}{4\,\Gamma(m+1/2)}\sum_{k=1}^\infty\frac{\Gamma(m+1/2)}{2\,k^{m+1/2}} \;=\; \frac{\pi^{m-5/2}}{8}\,\zeta\Big(m+\frac{1}{2}\Big)\ .\end{aligned}$$

Substituting (100) into (85), we obtain the formula of Theorem 1.2, which completes the proof for the case where ∂U is smooth.

We now consider the general case where ∂U is piecewise smooth (without cusps). Let S denote the set of singular points ('corners') of ∂U, and let S_N be the small neighborhood of S given by

$$S_N = \left\{z\in\partial U : \mathrm{dist}(z,S) < \frac{b'\sqrt{\log N}}{\sqrt N}\right\}\ ,$$

where $b' > 0$ is to be chosen below. We shall show that:

i) (100) holds uniformly for $z_0 \in \partial U \smallsetminus S_N$;
ii) $\sup\limits_{z\in\partial U\smallsetminus S} |\Upsilon(z)| = O\left(N^{-m+3/2+\varepsilon}\right)$.

Since $\mathrm{Vol}_{2m-1} S_N = O\left(\frac{\sqrt{\log N}}{\sqrt{N}}\right)$, the estimate (ii) implies that the integral in (85) over the small set S_N is negligible and hence

$$\mathrm{Var}\big(\mathrm{Vol}_{2m-2}[Z_{s_N}\cap U]\big) = \int_{\partial U\smallsetminus S_N} \Upsilon\wedge\Phi + O(N^{-m+1+2\varepsilon}) \,. \tag{101}$$

It then follows from (i) and (101) that

$$\begin{aligned}\mathrm{Var}\big(\mathrm{Vol}_{2m-2}[Z_{s_N}\cap U]\big) &= N^{-m+3/2}\left[\nu_m\,\mathrm{Vol}_{2m-1}(\partial U\smallsetminus S_N) + O(N^{-\frac12+2\varepsilon})\right] \\ &= N^{-m+3/2}\left[\nu_m\,\mathrm{Vol}_{2m-1}(\partial U) + O(N^{-\frac12+2\varepsilon})\right] \,,\end{aligned}$$

which is our desired formula.

It remains to prove (i)–(ii). To verify (ii), for each point $z_0 \in \partial U \smallsetminus S$, we choose holomorphic coordinates $\{z_j\}$ and non-holomorphic coordinates $\{v_j\}$ as above. We can choose these coordinates on a geodesic ball V_{z_0} about z_0 of a fixed radius $R > 0$ independent of the point z_0, but if z_0 is near a corner, ∂U will coincide with $\{\mathrm{Im}\, v_1 = 0\}$ only in a small neighborhood of z_0. To be precise, we let K_{z_0} denote the connected component of $V_{z_0}\cap\partial U\smallsetminus S$ containing z_0. Then we choose $\varphi\in\mathcal{C}^2(V_{z_0})$ with $\varphi(0)=0,\ d\varphi(0)=0$, such that

$$\{z\in V : \mathrm{Im}\, z_1 + \varphi(z) = 0\} \supset K_{z_0} \,. \tag{102}$$

Choose $N_0 > 0$ such that $b'\sqrt{\frac{\log N_0}{N_0}} < R$; then

$$\left\{w\in\partial U : d(z_0,w) < b'\sqrt{\tfrac{\log N}{N}}\right\} \subset V_{z_0}\,, \quad \text{for}\ \ N\ge N_0 \,.$$

Then for $N \ge N_0$, the integrals (96) and (98) hold, except now they are over a piecewise smooth hypersurface $\widetilde{B}_N$ of the $(b'\sqrt{\log N})$-ball in $\mathbb{C}^m$ instead of the linear hypersurface B_N^{2m-1}. Since

$$\left|\frac{v_j\,\bar\partial|v|^2}{e^{|v|^2}-1}\right| \le \frac{|v|^2}{e^{|v|^2}-1} \le 1 \,,$$

where the above norm is respect to the Euclidean metric on $T^*(\mathbb{C}^m_{\{v\}})$, it follows from (98) (with B_N replaced by $\widetilde{B}_N$) that

$$|\Upsilon(z_0)| \le \frac{m+o(1)}{4\pi^2}\, N^{-m+3/2}\,\mathrm{Vol}^E_{2m-1}(\widetilde{B}_N) \,, \tag{103}$$

where Vol^E denotes Euclidean volume. Since ∂U is piecewise smooth, we see that

$$\begin{aligned}\mathrm{Vol}^E_{2m-1}(\widetilde{B}_N) &\le N^{m-1/2}\,[1+o(1)]\,\mathrm{Vol}_{2m-1}\left\{w\in\partial U : d(z_0,w) < b'\sqrt{\tfrac{\log N}{N}}\right\} \\ &= N^{m-1/2}\, O\left(\left[\tfrac{\log N}{N}\right]^{m-1/2}\right) = O\left((\log N)^{m-1/2}\right) \,. \end{aligned}\tag{104}$$

Combining (103)–(104), we obtain the bound (ii).

To verify (i), we let

$$C = \sup_{z\in\partial U\smallsetminus S} \frac{\mathrm{dist}(z,S)}{\mathrm{dist}(z,\partial U\smallsetminus K_z)}\ .$$

We recall that our assumption that ∂U 'has no cusps' means that $\overline{U}$ is locally $\mathcal{C}^2$ diffeomorphic to a polyhedral cone, which implies that $C<+\infty$. We now let $b'=Cb$, where $b=\sqrt{2m+3}$ as before.

Consider any point $z_0\in\partial U\smallsetminus S_N$, $N\ge N_0$. Then

$$\mathrm{dist}(z_0,\partial U\smallsetminus K_{z_0}) \ge \frac{\mathrm{dist}(z,S)}{C} \ge \frac{b'\sqrt{\log N}}{C\sqrt{N}} = \frac{b\sqrt{\log N}}{\sqrt{N}}\ .$$

Thus by our far-off-diagonal decay estimate (87), the points in $\partial U\smallsetminus K_{z_0}$ contribute negligibly to the integral in (96), so that integral can be taken over $\tau_N^{-1}(K_{z_0})$, or over the linear $(b\sqrt{\log N})$-ball B_N. Then (100) follows as before.

Thus we have verified (i)–(ii), which completes the proof of Theorem 1.2 for the general case where ∂U has corners. □

7. Asymptotic normality: Proof of Theorem 1.4

The proof is a combination of Propositions 4.2 and 4.3 with a general result of Sodin-Tsirelson [ST] on asymptotic normality of nonlinear functionals of Gaussian processes. Following [ST], we define a *normalized complex Gaussian process* to be a random function $w(t)$ on a measure space (T,μ) of the form

$$w(t)=\sum c_j f_j(t)\ ,$$

where the c_j are i.i.d. complex Gaussian random variables (of mean 0, variance 1), and the f_j are (fixed) complex-valued measurable functions such that

$$\sum |f_j(t)|^2 = 1 \quad \text{for all} \quad t\in T.$$

We let $w_1,w_2,w_3,\dots$ be a sequence of normalized complex Gaussian processes on a finite measure space (T,μ). Let $f(r)\in\mathcal{L}^2(\mathbb{R}^+,e^{-r^2/2}rdr)$ and let $\psi:T\to\mathbb{R}$ be bounded measurable. We write

$$Z_N^\psi(w_N)=\int_T f(|w_N(t)|)\psi(t)d\mu(t).$$

Theorem **7.1.** [ST, Theorem 2.2] *Let $\rho_N(s,t)$ be the covariance functions for the Gaussian processes $w_N(t)$. Suppose that*

i) $\displaystyle\liminf_{N\to\infty}\frac{\int_T\int_T|\rho_N(s,t)|^{2\alpha}\psi(s)\psi(t)d\mu(s)d\mu(t)}{\sup_{s\in T}\int_T|\rho_N(s,t)|d\mu(t)}>0\ ,$
for $\alpha=1$ if f is monotonically increasing, or for all $\alpha\in\mathbb{Z}^+$ otherwise;

ii) $\displaystyle\lim_{N\to\infty}\sup_{s\in T}\int_T|\rho_N(s,t)|d\mu(t)=0.$

Then the distributions of the random variables

$$\frac{Z_N^\psi-\mathbf{E}Z_N^\psi}{\sqrt{\mathrm{Var}(Z_N^\psi)}}$$

converge weakly to $\mathcal{N}(0,1)$ *as* $N \to \infty$.

We apply this result in the case $f(r) = \log r$ and $(T, \mu) = (M, \Omega)$, with the normalized Gaussian processes

$$w_N(z) := \frac{s_N(z)}{\sqrt{\Pi_N(z,z)}},$$

where s_N is a random holomorphic section in $H^0(M, L^N)$ with respect to its Hermitian Gaussian measure. The covariance kernel of this Gaussian process is $P_N(z,w)$. Further, we let φ be a $\mathcal{C}^3$ real $(2m-2)$-form and we write $\partial\bar{\partial}\varphi = \psi\Omega$ as before (and hence $\psi \in \mathcal{C}^1$), so that

$$Z_N^{\psi}(w_N) = (Z_{s_N}, \varphi) = \int_M \log|s_N(z)| \partial\bar{\partial}\varphi(z)$$

is the smooth linear statistic of integration of the fixed test form φ over the random zero set. (This was the application of interest in [ST], where they considered random functions on $\mathbb{C}$, $\mathbb{CP}^1$, and the disk.)

By Proposition 2.1, we have

$$\mathbf{E}Z_N^{\psi} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log \Pi_N(z,z) + N\omega,$$

hence

$$Z_N^{\psi}(w_N) - \mathbf{E}Z_N^{\psi} = \int_M \log\frac{|s_N(z)|}{\sqrt{\Pi_N(z,z)}}\partial\bar{\partial}\varphi(z), \quad \varphi \in \mathcal{D}^{m-1,m-1}(M).$$

To apply the theorem it suffices to check that $P_N(z,w)$ satisfies conditions (i)–(ii). We start with (ii): by Proposition 4.2,

$$\lim_{N\to\infty} \sup_{z\in M} \int_{d(z,w)>\sqrt{\frac{b\log N}{N}}} |P_N(z,w)| dV_\omega(w) = 0.$$

On the other hand, since $|P_N(z,w)| \le 1$ it is obvious that the same limit holds for $d(z,w) \le \sqrt{\frac{b\log N}{N}}$.

To check (i), we again break up the integral into the near diagonal $d(z,w) \le \sqrt{\frac{b\log N}{N}}$ and the off-diagonal $d(z,w) > \sqrt{\frac{b\log N}{N}}$. As before, the integrals over the off-diagonal set tend to zero rapidly and can be ignored in both the numerator and denominator.

On the near diagonal, we replace P_N by its asymptotics in Proposition 4.3. The asymptotics are constant in z and with uniform remainders, so the condition becomes

$$\liminf_{N\to\infty} \frac{\int_M \int_{|u|<\sqrt{b\log N}} e^{-|u|^2}|1+R_N(u)|^2 \psi(z+\frac{u}{\sqrt{N}})\psi(z)\, du\, dV_\omega(z)}{\int_{|u|<\sqrt{b\log N}} e^{-\frac{1}{2}|u|^2}|1+R_N(u)|\, du} > 0.$$

Since $\psi \in \mathcal{C}^1$, the ratio clearly tends to $2^{-m}\int_M \psi(z)^2 dV_\omega > 0$, completing the proof. □

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Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail address: bshiffman@jhu.edu, szelditch@jhu.edu