COMPLEX ZEROS OF REAL ERGODIC EIGENFUNCTIONS

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Abstract. We determine the limit distribution (as $\lambda \to \infty$) of complex zeros for holomorphic continuations $\varphi_\lambda^{\mathbb{C}}$ to Grauert tubes of real eigenfunctions of the Laplacian on a real analytic compact Riemannian manifold (M, g) with ergodic geodesic flow. If $\{\varphi_{j_k}\}$ is an ergodic sequence of eigenfunctions, we prove the weak limit formula $\frac{1}{\lambda_j}[Z_{\varphi_{j_k}^{\mathbb{C}}}] \to \frac{i}{\pi}\overline{\partial}\partial|\xi|_g$, where $[Z_{\varphi_{j_k}^{\mathbb{C}}}]$ is the current of integration over the complex zeros and where $\overline{\partial}$ is with respect to the adapted complex structure of Lempert-Szöke and Guillemin-Stenzel.

1. Introduction

A well-known problem in the geometry of Laplace eigenfunctions is to determine the asymptotics of the volume of their nodal hypersurfaces in the limit of large eigenvalues. At the present time, the best result for compact real analytic Riemannian manifolds (M, g) of dimension m is the estimate

$$c_1\lambda \leq \mathcal{H}^{m-1}(Z_{\varphi_\lambda}) \leq C_2\lambda. \tag{1}$$

due to Donnelly-Fefferman [DF]. Further background and references can be found in [DF, JL, NPSo]. In this article we are concerned with the yet more difficult problem of the asymptotic distribution of nodal hypersurfaces, i.e. with integrals of continuous functions over nodal hypersurfaces (cf. (10)). This problem is much too difficult for real hypersurfaces, but it turns out to simplify quite a bit if we complexify the problem, i.e. analytically continue the eigenfunctions into the complexification of M. Our main results determine the asymptotic distribution of complex nodal hypersurfaces for eigenfunctions of real analytic Riemannian manifolds with ergodic geodesic flow, or more generally for any sequence of quantum ergodic eigenfunctions.

To state our results, we need some notation. Let (M, g) be a real analytic Riemannian manifold of dimension m, and consider an orthonormal basis of real eigenfunctions

$$\Delta_g\varphi_j = \lambda_j^2\varphi_j, \quad \langle\varphi_j, \varphi_k\rangle_{L^2(M,dvol_g)} = \int_M \varphi_j(x)\varphi_k(x)dvol_g(x) = \delta_{jk}$$

of its Laplacian Δ_g. We use the sign convention for which the Laplacian is positive; we often write Δ_g as Δ when the metric is understood. As reviewed in §2, a real analytic manifold possesses a Bruhat-Whitney complexification $M_{\mathbb{C}}$, that is, a complex manifold in which M embeds as a totally real submanifold. This complex manifold may be identified (by means of the complexified exponential map) with a ball bundle $B^*_{\epsilon_0}M$ inside the cotangent bundle T^*M, equipped with a complex structure J_g adapted to the metric in the sense of Guillemin-Stenzel [GS1, GS2] and Lempert-Szöke [LS1, LS2]. We denote by $|\xi|_g^2 = \sum_{ij} g^{ij}\xi_i\xi_j$

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the norm-squared of $\xi \in T_x^*M$ with respect to the metric, and by $\overline{\partial}$ the Cauchy-Riemann operator with respect to J_g. The maximal ball bundle $B^*_{\epsilon_0}M$ to which this complex structure extends is known as the Grauert tube of (M, g). The natural symplectic form ω and the complex structure J_g endow $B^*_{\epsilon_0}M$ with the Kähler metric $\omega_g = \frac{1}{i}\overline{\partial}\partial|\xi|^2_g$.

By a theorem due to Boutet de Monvel [Bou] (see also [GS2, GLS]) the eigenfunctions possess analytic continuations $\varphi_\lambda^{\mathbb{C}}$ to the maximal Grauert tube. The complex nodal hypersurface of an eigenfunction is defined by

$$Z_{\varphi_\lambda^{\mathbb{C}}} = \{\zeta \in B^*_{\epsilon_0}M : \varphi_\lambda^{\mathbb{C}}(\zeta) = 0\}. \tag{2}$$

There exists a natural current of integration over the nodal hypersurface in any ball bundle $B^*_\epsilon M$ with $\epsilon < \epsilon_0$, given by

$$\langle [Z_{\varphi_\lambda^{\mathbb{C}}}], \varphi\rangle = \frac{i}{2\pi}\int_{B^*_\epsilon M} \partial\bar{\partial}\log|\varphi_\lambda^{\mathbb{C}}|^2 \wedge \varphi = \int_{Z_{\varphi_\lambda^{\mathbb{C}}}} \varphi, \quad \varphi \in \mathcal{D}^{(m-1,m-1)}(B^*_\epsilon M). \tag{3}$$

In the second equality we used the Poincaré-Lelong formula. The notation $\mathcal{D}^{(m-1,m-1)}(B^*_\epsilon M)$ stands for smooth test $(m-1, m-1)$-forms with support in $B^*_\epsilon M$.

The nodal hypersurface $Z_{\varphi_\lambda^{\mathbb{C}}}$ also carries a natural volume form $|Z_{\varphi_\lambda^{\mathbb{C}}}|$ as a complex hypersurface in a Kähler manifold. By Wirtinger's formula, it equals the restriction of $\frac{\omega_g^{m-1}}{(m-1)!}$ to $Z_{\varphi_\lambda^{\mathbb{C}}}$. Hence, one can regard $Z_{\varphi_\lambda^{\mathbb{C}}}$ as defining the measure

$$\langle |Z_{\varphi_\lambda^{\mathbb{C}}}|, \varphi\rangle = \int_{Z_{\varphi_\lambda^{\mathbb{C}}}} \varphi\frac{\omega_g^{m-1}}{(m-1)!}, \quad \varphi \in C(B^*_\epsilon M). \tag{4}$$

We prefer to state results in terms of the current $[Z_{\varphi_\lambda^{\mathbb{C}}}]$ since it carries more information.

We will say that a sequence $\{\varphi_{j_k}\}$ of L^2-normalized eigenfunctions is *quantum ergodic* if

$$\langle A\varphi_{j_k}, \varphi_{j_k}\rangle \to \frac{1}{\mu(S^*M)}\int_{S^*M} \sigma_A d\mu, \quad \forall A \in \Psi^0(M). \tag{5}$$

Here, $\Psi^s(M)$ denotes the space of pseudodifferential operators of order s, and $d\mu$ denotes Liouville measure on the unit cosphere bundle S^*M of (M, g). More generally, we denote by $d\mu_r$ the (surface) Liouville measure on $\partial B^*_r M$, defined by

$$d\mu_r = \frac{\omega^m}{d|\xi|_g} \text{ on } \partial B^*_r M. \tag{6}$$

Our main result is:

THEOREM **1.1.** *Let (M, g) be real analytic, and let $\{\varphi_{j_k}\}$ denote a quantum ergodic sequence of eigenfunctions of its Laplacian Δ. Let $(B^*_{\epsilon_0}M, J)$ be the maximal Grauert tube around M with complex structure J_g adapted to g. Let $\epsilon < \epsilon_0$. Then:*

$$\frac{1}{\lambda_j}[Z_{\varphi_{j_k}^{\mathbb{C}}}] \to \frac{i}{\pi}\overline{\partial}\partial|\xi|_g = \frac{1}{2\pi|\xi|_g}\omega_g + \frac{d|\xi|_g^2 \wedge \alpha}{4\pi|\xi|_g^3}, \quad \textit{weakly in} \quad \mathcal{D}'^{(1,1)}(B^*_\epsilon M).$$

In other words, for any continuous test form $\psi \in \mathcal{D}'^{(m-1,m-1)}(B^*_\epsilon M)$, we have

$$\frac{1}{\lambda_j}\int_{Z_{\varphi_{j_k}^{\mathbb{C}}}} \psi \to \frac{i}{\pi}\int_{B^*_\epsilon M} \psi \wedge \overline{\partial}\partial|\xi|_g.$$

The limit $(1,1)$ form $\frac{i}{\pi}\bar{\partial}\partial|\xi|_g$ first arose in [LS1, GS1] for a different reason which is reviewed in §2.1. As a corollary we obtain a similar result on the integrals of scalar functions against the measures $|Z_{\varphi_{j_k}^{\mathbb{C}}}|$: for any $\varphi \in C(B^*_\epsilon M)$,

$$\frac{1}{\lambda_j}\int_{Z_{\varphi_{j_k}^{\mathbb{C}}}} \varphi \frac{\omega_g^{m-1}}{(m-1)!} \to \frac{i}{\pi}\int_{B^*_\epsilon M} \varphi \bar{\partial}\partial|\xi|_g \wedge \frac{\omega_g^{m-1}}{(m-1)!}.$$

As is well-known, ergodic sequences of density one in the spectrum arise when the geodesic flow is ergodic, and an entire orthonormal basis is ergodic when the Laplacian is quantum uniquely ergodic [CV, Shn, Z, Z2]. Thus, we obtain the titled result:

Corollary **1.2.** *Let (M,g) be a real analytic with ergodic geodesic flow. Let $\{\varphi_{j_k}\}$ denote a full density ergodic sequence. Then for all $\epsilon < \epsilon_0$,*

$$\frac{1}{\lambda_{j_k}}[Z_{\varphi_{j_k}^{\mathbb{C}}}] \to \frac{i}{\pi}\bar{\partial}\partial|\xi|_g, \quad \textit{weakly in } \mathcal{D}'^{(1,1)}(B^*_\epsilon M).$$

By the unique quantum ergodicity result of E. Lindenstrauss [Lin], the zero currents of the full sequence of Hecke eigenfunctions on arithmetic hyperbolic surfaces satisfy the limit formula in Corollary 1.2. The Rudnick-Sarnak conjecture [RS] that negatively curved compact manifolds are quantum uniquely ergodic would imply that the zero currents of the full sequence of eigenfunctions should satisfy the limit formula on such spaces.

Theorem 1.1 also has implications for a general Riemannian manifold (M,g). The orthonormal basis of a general Riemannian manifold is not quantum ergodic and moreover the complex zeros do not generally tend to the limit $\frac{i}{\pi}\bar{\partial}\partial|\xi|_g$ (a flat torus provides a simple example where zeros concentrate on complex hypersurface). However, in a precise sense, a *random orthonormal basis* of $L^2(M)$ (adapted to Δ_g) has the quantum ergodic property, and hence the complex zeros of the basis functions will satisfy the limit formula of Theorem 1.1.

To state the result, we recall the definition and results on these random orthonormal bases from [Z3]. We partition the spectrum of $\sqrt{\Delta_g}$ into the intervals $I_k = [k, k+1]$ and denote by $\Pi_k = E(k+1) - E(k)$ the spectral projections for $\sqrt{\Delta_g}$ corresponding to the interval I_k. We denote by $N(k)$ the number of eigenvalues in I_k and put $\mathcal{H}_k = \mathrm{ran}\Pi_k$ (the range of Π_k). $\mathcal{H}_k$ consists of linear combinations $\sum_{j:\lambda_j \in I_k} c_j\varphi_j$ of the eigenfunctions of $\sqrt{\Delta_g}$ with eigenvalues in I_k. We define a *random* orthonormal basis $\{U_k\varphi_j\}$ of $\mathcal{H}_k$ by changing the basis of $\sqrt{\Delta}$-eigenfunctions $\{\varphi_j\}$ of Δ in $\mathcal{H}_k$ by a random element U_k of the unitary group $U(\mathcal{H}_k)$ (equipped with its normalized Haar measure $d\nu_k$ of the finite dimensional Hilbert space $\mathcal{H}_k$.

We then define a *random orthonormal basis* of $L^2(M)$ (adapted to Δ_g) by taking the product over all the spectral intervals in our partition. That is, we define the infinite dimensional unitary group

$$U(\infty) = \Pi_{k=1}^\infty U(\mathcal{H}_k)$$

of sequences $(U_1, U_2, \dots)$, with $U_k \in U(\mathcal{H}_k)$, and equip $U(\infty)$ with the product measure

$$d\nu_\infty = \Pi_{k=1}^\infty d\nu_k.$$

A *random* orthonormal basis $\Psi = \{(U_k\varphi_j)\}$ of $L^2(M)$ is thus an orthonormal basis obtained by applying a random element $U \in U(\infty)$ to the orthonormal basis $\Phi = \{\varphi_j\}$ of eigenfunctions of $\sqrt{\Delta}$.

In [Z3], it is proved that random orthonormal bases satisfy the following variance asymptotics:

$$\mathbf{E}\left(\sum_{j:\lambda_j\in I_k}|(AU\varphi_j, U\varphi_j)-\omega(A)|^2\right)\sim(\omega(A^*A)-\omega(A)^2).$$

To be precise, in [Z3] it is assumed that the widths of the intervals I_k increase to infinity, but the more recent strong Szegö asymptotics of [GO, LRS] allow one to prove the same result for intervals of bounded width such as the I_k above. By the strong law of large numbers (see [Z3]) it follows that with probability one, a random orthonormal basis of $L^2(M)$ is quantum ergodic. We thus have:

Corollary **1.3.** *Let (M, g) be any real analytic compact Riemannian manifold. Then with probability one, a random orthonormal basis $\{\psi_j = U\varphi_j\}$ of $L^2(M)$ as defined above satisfies*

$$\frac{1}{\lambda_{j_k}}[Z_{\psi^{\mathbb{C}}_{j_k}}]\to\frac{i}{\pi}\overline{\partial}\partial|\xi|_g,\quad \textit{weakly in } \mathcal{D}'^{(1,1)}(B^*_\epsilon M),$$

for a full density subsequence $\{\psi_{j_k}\}$.

This gives a kind of almost sure improvement of the complexification of Theorem 14.3 of Jerison-Lebeau [JL] from an inequality to an asymptotic formula.

1.1. **Discussion and outline of the proof.** In summary, the complex zeros of a quantum ergodic sequence become equidistributed with respect to the $(1,1)$ form $\frac{i}{\pi}\overline{\partial}\partial|\xi|_g$. As mentioned above, the Kähler form on $B^*_{\epsilon_0}M$ associated to g is the (1,1)-form $\omega_g=\frac{1}{i}\overline{\partial}\partial|\xi|^2_g$. We observe that $\overline{\partial}\partial|\xi|_g$ is singular relative to the Kähler form along the zero section, i.e. the totally real submanifold M (the geometry will be reviewed in §2). This singular concentration could be attributed to the fact that the Laplacian is time reversal invariant (i.e. invariant under complex conjugation), so that the eigenfunctions are usually real-valued on the real submanifold M. Hence their complex zero sets are invariant under the (time reversal) involution $\sigma:(x,\xi)\to(x,-\xi)$ (the classical limit of complex conjugation).

As a simple example of Theorem 1.1, consider the circle S^1. The geodesic flow is ergodic modulo the symmetry $(x,\xi)\to(x,-\xi)$. The real eigenfunctions $\sin 2\pi kx, \cos 2\pi kx$ are therefore quantum ergodic. They complexify to the cylinder as $\sin 2\pi kz, \cos 2\pi kz$. The complex zero set of these holomorphic functions lies entirely on the set $\Im z = 0$ and become uniformly distributed with respect to $2d\theta$ as $k\to\infty$. It will be checked in §5 that the coefficient agrees with the result of Theorem 4.1. Note however that the complex eigenfunctions $e^{\pm 2\pi ikx}$ are not quantum ergodic and have no complex zeros (see §5 for further discussion).

We now outline the main steps of the proof. As mentioned above, eigenfunctions φ_λ of Laplcians of real analytic Riemannian manifolds admit holomorphic extensions $\varphi^{\mathbb{C}}_\lambda$ to a maximal Grauert tube in the complexification $M_{\mathbb{C}}$ of M, which we will identify with a maximal ball bundle $B^*_{\epsilon_0}M$ on which the adapted complex structure is defined. The square $|\xi|^2_g$ of metric norm function $|\xi|_g=\sqrt{\sum_{i,j=1}^m g^{ij}(x)\xi_i\xi_j}$ is a smooth, strictly plurisubharmonic exhaustion function on $B^*_{\epsilon_0}M$. For $0<\epsilon\le\epsilon_0$ the sphere bundles $S^*_\epsilon M=\partial B^*_\epsilon M$ are strictly pseudoconvex CR manifolds. We denote by $\mathcal{O}(B^*_\epsilon(M))$ the class of holomorphic functions on this domain, and by $\mathcal{O}(\partial B^*_\epsilon(M))$ the space of boundary values of holomorphic functions, i.e. the CR holomorphic functions. For each $0<\epsilon<\epsilon_0$, the restriction $\varphi^{\mathbb{C}}_\lambda|_{\partial B^*_\epsilon(M)}$ thus lies in the Hardy space $\mathcal{O}^0(\partial B^*_\epsilon(M))$ of square integrable CR functions.

A key object in the proof is the sequence of functions $U_\lambda(x,\xi) \in C^\infty(B^*_\epsilon M)$ defined by

$$(7) \qquad \begin{cases} U_\lambda(x,\xi) := \frac{\varphi_\lambda^{\mathbb{C}}(x,\xi)}{\rho_\lambda(x,\xi)}, \quad (x,\xi) \in B^*_\epsilon M, \quad \text{where} \\ \\ \rho_\lambda(x,\xi) := ||\varphi_\lambda^{\mathbb{C}}|_{\partial B_{|\xi|_g}}||_{L^2(\partial B^*_{|\xi|_g} M)} \end{cases}$$

Thus, $\rho_\lambda(x,\xi)$ is the 'moving' L^2-norm of $\varphi_\lambda^{\mathbb{C}}$ as it is restricted to the one-parameter family $\{\partial B^*_\epsilon M\}$ of strictly pseudo-convex CR manifolds. U_λ is of course not holomorphic, but its restriction to each sphere bundle is CR holomorphic there, i.e.

$$(8) \qquad u_\lambda^\epsilon = U_\lambda|_{\partial B^*_\epsilon M} \in \mathcal{O}^0(\partial B^*_\epsilon(M).$$

Our first result gives an ergodicity property of holomorphic continuations of ergodic eigenfunctions.

LEMMA **1.4.** *Assume that $\{\varphi_\lambda\}$ is a quantum ergodic sequence of Δ-eigenfunctions on M in the sense of (5). Then for each $0 < \epsilon < \epsilon_0$,*

$$|U_\lambda|^2 \to \frac{1}{\mu_1(S^*M)} |\xi|_g^{-m+1}, \quad \textit{weakly in } \; L^1(B^*_\epsilon M, \omega^m).$$

We note that $\omega^m = r^{m-1} dr d\omega dvol(x)$ in polar coordinates, so the right side indeed lies in L^1. The actual limit function is otherwise irrelevant. The next step is to use a compactness argument from [SZ] (see also [NV]) to obtain strong convergence of the normalized logarithms of the sequence $\{|U_\lambda|^2\}$. The first statement of the following lemma immediately implies the second.

LEMMA **1.5.** *Assume that $|U_\lambda|^2 \to \frac{1}{\mu_1(S^*M)} |\xi|_g^{-m+1}$, weakly in $L^1(B^*_\epsilon M, \omega^m)$. Then:*

1. $\frac{1}{\lambda_j} \log |U_j|^2 \to 0$ *strongly in* $L^1(B^*_\epsilon M)$.
2. $\frac{1}{\lambda_j} \partial\bar{\partial} \log |U_j|^2 \to 0$, *weakly in* $\mathcal{D}'(1,1)(B^*_\epsilon M)$.

Separating out the numerator and denominator of $|U_j|^2$, we obtain that

$$(9) \qquad \frac{1}{\lambda_j} \partial\bar{\partial} \log |\varphi_\lambda^{\mathbb{C}}|^2 - \frac{2}{\lambda_j} \partial\bar{\partial} \log \rho_{\lambda_j} \to 0, \quad (\lambda_j \to \infty).$$

The next lemma shows that the second term has a weak limit:

LEMMA **1.6.** *For $0 < \epsilon < \epsilon_0$,*

$$\frac{1}{\lambda} \log \rho_\lambda(x,\xi) \to \; |\xi|_{g_x}, \quad \textit{in} \quad L^1(B^*_\epsilon M) \;\; \textit{as} \;\; \lambda \to \infty.$$

Hence,

$$\frac{1}{\lambda_j} \partial\bar{\partial} \log \rho_{\lambda_j} \to \partial\bar{\partial} |\xi|_{g_x}, \quad (\lambda_j \to \infty) \;\; \textit{weakly in} \quad \mathcal{D}'(B^*_\epsilon M).$$

It follows that the left side of (9) has the same limit, and that will complete the proof of Theorem 1.1.

The proofs of the lemmas are based on the properties of the analytic continuation of the wave kernel as a complex Fourier integral associated to the complexified exponential map [Bou, GS2, GLS]. Since $\varphi_\lambda^{\mathbb{C}}|_{\partial B^*_{\sqrt{\rho}} M} = e^{-\lambda\sqrt{\rho}} E(i\sqrt{\rho}) \varphi_\lambda$, the analytic continuation of φ_λ is obtained by applying a complex Fourier integral operator of known order and symbol.

This allows us to connect growth and distribution of zeros in tubes to the dynamics of the geodesic flow. Although this paper treats only the ergodic case, it would be of interest to investigate complex zeros of Δ-eigenfunctions under other dynamical hypotheses such as complete integrability. It would also be interesting to investigate analogues for boundary value problems.

Let us compare the results of this paper to earlier results of B. Shiffman-S. Zelditch [SZ] and of S.Nonnemacher- A. Voros [NV] (see also [R]) on complex zeros of eigenfunctions of ergodic quantum maps in Kähler phase spaces. These articles were concerned with the complex zeros of eigenfunctions of ergodic quantum maps acting on spaces $H^0(M, L^N)$ of sections of powers of a holomorphic line bundle over a a Kähler manifold (M, ω). The role of the Grauert tube was played by the disc bundle $D^* \subset L^*$ of the dual line bundle, and the role of the CR manifolds $S^*_\epsilon M$ was played by the circle bundle $X = \partial D^*$, which is a strictly pseudoconvex CR submanifold of L^*. The norm function $|\xi|_g$ of (M, g) is thus analogous to the hermitian metric of (L, h), but the analogy is not very close. Indeed, the role of the geodesic role in the (M, g) setting was split in the Kähler setting between two dynamical systems: the S^1 action on the circle bundle $X \to M$ (which is the direct analogue of the geodesic flow in the Riemannian setting but is of course not ergodic); and an auxiliary ergodic symplectic transformation χ of the Kähler manifold M. Ergodicity of joint eigenfunctions $\{s_{N,j}\}$ of the S^1 action and of the quantum map associated to χ gave $||s_{N,j}(z)||^2_{h^N} \to 1$ in the weak sense (as in Theorem 1.4), where the norm is the pointwise norm relative to a hermitian metric on L with curvature equal to the Kähler form ω. Weak convergence of the zero currents as in Lemma 1.5 showed that $\frac{1}{N} Z_{s_{N,j}} \to \omega$, proving equidistribution of zeros relative to the Kähler form. In comparison, Theorem 1.1 shows that complex zeros of real ergodic eigenfunctions are not equidistributed relative to the analogous Kähler form ω_g in the Riemannian setting, but rather to the relatively singular form stated in the theorem. The difference can be traced to Lemma 1.6, which indicates that the moving L^2-norm ρ_λ ends up playing the key role of the hermitian metric.

1.2. **A conjecture on ergodic real nodal hypersurfaces.** We close the introduction by stating a conjecture on the real nodal hypersurfaces $Z_{\varphi_j} = \{x \in M : \varphi_j(x) = 0\}$ on Riemannian manifolds with ergodic geodesic flow. We define the distribution of real zeros of φ_λ by integration

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{m-1}, \tag{10}$$

with respect to $(m-1)$-dimensional Haussdorf measure $d\mathcal{H}^{m-1}$ on the nodal hypersurface induced by the Riemannian metric of (M, g).

CONJECTURE **1.7.** *Let (M, g) be a real analytic Riemannian manifold with ergodic geodesic flow, and let $\{\varphi_j\}$ be the density one sequence of ergodic eigenfunctions. Then,*

$$\langle [Z_{\varphi_j}], f \rangle \sim \{ \int_M f dVol_g \} \lambda.$$

More generally, we conjecture the same limit result for any quantum ergodic sequence of eigenfunctions. In the case of random spherical harmonics, the limit formula was proved in the PhD thesis of J. Neuheisel [Ne]. We believe that (in a straightforward way) it can be extended to random orthonormal bases on any compact Riemannian manifold as defined above, which would give the asymptotic strengthening of [JL] in the real domain.

There is of course a very wide gap between the equidistribution Conjecture 1.7 and the best known result on volumes (1). Due to the singular concentration along the real zero set, it seems possible that Corollary 1.4 could have implications for the distribution of real zeros in the ergodic case.

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2. Analytic continuation to a Grauert tube

In this section, we recall the relevant known results on analytic continuation of the wave kernel and Δ- eigenfunctions of a real analytic (M, g) to a Grauert tube.

A real analytic manifold M always possesses a complexification $M_{\mathbb{C}}$, i.e. a complex manifold of which M is a totally real submanifold (Bruhat-Whitney [BW]). The germ of $M_{\mathbb{C}}$ along M is unique. In [G], Grauert constructed plurisubharmonic exhaustion functions ρ on $M_{\mathbb{C}}$, which define the Grauert tubes $M_\epsilon = \{\rho < \epsilon\}$ relative to ρ.

In [GS1, GS2, LS1, LS2, GLS], the complex geometry of Grauert tubes was brought into contact with the symplectic geometry of the cotangent bundle T^*M and with the Riemannian geometry of real analytic metrics g. The key results (for the purposes of this article) are the following: First, a metric g determines a canonical plurisubharmonic function ρ_g on $M_{\mathbb{C}}$. It is defined on a maximal Grauert tube whose radius ϵ_0 is often called the 'radius of the Grauert tube'. There is a symplectic diffeomorphism

$$\psi : M_\epsilon \to B^*_\epsilon M$$

of the Grauert tubes with respect to ρ_g to the ball bundles $B^*_\epsilon M \subset T^*M$ with respect to g, which identifies ρ_g with $|\xi|^2_g$. As pointed out in [LS1] (see also [GLS]), one may take ψ^{-1} to be the complexified exponential map

$$(x, \xi) \in B^*M \to \exp_x \sqrt{-1}\xi \in M_\epsilon.$$

The map ψ endows $B^*_\epsilon M$ with a complex structure J_g *adapted* to g.

Proposition **2.1.** [LS1, GLS] *The adapted complex structure is uniquely characterized by the property that the complexified exponential map,*

$$(x, \xi) \in B^*_\epsilon M \to \exp_x \sqrt{-1}\xi \in M_\epsilon$$

is a biholmorphism for $0 < \epsilon < \epsilon_0$.

The domains $B^*_\epsilon M$ are strictly pseudoconvex for $\epsilon < \epsilon_0$, hence their boundaries $S^*_\epsilon M = \partial B^*_\epsilon M$ are strictly pseudoconvex CR manifolds. We use the notation $\partial B^*_\epsilon M$ to emphasize the role of sphere bundles as boundaries of domains. The restrictions

$$\psi_\epsilon = \exp(i\epsilon)^{-1} : \partial M_\epsilon \to \partial B^*_\epsilon$$

of ψ are then CR holomorphic diffeomorphisms.

As mentioned above, the metric norm function $|\xi|_g$ pulls back under ψ_ϵ to the function $\sqrt{\rho}$ on $M_{\mathbb{C}}$, which is known as the Monge-Ampére function [GS1]. It equals $r_{\mathbb{C}}(z, \bar{z})$ where $r_{\mathbb{C}}$ is the holomorphic extension of the distance function. In the cotangent picture, the metric norm function $|\xi|_g$ is smooth on $B^*_{\epsilon_0} M \backslash M$ and solves the homogeneous complex Monge-Ampère equation $(\partial\bar{\partial}|\xi|_g)^m = 0$ there. In fact, the form $\partial\bar{\partial}|\xi|_g$ has rank $m-1$ on $B^*_{\epsilon_0} M \backslash M$, and its kernel is a smooth rank 1 sub-bundle of $T(B^*_{\epsilon_0} M \backslash M)$. The leaves of the associated ('Monge-Ampère' or Riemann) foliation are the complex curves $t + i\tau \to \tau\dot{\gamma}(t)$, where γ is a geodesic, where $\tau > 0$ and where $\tau\dot{\gamma}(t)$ denotes multiplication of the tangent vector to γ by τ. We refer to [LS1] for further discussion.

2.1. **Model examples.** To better understand complexifications and Monge-Ampére functions, and in particular the limit form in Theorem 1.1, we go over several (well-known) model examples. We note that these examples do not have ergodic geodesic flow, so they do not exemplify Theorem 1.1, but only the objects involved in it.

(i) Complex tori:

The complexification of the torus $M = \mathbb{R}^m/\mathbb{Z}^m$ is $M_{\mathbb{C}} = \mathbb{C}^m/\mathbb{Z}^m$. The adapted complex structure to the flat metric on M is the standard (unique) complex structure on $\mathbb{C}^m$. The complexified exponential map is $\exp_{\mathbb{C}x}(i\xi) = z := x + i\xi$, while the distance function $r(x,y) = |x-y|$ extends to $r_{\mathbb{C}}(z,w) = \sqrt{(z-w)^2}$. Then $\sqrt{\rho}(z,\bar{z}) = \sqrt{(z-\bar{z})^2} = \pm 2i|\Im z| = \pm 2i|\xi|$. Thus, the limit form is $\frac{i}{\pi}\partial\bar{\partial}|\Im z|$.

(ii) $\mathbb{S}^n$ [PW, GS1] The unit sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$ is complexified as the complex quadric

$$S^2_{\mathbb{C}} = \{(z_1, \dots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1\}.$$

If we write $z_j = x_j + i\xi_j$, the equations become $|x|^2 - |\xi|^1 = 1, \langle x, \xi\rangle = 0$. The geodesic flow $G^t(x,\xi) = (\cos tx + \sin t\xi, -\sin tx + \cos t\xi)$ induces the exponential map $exp_x \xi = (\cos|\xi|)x + (\sin|\xi|)\xi$, which complexifies to

$$\exp_{\mathbb{C},x}\sqrt{-1}\xi = (\cosh|\xi|)x + \sqrt{-1}(\sinh|\xi|)\frac{\xi}{|\xi|}.$$

The distance function of $\mathbb{S}^n$ of constant curvature 1 is given by:

$$r(x,y) = 2\sin^{-1}\frac{|x-y|}{2} = 2\sin^{-1}(\frac{1}{2}\sqrt{(x-y)^2}),$$

whose analytic continuation to $\mathbb{S}^n_{\mathbb{C}} \times \mathbb{S}^n_{\mathbb{C}}$ is the doubly-branched holomorphic function:

$$r_{\mathbb{C}}(z,w) = 2\sin^{-1}\frac{1}{2}\sqrt{(z-w)^2}.$$

One branch gives the pluri-subharmonic function

$$\sqrt{\rho}(z) = r_{\mathbb{C}}(z,\bar{z}) = 2\sin^{-1} i|\Im z|) = 2i\sinh^{-1}|\Im z|) = i\cosh^{-1}|z|^2, \quad (z \in \mathbb{S}^n_{\mathbb{C}}).$$

Since

$$\begin{array}{rcl} \exp^*_{\mathbb{C}}\sqrt{\rho}(x,\xi) & = & \cosh^{-1}|(\cosh|\xi|)x + i(\sinh|\xi|)\frac{\xi}{|\xi|}|^2 \\ & = & \cosh^{-1}\{(\cosh|\xi|)^2 - (\sinh|\xi|)^2\} \\ & = & \cosh^{-1}\cosh 2|\xi| = 2|\xi|, \end{array}$$

the limit form is $\frac{i}{2\pi}\partial\bar{\partial}\cosh^{-1}|z|^2$ on $\mathbb{S}^n_{\mathbb{C}}$.

(iii) (See e.g. [KM]**).** $\mathbb{H}^n$ The hyperboloid model of hyperbolic space is the hypersurface in $\mathbb{R}^{n+1}$ defined by

$$\mathbb{H}^n = \{x_1^2 + \cdots x_n^2 - x_{n+1}^2 = -1, \;\; x_n > 0\}.$$

Then,

$$H^n_{\mathbb{C}} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \cdots z_n^2 - z_{n+1}^2 = -1\}.$$

In real coordinates $z_j = x_j + i\xi_j$, this is:

$$\langle x, x\rangle_L - \langle \xi, \xi\rangle_L = -1, \;\; \langle x, \xi\rangle_L = 0$$

where $\langle , \rangle_L$ is the Lorentz inner product of signature $(n, 1)$. The complexified exponential map is given by

$$\exp_{\mathbb{C}x}(\sqrt{-1}\xi) = \cos(\frac{||\xi||_L}{\sqrt{2}})x + \sqrt{-1}(\frac{\sin\frac{||\xi||_L}{\sqrt{2}}}{||\xi||_L})\xi.$$

Let

$$M_\epsilon = \{z \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_n^2 - z_{n+1}^2 = -1, \;\; |z_1|^2 + \cdots + |z_n|^2 - |z_n|^2 < \epsilon\}.$$

We note that M_ϵ has two components according to the sign of $\Re z_{n+1}$. The Monge-Ampere function is:

$$\sqrt{\rho}(z) = \cos^{-1}(||x||_L^2 + ||\xi||_L^2 - \pi)/\sqrt{2}.$$

The radius of maximal Grauert tube is $\epsilon = 1$ or $r = \pi/\sqrt{2}$. Hence the limit form is $\frac{i}{\pi}\partial\bar{\partial}\cos^{-1}(||x||_L^2 + ||\xi||_L^2 - \pi)/\sqrt{2})$ on M_1.

2.2. **Analytic Continuation of the wave kernel.** By the wave kernel of (M, g) we mean the kernel

$$E(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j}\varphi_j(x)\varphi_j(y)$$

of $e^{it\sqrt{\Delta}}$. As discussed in [Bou, GS2, GLS], the wave kernel at imaginary times admits a holomorphic extension to $M_\epsilon \times M$ as

$$E(i\epsilon, \zeta, y) = \sum_{j=0}^{\infty} e^{-\epsilon\lambda_j}\varphi_{Cj}(\zeta)\varphi_j(y), \quad (\zeta, y) \in M_\epsilon \times M. \tag{11}$$

In the simplest (albeit non-compact) case of $\mathbb{R}^n$, the wave kernel $E(t, x, y) = \int_{\mathbb{R}^n} e^{it|\xi|}e^{i\langle\xi, x-y\rangle}d\xi$ analytically continues to $t + i\tau, \zeta = x + ip \in \mathbb{C}_+ \times \mathbb{C}^n$ as the integral

$$E(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|}e^{i\langle\xi, x+ip-y\rangle}d\xi,$$

which converges absolutely for $|p| < \tau$. At positive imaginary times and for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $E(i\tau, x, y)$ is the Poisson kernel of the upper half space $\mathbb{R}^n_x \times \mathbb{R}^+_\tau$,

$$K(\tau, x, y) = \tau^{-n}\left(1 + (\frac{x-y}{\tau})^2\right)^{-\frac{n+1}{2}} = \tau\left(\tau^2 + (x-y)^2)\right)^{-\frac{n+1}{2}},$$

which visibly has a holomorphic continuation to $\zeta = x + ip$ in the x variable for $|p| < \tau$.

On a general analytic Riemannian manifold, one has a similar integral formula for the wave kernel of the form

$$E(t,x,y) = \int_{T_y^*M} e^{it|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(x)\rangle} A(t,x,y,\xi) d\xi \tag{12}$$

where $|\xi|_{g_x}$ is the metric norm function at x, and where $A(t,x,y,\xi)$ is a polyhomogeneous amplitude of order 0. The analytic continuation of the wave group at imaginary times is the Poisson operator $e^{-\tau\sqrt{\Delta}}$, a Fourier integral operator with complex phase; for background on the microlocal analysis of such kernels, we refer to [T] (Chapter XI). The analytic continuation,

$$E(i\tau,\zeta,y) = \int_{T_y^*} e^{-\tau|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(\zeta)\rangle} A(t,\zeta,y,\xi) d\xi \quad (\zeta = x + ip). \tag{13}$$

of the Poisson kernel to the Grauert tube $|\zeta| < \tau$ defines a complex Fourier integral operator from $L^2(M)$ with values in holomorphic functions in M_τ where $\tau < \epsilon_0$. Its canonical relation is the complexification of the canonical relation of the real wave group, i.e. 'graph' of the complexified geodesic flow at imaginary times

$$\Gamma_{i\tau} = \{(x,\xi,z,\zeta) \in T^*M\backslash 0 \times T^*M_\epsilon\backslash 0 : G^{i\tau}(x,\xi) = (z,\zeta).\} \tag{14}$$

We may (and will) also express the adjoint operator in the modified form

$$E^*(i\tau,x,\zeta) = \int_{\mathbb{R}^n} e^{-\tau|\xi|_{g_x}} e^{i\langle \xi, \exp_x^{-1}(\zeta)\rangle} A^*(t,\zeta,x,\xi) d\xi \quad (\zeta = x + ip). \tag{15}$$

Here, we use the phase $-\tau|\xi|_{g_x} + i\langle \xi, \exp_x^{-1}(\bar{\zeta})\rangle$ instead of $-\tau|\xi|_{g_x} - i\langle \xi, \exp_{\bar{\zeta}}^{-1}(x)\rangle$.

We can transport the complexified wave kernel to $B_\epsilon^*M \times M$ using the complexified exponential map (or ψ):

$$\tilde{E}(i\epsilon,(x,\xi),y) := E(i\epsilon, \exp_x^{\mathbb{C}} \sqrt{-1}\xi, y). \tag{16}$$

It is again a complex Fourier integral operator whose canonical relation $\tilde{\Gamma}_{i\tau}$ can be obtained from (14) by composing with ψ.

2.3. **Szego projector.** We denote by $\mathcal{O}^{s+\frac{n-1}{4}}(\partial B_\epsilon^*M)$ the subspace of the Sobolev space $W^{s+\frac{n-1}{4}}(\partial B_\epsilon^*M)$ consisting of CR holomorphic functions, i.e.

$$\mathcal{O}^{s+\frac{m-1}{4}}(\partial B_\epsilon^*M) = W^{s+\frac{m-1}{4}}(\partial B_\epsilon^*M) \cap \mathcal{O}(\partial B_\epsilon^*M).$$

The inner product on $\mathcal{O}^0(\partial B_\epsilon^*M)$ is with respect to the Liouville measure $d\mu_\epsilon$. There are similar spaces for ∂M_ϵ and composition with ψ_ϵ defines an isomorphism

$$\psi_\epsilon^* : \mathcal{O}^{s+\frac{m-1}{4}}(\partial B_\epsilon^*M) \to \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\epsilon).$$

We further denote by

$$\tilde{\Pi}_\epsilon : L^2(\partial B_\epsilon^*M) \to \mathcal{O}^0(\partial B_\epsilon^*M)$$

the Szegö projector for the tube B_ϵ^*M, i.e. the orthogonal projection onto boundary values of holomorphic functions in the tube. It is well-known (cf. [BoSj, MS, GS2]) that $\tilde{\Pi}_\epsilon$ is a complex Fourier integral operator, whose real canonical relation is the graph Δ_Σ of the identity map on the symplectic cone $\Sigma_\epsilon \subset T^*(\partial B^*_\epsilon M)$ spanned by the contact form $\alpha = \xi \cdot dx$, i.e.

$$\Sigma_\epsilon = \{(x, \xi; r\alpha_\epsilon), \quad (x, \xi) \in \partial B^*_\epsilon M), \ r > 0\} \subset T^*(\partial B^*_\epsilon M).$$

Alternatively it is a Toeplitz operator in the sense of Boutet de Monvel-Guillemin [BG]. The analogous Szegö projector $\Pi_\epsilon : L^2(\partial M_\epsilon) \to \mathcal{O}^0(\partial M_\epsilon)$ is conjugate to $\tilde{\Pi}_\epsilon$ in the sense that $\tilde{\Pi}_\epsilon = (\psi^*_\epsilon)^{-1} \Pi_\epsilon \psi^*_\epsilon$.

We now consider the restrictions $\tilde{\Pi}_\epsilon \circ \tilde{E}(i\epsilon)$ from $L^2(M)$ to $\mathcal{O}(\partial B^*_\epsilon M)$. Since Σ_ϵ is an $\mathbb{R}_+$-bundle over $\partial B^*_\epsilon M$, we can define the symplectic equivalence of cones:

$$\iota_\epsilon : T^*M \to \Sigma_\epsilon, \ \ \iota(x, \xi) = (x, \epsilon\xi, |\xi| \alpha_{(x, \epsilon\xi)}). \tag{17}$$

The following result is the transport under ψ_ϵ to $\partial B^*_\epsilon M$ of results due to Boutet de Monvel (see also [GS2]).

Theorem **2.2.** [Bou, GS2] $\tilde{\Pi}_\epsilon \circ \tilde{E}(i\epsilon) : L^2(M) \to \mathcal{O}(\partial B^*_\epsilon M)$ *is a complex Fourier integral operator of order* $-\frac{m-1}{4}$ *associated to the canonical relation*

$$\Gamma = \{(y, \eta, \iota_\epsilon(y, \eta)\} \subset T^*M \times \Sigma_\epsilon.$$

Moreover, for any s*,*

$$\tilde{\Pi}_\epsilon \circ \tilde{E}(i\epsilon) : W^s(M) \to \mathcal{O}^{s + \frac{m-1}{4}}(\partial B^*_\epsilon M)$$

is a continuous isomorphism.

2.4. **Analytic continuation of eigenfunctions.** We obtain the holomorphic extension of the eigenfunctions φ_λ by applying the complex Fourier integral operator $E(i\tau)$:

$$E(i\tau)\varphi_\lambda = e^{-\tau\lambda} \varphi^{\mathbb{C}}_\lambda. \tag{18}$$

As usual, we can use the complexified exponential map to transport the $\varphi^{\mathbb{C}}_\lambda$ to $B^*_{\epsilon_0}(M)$:

$$\tilde{\varphi}^{\mathbb{C}}_\lambda(x, \xi) = \varphi^{\mathbb{C}}_\lambda(\exp_x \sqrt{-1}\xi). \tag{19}$$

By Theorem 2.2,we obtain:

Corollary **2.3.** [Bou, GLS] *Each eigenfunction* φ_λ *has a holomorphic extension to* $B^*_\epsilon M$ *satisfying*

$$\sup_{(x,\xi) \in B^*_\epsilon M} |\tilde{\varphi}^{\mathbb{C}}_\lambda(x, \xi)| \leq C_\epsilon \lambda^{m+1} e^{\epsilon\lambda}.$$

The fact that the holomorphic continuations of eigenfunctions can be obtained by applying a complex Fourier integral operator is the crucial link connecting the geodesic flow and the growth rate and zeros of $\varphi^{\mathbb{C}}_\lambda$.

2.5. **Examples.** We pause to consider some basic examples of holomorphic continuations of eigenfunctions. The simplest example is the flat torus $\mathbb{R}^m/\mathbb{Z}^m$, where the real eigenfunctions are $\cos\langle k, x\rangle, \sin\langle k, x\rangle$ with $k \in 2\pi\mathbb{Z}^m$. The complexified torus is $\mathbb{C}^m/\mathbb{Z}^m$ and the complexified eigenfunctions are $\cos\langle k, \zeta\rangle, \sin\langle k, \zeta\rangle$ with $\zeta = x + i\xi$.

No such explicit examples exist in the ergodic case, but one can see the uniform analytic continuation of eigenfunctions very clearly in the case of compact hyberbolic quotients $\mathbf{H}^m/\Gamma$.

For simplicity we consider the two-dimensional case. Eigenfunctions can be then represented by Helgason's generalized Poisson integral formula [H],

$$\varphi_\lambda(z) = \int_B e^{(i\lambda+1)\langle z,b\rangle} dT_\lambda(b).$$

Here, $z \in D$ (the unit disc), $B = \partial D$, and $dT_\lambda \in \mathcal{D}'(B)$ is the boundary value of φ_λ, taken in a weak sense along circles centered at the origin 0. Also, $\langle z, b\rangle$ is the (signed) hyperbolic distance of the horocycle passing through z and b to 0. To analytically continue φ_λ it suffices to analytically continue $\langle z, b\rangle$. Writing the latter as $\langle \zeta, b\rangle$, we have:

$$\varphi_\lambda^{\mathbb{C}}(\zeta) = \int_B e^{(i\lambda+1)\langle \zeta,b\rangle} dT_\lambda(b).$$

Using this representation, one could verify directly the estimates on growth of complexified eigenfunctions.

3. The operators $\tilde{E}(i\epsilon)^*\Pi_\epsilon a \Pi_\epsilon \tilde{E}(i\epsilon)$

The following lemma will be used to reduce ergodicity properties and norm estimates of the complexified eigenfunctions $\varphi_\lambda^{\mathbb{C}}$ to properties of the initial real eigenfunctions φ_λ. We recall that the $\tilde{E}, \tilde{\Pi}$ notation refers to the $B^*_\epsilon M$ setting while E, Π refers to the M_ϵ setting.

Lemma **3.1.** *Let $a \in S^0(T^*M - 0)$. Then for all $0 < \epsilon < \epsilon_0$, we have:*

$$\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon a \tilde{\Pi}_\epsilon \tilde{E}(i\epsilon) \in \Psi^{-\frac{m-1}{2}}(M),$$

with principal symbol equal to $a(x,\xi)\,|\xi|_g^{-(\frac{m-1}{2})}$.

Proof. We observe that $\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon a \tilde{\Pi}_\epsilon \tilde{E}(i\epsilon)$ is a complex Fourier integral operator on $L^2(M)$ associated to the canonical relation,

$$\Gamma^* \circ \Delta_{\Sigma_\epsilon} \circ \Gamma = \{(y,\eta,x,\xi)\} \subset T^*M \times T^*M : \iota_\epsilon(x,\xi) = \iota_\epsilon(y,\eta)\}.$$

Since ι_ϵ is a conic symplectic isomorphism, it follows that $\Gamma^* \circ \Delta_\epsilon \circ \Gamma = \Delta_{T^*M \times T^*M}$, i.e. that $\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon a \tilde{\Pi}_\epsilon \tilde{E}(i\epsilon)$ is a pseudodifferential operator. It follows from Theorem 2.2 that

$$\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon a \tilde{\Pi}_\epsilon \tilde{E}(i\epsilon) : W^s(M) \to W^{s+\frac{m-1}{2}}(M),$$

is a continuous linear map and hence that the order is $-\frac{m-1}{2}$.

The principal symbol of $\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon a \tilde{\Pi}_\epsilon \tilde{E}(i\epsilon)$ equals $a(x,\xi)$ times the principal symbol of $\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon \tilde{E}(i\epsilon)$. Indeed, by the calculus and Egorov theorem for complex Fourier integral operators, it equals the the principal symbol of $\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon \tilde{E}(i\epsilon)$ times the translate of a under the canonical relation underlying $\tilde{\Pi}_\epsilon \tilde{E}(it)$. As discussed above, this relation is the symplectic identification of $T^*M - 0 \equiv \Sigma_\epsilon$.

Thus, it suffices to show that the principal symbol of $\tilde{E}(i\epsilon)^*\tilde{\Pi}_\epsilon \tilde{E}(i\epsilon)$ equals $|\xi|_g^{-\frac{m-1}{2}}$. Since it equals the principal symbol of $E(it)^*\Pi_\epsilon E(it)$ transported under ψ, we can (and will) do the computation in the M_ϵ setting. The calculation could be done using the calculus of complex Fourier integral operators, but that would require a digression on symbols of Szegö projectors and on the composition of the symbols of the three factors. It seems quicker to calculate the symbol from scratch by applying stationary phase for complex phase functions with only real critical points.

The calculation is well illustrated by the simplest case of the Euclidean wave kernel on $\mathbb{R}^m$. We then have:

$$E(i\tau)^*\Pi_\tau E(i\tau)(x,y) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m\times S^{m-1}} e^{-i(-i\tau)|\xi_1|}e^{-i\langle \xi_1, x-(x_1-ip)\rangle}e^{i(i\tau)|\xi_2|}e^{i\langle\xi_2, x_1+ip-y\rangle}d\xi_1 d\xi_2 dp dx_1.$$

This is a complex Fourier integral operator with phase

$$\Psi_0(x,y,\xi,\zeta,\tau) = -\tau(|\xi_1|+|\xi_2|) + i\langle \xi_2, x_1+ip-y\rangle - i\langle\xi_1, x-(x-ip)\rangle. \tag{20}$$

The dx_1 integral produces $\delta(\xi_1+\xi_2)$, and then the $d\xi_1$ integral gives

$$\begin{array}{lll} E(i\tau)^*\Pi_\tau E(i\tau)(x,y) & = & \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{S^{m-1}} e^{-i(t-i\tau)|\xi|}e^{-i\langle\xi, x-(ip)\rangle}e^{i(t+i\tau)|\xi|}e^{i\langle\xi,\ ip-y\rangle}d\xi dp \\ & = & \int_{\mathbb{R}^m}\int_{S^{m-1}} e^{-2\tau|\xi|}e^{-2\langle\xi,p\rangle}e^{-i\langle\xi,y\rangle}d\xi dp \\ & \sim & 2\tau^{\frac{m-1}{2}}\int_{\mathbb{R}^m}|\xi|^{-\frac{m-1}{2}}\sinh 2\tau|\xi|e^{-2\tau|\xi|}e^{-i\langle\xi,x-y\rangle}d\xi \end{array}$$

In the last step we used the asymptotics of the inner integral

$$\begin{array}{lll} \int_{\tau S^{m-1}} e^{-2\langle\xi,p\rangle}d\mu(p) & = & \tau^{m-1}\int_{S^{m-1}} e^{-2\tau|\xi|\langle\omega,e_1\rangle}d\mu(\omega) \\ & \sim & 2\tau^{\frac{m-1}{2}}|\xi|^{-\frac{m-1}{2}}\sinh 2\tau|\xi|e^{-2\tau|\xi|}. \end{array} \tag{21}$$

Thus, $E(i\tau)^*\Pi_\tau E(i\tau)$ is a Fourier multiplier by a polyhomogeneous function with leading term $|\xi|^{-\frac{m-1}{2}}$.

We now show that the same result holds on a general compact analytic Riemannian manifold by reducing to the Euclidean case. Using the analytic continuation of the parametrix (13), we have

$$\begin{array}{lll} E(i\tau)^*\Pi_\tau E(i\tau)(x,y) & = & \int_{T_x^*M}\int_{T_y^*M}\int_{S^*M} e^{-\tau(|\xi_1|_{g,x}+|\xi_2|_{g,y})}e^{-i\langle\xi_1,\overline{\exp_x^{-1}(\zeta)}\rangle}e^{i\langle\xi_2,\exp_y^{-1}(\zeta)\rangle} \\ & \times & A^*(t,\zeta,x,\xi_1)A(t,\zeta,y,\xi_2)d\xi_1 d\xi_2 dV(z)d\omega \end{array} \tag{22}$$

The phase

$$\Psi = -\tau(|\xi_1|_{g,\bar\zeta}+|\xi_2|_{g,y}) - i\langle\xi_1,\overline{\exp_x^{-1}(\zeta)}\rangle + i\langle\xi_2,\exp_y^{-1}(\zeta)\rangle \tag{23}$$

is complex but has only real critical points given by the non-degenerate critical manifold $C_\psi \simeq T^*M$:

$$C_\Psi = \{(x,y,\xi_1,\xi_2,\zeta = z+i\tau\omega) : y = x,\quad \xi_1 = -\xi_2,\ z = 0,\ \omega = \frac{\xi_1}{|\xi_1|}\}. \tag{24}$$

The Lagrange immersion

$$\iota_\Psi : C_\Psi \to T^*M\times T^*M,\quad \iota_\Psi(x,y,\xi_1,\xi_2,\zeta = z+i\tau\omega) = (x, d_x\Psi, y, -d_y\Psi)$$

is then real valued and, as mentioned above, we may apply stationary phase for complex phase functions with only real critical points (cf. [Ho], Vol. 1) to evaluate the principal symbol.

We recall that the principal symbol is the transport to $\Delta_{T^*M\times T^*M}$ by ι_Ψ of the 1/2-density $\sigma_0\sqrt{d_{C_\Psi}}$ where σ_0 is the principal term of the amplitude restricted to C_Ψ, and where d_{C_Ψ} is the Leray density on C_Ψ induced by the map i_Ψ and the coordinate volume density

$d\xi_1 d\xi_2 dV(z) d\omega$. Above, σ_0 is the principal term of $A^*(t,\zeta,x,\xi_1)A(t,\zeta,y,\xi_2) \equiv 1$ on C_Ψ. The Leray density is given by

$$d_{C_\Psi} = \left\| \frac{D(x,\xi_1,\frac{\partial \Psi}{\partial \xi_1},\frac{\partial \Psi}{\partial \xi_2},\frac{\partial \Psi}{\partial \zeta})}{D(x,\xi_1,y,\xi_2,\zeta)} \right\|^{-1} dx d\xi_1 = \left\| \frac{D(\frac{\partial \Psi}{\partial \xi_1},\frac{\partial \Psi}{\partial \xi_2},\frac{\partial \Psi}{\partial \zeta})}{D(y,\xi_2,\zeta)} \right\|^{-1} dx d\xi_1,$$

where we regard (x,ξ_1) as coordinates on C_Ψ. We write $\zeta = x_1 + ip$ in Riemannian normal coordinates based at x. Expanding first along the D_y rows and then along the Ψ'_{x_1} columns, we obtain

$$d_{C_\Psi} = \left\| \frac{D(\frac{\partial \Psi}{\partial p})}{D(p)} \right\|^{-1} dx d\xi_1. \tag{25}$$

Indeed, simple calculations show that the D_y derivatives only act non-trivially on Ψ'_{ξ_2}, and that Ψ'_{x_1} only has non-trivial derivatives under D_{ξ_2}, D_{x_1}. To eliminate the D_y rows and the Ψ'_{x_1} column, it suffices to show that $\Psi''_{x_1 x_1} = 0$ and that $\Psi''_{y\xi_2} = Id = \Psi''_{x_1\xi_2}$ on the critical set. In the calculation of $\Psi''_{x_1x_1}$ on the critical set we may put $\xi_1 = -\xi_2 = \xi, x = y, p = \frac{\xi_1}{|\xi_1|}$, and then the calculation reduces to $D^2_{x_1}(\langle \xi, \exp_x^{-1}(x_1 - ip) - \exp_x^{-1}(x_1 + ip)\rangle)|_{x_1=0} = 0$, which holds since $\exp_x^{-1}$ is the identity map when $\zeta = x + ip$ is expressed in Riemannian normal coordinates at x. In calculating $\Psi''_{y\xi_2} = D^2_{y\xi_2}(|\xi_2|_{g_y} + i\langle \xi_2, \exp_y^{-1}(\zeta)\rangle)|_{x=y,\xi_2=-\xi_1}$, the first term has a zero Hessian since all metric coefficients vanish in normal coordinates. The second is easily seen to be a constant multiple of the identity operator (the multiple is the same as in the Euclidean case). Finally, $\Psi''_{x_1\xi_2} = iD_{x_1}\exp_y^{-1}(x_1 + ip)|_{x_1=0,y=x} = CId$. Taking the determinant of the result gives (25). Finally, we calculate this last determinant after restricting to the critical set, obtaining a constant multiple of

$$\det D^2_{pp}\langle \xi, \exp_x^{-1}(ip)\rangle|_{p=\frac{\xi}{|\xi|}} = C \det D^2_{pp}\langle \xi, p\rangle|_{p=\frac{\xi}{|\xi|}}.$$

The last determinant is precisely the one that arises as the Hessian determinant in the stationary phase formula in (21). The derivatives are taken on S^{m-1}, hence we obtain $|\xi|^{m-1}$ times the normalized determinant, which is invariant under rotations and hence constant. Raising to the power $-\frac{1}{2}$ completes the calculation.

□

4. Proof of Lemma 1.4 and Lemma 1.6

We now use Lemma 3.1 to reduce the quantum ergodicity and norm properties of the complexified eigenfunctions to properties of the original real eigenfunctions.

4.1. **Proof of Lemma 1.4.** We begin by proving a weak limit formula for the CR holomorphic functions u_λ^ϵ defined in (8) for fixed ϵ. For notational simplicity, we drop the tilde notation although we work in the $B^*_\epsilon M$ setting.

Lemma **4.1.** *Assume that $\{\varphi_\lambda\}$ is a quantum ergodic sequence. Then for each $0 < \epsilon < \epsilon_0$,*

$$|u_\lambda^\epsilon|^2 \to \frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)}, \quad \textit{weakly in } L^1(\partial B^*_\epsilon M, d\mu_\epsilon).$$

That is, for any $a \in C(\partial B^*_\epsilon M)$,

$$\int_{\partial B^*_\epsilon M} a(x,\xi)|u_\lambda^\epsilon((x,\xi)|^2 d\mu_\epsilon \to \frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)} \int_{\partial B^*_\epsilon M} a(x,\xi) d\mu_\epsilon.$$

Proof. It suffices to consider $a \in C^\infty(\partial B^*_\epsilon M)$. We then consider the Toeplitz operator $\Pi_\epsilon a \Pi_\epsilon$ on $\mathcal{O}^0(\partial B^*_\epsilon M)$. We have,

$$\begin{array}{lll} \langle \Pi_\epsilon a \Pi_\epsilon u_j^\epsilon, u_j^\epsilon \rangle & = & e^{2\epsilon\lambda_j}\ ||\varphi_\lambda^{\mathbb{C}}||^{-2}_{L^2(\partial B^*_\epsilon M)} \langle \Pi_\epsilon a \Pi_\epsilon E(i\epsilon)\varphi_j, E(i\epsilon)\varphi_j \rangle_{L^2(\partial B^*_\epsilon M)} \\ & = & e^{2\epsilon\lambda_j}\ ||\varphi_\lambda^{\mathbb{C}}||^{-2}_{L^2(\partial B^*_\epsilon M)} \langle E(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon E(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}. \end{array} \tag{26}$$

By Lemma 3.1, $E(i\epsilon)^*\Pi_\epsilon a \Pi_\epsilon E(i\epsilon)$ is a pseudodifferential operator on M of order $-\frac{m-1}{2}$ with principal symbol $\tilde{a}|\xi|_g^{-\frac{m-1}{2}}$, where $\tilde{a}$ is the (degree 0) homogeneous extension of a to $T^*M - 0$. The normalizing factor $e^{2\epsilon\lambda_j}\ ||\varphi_\lambda^{\mathbb{C}}||^{-2}_{L^2(\partial B^*_\epsilon M)}$ has the same form with $a = 1$. Hence, the expression on the right side of (26) may be written as

$$\frac{\langle E(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon E(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}}{\langle E(i\epsilon)^* \Pi_\epsilon E(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}}. \tag{27}$$

By the standard quantum ergodicity result on compact Riemannian manifolds with ergodic geodesic flow (see [Shn, Z2, Z, CV] for proofs and references) we have

$$\frac{\langle E(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon E(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}}{\langle E(i\epsilon)^* \Pi_\epsilon E(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}} \to \frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)} \int_{\partial B^*_\epsilon M} a d\mu_\epsilon. \tag{28}$$

More precisely, the numerator is asymptotic to the right side times $\lambda^{-\frac{m-1}{2}}$, while the denominator has the same asymptotics when a is replaced by 1. We also use that $\frac{1}{\mu_\epsilon(\partial B^*_\epsilon M)} \int_{\partial B^*_\epsilon M} a d\mu_\epsilon$ equals the analogous average of $\tilde{a}$ over ∂B_1 (see the discussion around (30)). Taking the ratio produces (28).

Combining (26), (28) and the fact that

$$\langle \Pi_\epsilon a \Pi_\epsilon u_j^\epsilon, u_j^\epsilon \rangle = \int_{\partial B^*_\epsilon M} a |u_j^\epsilon|^2 d\mu_\epsilon$$

completes the proof of the lemma.

□

We now complete the proof of Lemma 1.4, i.e. we prove that

$$\int_{B^*_\epsilon M} a |U_\lambda|^2 \omega^m \to \frac{1}{\mu_1(S^*M)} \int_{B^*_\epsilon M} a |\xi|_g^{-m+1} \omega^m \tag{29}$$

for any $a \in C(B^*_\epsilon M)$. It is only necessary to relate the Liouville measures $d\mu_r$ (6) to the symplectic volume measure. One may write $d\mu_r = \frac{d}{dt}|_{t=r} \chi_t \omega^m$, where χ_t is the characteristic function of $B^*_t M = \{|\xi|_g \leq t\}$. By homogeneity of $|\xi|_g$, $\mu_r(\partial B^*_r M) = r^{m-1}\mu_1(\partial B^*_1 M)$. If $a \in C(B^*_\epsilon)$, then $\int_{B^*_\epsilon M} a\omega^m = \int_0^\epsilon \{\int_{\partial B^*_r M} a d\mu_r\} dr$. By Lemma 4.1, we have

(30)

$$\begin{array}{lll} \int_{B^*_\epsilon M} a |U_\lambda|^2 \omega^m = \int_0^\epsilon \{\int_{\partial B^*_r M} a |u_\lambda^r|^2 d\mu_r\} dr & \to & \int_0^\epsilon \{\frac{1}{\mu_r(\partial B^*_r)} \int_{\partial B^*_r M} a d\mu_r\} dr \\ & = & \frac{1}{\mu_1(\partial B^*_1 M)} \int_{B^*_\epsilon M} a r^{-m+1} \omega^m, \\ & \implies & w^* - \lim_{\lambda \to \infty} |U_\lambda|^2 = \frac{1}{\mu_1(\partial B^*_1 M)} |\xi|_g^{-m+1}. \end{array}$$

4.2. **Proof of Lemma 1.6.** We actually prove the stronger result that

$$\frac{1}{\lambda}\log \rho_\lambda(x,\xi) \to |\xi|_{g_x}, \quad \text{uniformly in} \quad B^*_\epsilon M \;\text{ as }\; \lambda \to \infty.$$

We state the weaker form because that is what we need in the proof of Theorem 1.1.
Again we drop the tilde notation for simplicity.

Proof. Again using $E(i\epsilon)\varphi_\lambda = e^{-\lambda\epsilon}\varphi_\lambda^{\mathbb{C}}$, we have:

$$\begin{aligned} \rho^2_\lambda(x,\xi) &= \langle \Pi_\epsilon \varphi_\lambda^{\mathbb{C}}, \Pi_\epsilon \varphi_\lambda^{\mathbb{C}} \rangle_{L^2(\partial B^*_\epsilon M)} \;\; (\epsilon = |\xi|_{g_x}) \\ &= e^{2\lambda\epsilon}\langle \Pi_\epsilon E(i\epsilon)\varphi_\lambda, \Pi_\epsilon E(i\epsilon)\varphi_\lambda\rangle_{L^2(\partial B^*_\epsilon M)} \\ &= e^{2\lambda\epsilon}\langle E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)\varphi_\lambda, \varphi_\lambda\rangle. \end{aligned} \tag{31}$$

Hence,

$$\frac{2}{\lambda}\log\rho_\lambda(x,\xi) = 2|\xi|_{g_x} + \frac{1}{\lambda}\log\langle E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)\varphi_\lambda,\varphi_\lambda\rangle. \tag{32}$$

To complete the proof, we observe that

$$\frac{1}{\lambda}\log\langle E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)\varphi_\lambda,\varphi_\lambda\rangle \leq C\,\frac{\log\lambda}{\lambda}, \;\text{ uniformly in }\; \epsilon, \tag{33}$$

where C is a constant independent of (ϵ,λ). Indeed, by Lemma 3.1, $E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)$ is a pseudodifferential operator of order $-\frac{m-1}{2}$ with principal symbol $|\xi|^{-\frac{m-1}{2}}$. To obtain a uniform bound on $\langle E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)\varphi_\lambda,\varphi_\lambda\rangle$ in ϵ, any of the standard bounds for the norm of a pseudodifferential operator in terms of derivatives of the complete symbol would suffice.

To take one such bound with a convenient reference, we write $\langle E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)\varphi_\lambda,\varphi_\lambda\rangle$ as $(1+\lambda^2)^{\frac{m+1}{2}}\langle (I-\Delta)^{-\frac{m+1}{2}}E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)\varphi_\lambda,\varphi_\lambda\rangle$. Put $A_\epsilon := (I-\Delta)^{-\frac{m+1}{2}}E(i\epsilon)^*\Pi_\epsilon E(i\epsilon)$. Since $A_\epsilon \in \Psi^{-(m-1)}$, we may apply the Schur-Young bound of [H] (Vol. III, Theorem 18.1.11) to obtain

$$\langle A_\epsilon\varphi_\lambda,\varphi_\lambda\rangle \leq ||A_\epsilon||_{L^2\to L^2} \leq C_m\left(\sup_x \int_{T^*_x M} |a_\epsilon(x,\xi)|d\xi\right), \tag{34}$$

where a_ϵ is the complete symbol of A_ϵ relative to some choice of quantization $a(x,D)$ of symbols. The complete symbol of A_ϵ may be obtained by applying $\langle (I-\Delta)^{-\frac{m+1}{2}}$ to the representation in (22). It is clear that the complete symbol is smooth in the parameter ϵ, hence the right side of (34) has uniform bound in ϵ, proving (34) and therefore the lemma. ∎

5. Proof of Lemma 1.5 and Theorem 1.1

The remaining step in the proof of Theorem 1.1 is the proof of Lemma 1.5.

5.1. **Proof of Lemma 1.5.**

Proof. The proof is similar to that of Lemma 1.4 of [SZ]. We wish to prove that

$$\psi_j := \frac{1}{\lambda_j} \log |U_j|^2 \to 0 \text{ in } L^1(B^*_\epsilon M).$$

We argue by contradiction. If the conclusion is not true, then there exists a subsequence ψ_{j_k} satisfying $||\psi_{j_k}||_{L^1(B^*_\epsilon M)} \geq \delta > 0$.

To obtain a contradiction, we first observe that ψ_j is quasi-plurisubharmonic (QPSH) on $B^*_\epsilon M$, i.e. may be locally written as the sum of a plurisubharmonic function v_j and a smooth function ρ_j; equivalently $i\partial\bar{\partial}\psi_j$ is locally bounded below by a negative smooth $(1,1)$ form. Indeed we put

$$v_j := \frac{1}{\lambda_j} \log |\varphi_j^{\mathbb{C}}|^2, \quad \rho_j := -\rho_{\lambda_j}.$$

We use the following fact about subharmonic functions (see [Ho, Theorem 4.1.9]):

Let $\{v_j\}$ be a sequence of subharmonic functions in an open set $X \subset \mathbb{R}^m$ which have a uniform upper bound on any compact set. Then either $v_j \to -\infty$ uniformly on every compact set, or else there exists a subsequence v_{j_k} which is convergent in $L^1_{loc}(X)$.

Since the proof is local, it also holds for open sets in manifolds, and in particular for $X = B^*_\epsilon M$.

We now verify that the hypotheses are satisfied in our example:

- (i) the functions v_j are uniformly bounded above on $B^*_\epsilon M$;
- (ii) $\limsup_{j\to\infty} v_j \leq 2|\xi|_g$.

It suffices to prove these statements on each surface $\partial B^*_\epsilon M$ with uniform constants independent of ϵ. On the surface $\partial B^*_\epsilon M$, $U_j = u^\epsilon_j$. By the Sobolev inequality in $\mathcal{O}^{\frac{m-1}{4}}(\partial B^*_\epsilon M)$, we have

$$\begin{aligned} \sup_{(x,\xi)\in\partial B^*_\epsilon M)} |u^\epsilon_j(x,\xi)| &\leq \lambda_j^m ||u^\epsilon_j(x,\xi)||_{L^2(\partial B^*_\epsilon M)} \\ &\leq \lambda_j^m. \end{aligned}$$

Taking the logarithm, dividing by λ_j, and combining with the limit formula of Lemma 1.6 proves (i) - (ii).

We now settle the dichotomy above by proving that the sequence $\{v_j\}$ does not tend uniformly to $-\infty$ on compact sets. That would imply that $\psi_j \to -\infty$ uniformly on the spheres $\partial B^* M_\epsilon$ for each $\epsilon < \epsilon_0$. Hence, for each ϵ, there would exist $K > 0$ such that for $k \geq K$,

$$\frac{1}{\lambda_{j_k}} \log |u^\epsilon_{j_k}(z)| \leq -1. \tag{35}$$

However, (35) implies that

$$|u_{j_k}(z)| \leq e^{-2\lambda_{j_k}} \quad \forall z \in \partial B^*_\epsilon M \ ,$$

which is inconsistent with the hypothesis that $|u^\epsilon_{j_k}(z)| \to 1$ in $\mathcal{D}'(\partial B^*_\epsilon M)$.

Therefore, the second half of the dichotomy holds, i.e. there must exist a subsequence, which we continue to denote by $\{v_{j_k}\}$, which converges in $L^1(B^*_{\epsilon_0})$ to some $v \in L^1(B^*_{\epsilon_0})$. By passing if necessary to a further subsequence, we may assume that $\{v_{j_k}\}$ converges pointwise almost everywhere to v in $B^*_{\epsilon_0}$. Then,

$$v(z) = \limsup_{k\to\infty} v_{j_k} \leq 2|\xi|_g \qquad \text{(a.e)} .$$

Now let

$$v^*(z) := \limsup_{w\to z} v(w) \leq 0$$

be the upper-semicontinuous regularization of v. Then v^* is plurisubharmonic on $B^*_\epsilon M$ and $v^* = v$ almost everywhere.

Put $\psi^* := v^* - 2|\xi|_g$. Then $\psi^* \leq 0$, and the assumption $||\psi_{j_k}||_{L^1(B^*_\epsilon M)} \geq \delta > 0$ implies that

$$U_\delta := \{\zeta \in B^*_{\epsilon_0} M : \ \psi^*(\zeta) \ < -\delta/2\}$$

has positive volume. Since $\psi_{j_k} \to \psi^*$ in $L^1(U_\delta)$, one has by [Ho] Theorem 4.1.9 (b) that

$$\limsup_{k\to\infty} \psi_{j_k}|_{U_\delta} \leq \psi^*|_{U_\delta} < -\delta/2. \tag{36}$$

Hence, there exists a positive integer K such that $\psi_{j_k}(\zeta) \leq -\delta/2$ for $\zeta \in U_\delta, \ k \geq K$; i.e.,

$$|\psi_{j_k}(\zeta)| \leq e^{-\delta\lambda_{j_k}}, \qquad \zeta \in U_\delta, \ \ k \geq K. \tag{37}$$

This again contradicts the weak convergence to 1.

Therefore $||\psi_{j_k}||_{L^1(B^*_\epsilon M)} \geq \delta > 0$ leads to a contradiction, and the Lemma is proved. □

To complete the proof of Theorem 1.1 it suffices to combine the results that

$$\frac{i}{2\pi\lambda}\partial\bar\partial \log |U_\lambda|^2 = \frac{1}{\lambda}\ [Z_\lambda] - \frac{i}{2\pi\lambda}\partial\bar\partial \log ||\varphi_\lambda^\epsilon||^2_{L^2(\partial M_\epsilon)} \to 0 \quad \text{(weakly)},$$

and that (by Lemma 1.6) the second term tends to $\frac{i}{\pi}\partial\bar\partial|\xi|_g$. □

5.2. **Final remarks. (i)** We check the numerical details in the case of the circle.

The zeros of $\sin 2\pi k z$ in the cylinder $\mathbb{C}/\mathbb{Z}$ all lie on the real axis at the points $z = \frac{n}{2k}$. Thus, there are $2k$ real zeros, and the Poincaré-Lelong formula gives

$$\begin{aligned} \lim_{k\to\infty} \tfrac{i}{2\pi k}\partial\bar\partial \log|\sin 2\pi k|^2 &= \lim_{k\to\infty} \tfrac{1}{k}\textstyle\sum_{n=1}^{2k} \delta_{\frac{n}{2k}} \\ &= \tfrac{1}{\pi}\delta_0(\xi) dx\wedge d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \tfrac{i}{\pi}\partial\bar\partial|\xi| &= \tfrac{i}{\pi}\tfrac{d^2}{4d\xi^2}|\xi| \ \ \tfrac{2}{i}dx\wedge d\xi \\ &= \tfrac{i}{\pi}\ \tfrac{1}{2}\ \delta_0(\xi)\ \ \tfrac{2}{i}dx\wedge d\xi, \end{aligned}$$

matching the other expression.

As mentioned in the introduction, the complex eigenfunctions $e^{2\pi i k x}$ have no complex zeros, hence Theorem 1.1 is false for them. The reason is that they are not quantum ergodic but rather localize on just one of the two components of the unit tangent bundle (the one with the same sign as k). Running through the previous calculation shows that the limit zero current for these eigenfunctions is $\frac{i}{\pi}\partial\bar\partial\xi = 0$ rather than $\frac{i}{\pi}\partial\bar\partial|\xi|$.

(ii) One can obtain other formulae for the distribution of zeros in the ergodic case using the fact that the maps $t+\sqrt{-1}s \to \exp_{\gamma(t)} s\dot{\gamma}(t)$ are holomorphic curves $\gamma_{\mathbb{C}}(t+\sqrt{-1}s)$ relative to the adapted complex structure for each geodesic γ. If one pulls back complexified eigenfunctions under $\gamma_{\mathbb{C}}$, then one obtains a holomorphic function in a strip around the real-axis. Its complex zeros are discrete and correspond to the intersection points $Z_{\varphi_\lambda^{\mathbb{C}}} \cap \gamma_{\mathbb{C}}$. Its real zeros are the intersection points of the real geodesic γ with the nodal hypersurface. In connection with Conjecture 1.7, it is natural to conjecture that these intersection points become uniformly distributed on (M,g) when γ is a uniformly distributed geodesic.

(iii) We can give a simpler form to $\frac{i}{\pi}\partial\bar{\partial}|\xi|_g$ in dimensions $m \geq 2$. Let $\rho(x,\xi) = |\xi|_g^2$. We note (with [GS2]) that

$$\partial\bar{\partial} f(\rho) = f'(\rho)\partial\bar{\partial}\rho + f''(\rho)\bar{\partial}\rho \wedge \partial\rho,$$

and that

$$\bar{\partial}\rho \wedge \partial\rho = i d\rho \wedge \alpha, \quad \partial\bar{\partial}\rho = -i\omega_g.$$

It follows that in dimensions $m \geq 2$, we have

$$\frac{i}{\pi}\partial\bar{\partial}\sqrt{\rho} = \frac{1}{2\pi\rho^{1/2}}\omega_g + \frac{d\rho\wedge\alpha}{4\pi\rho^{3/2}}. \tag{38}$$

References

[Be] P. Bérard, On the wave equation on a compact Riemannian manifold without conjugate points. Math. Z. 155 (1977), no. 3, 249–276.

[Bou] L. Boutet de Monvel, Convergence dans le domaine complexe des séries de fonctions propres. C. R. Acad. Sci. Paris Sr. A-B 287 (1978), no. 13, A855–A856.

[BG] L. Boutet de Monvel and V. Guillemin, The spectral theory of Toeplitz operators. Annals of Mathematics Studies, 99. Princeton University Press, Princeton, NJ (1981).

[BoSj] L. Boutet de Monvel and J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, *Asterisque* 34–35 (1976), 123–164.

[BW] F. Bruhat and H. Whitney, Quelques proprits fondamentales des ensembles analytiques-rels. Comment. Math. Helv. 33 1959 132–160.

[CV] Y. Colin de Verdiére, Ergodicité et fonctions propres du laplacien. Comm. Math. Phys. 102 (1985), no. 3, 497–502.

[DF] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93 (1988), 161-183.

[GLS] F. Golse, E. Leichtnam, and M. Stenzel, Intrinsic microlocal analysis and inversion formulae for the heat equation on compact real-analytic Riemannian manifolds. Ann. Sci. École Norm. Sup. (4) 29 (1996), no. 6, 669–736.

[Gr] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, *Math. Annalen* 146 (1962), 331–368.

[G] V. Guillemin, The homogeneous Monge-Ampere equation on a pseudoconvex domain, Asterisque 210 (1992), 97-113.

[GO] V. Guillemin and K. Okikiolu, Subprincipal terms in Szegö estimates, Math. Res. Lett. 4 (1997), 173–179.

[GS1] V. Guillemin and M. Stenzel, Grauert tubes and the homogeneous Monge-Ampre equation. J. Differential Geom. 34 (1991), no. 2, 561–570.

[GS2] V. Guillemin and M. Stenzel, Grauert tubes and the homogeneous Monge-Ampre equation. II. J. Differential Geom. 35 (1992), no. 3, 627–641.

[H] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions.* Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2000.

[Ho] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Grund. Math. Wiss. 256, 274 Springer-Verlag, New York, 1983.

[JL] D. Jerison and G. Lebeau, Nodal sets of sums of eigenfunctions, Ch. 14 of *Harmonic analysis and partial differential equations* (Chicago, IL, 1996), 223–239, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999.

[KM] S-J. Kan and D. Ma, On rigidity of Grauert tubes over Riemannian manifolds of constant curvature. Math. Z. 239 (2002), no. 2, 353–363.

[LRS] A. Laptev, D. Robert and Yu. Safarov, Remarks on the paper of V. Guillemin and K. Okikiolu: "Subprincipal terms in Szeg estimates" [Math. Res. Lett. 4 (1997), no. 1, 173–179. Math. Res. Lett. 5 (1998), no. 1-2, 57–61.

[LS1] L. Lempert and R. Szöke, Global solutions of the homogeneous complex Monge-Ampre equation and complex structures on the tangent bundle of Riemannian manifolds. Math. Ann. 290 (1991), no. 4, 689–712.

[LS2] L. Lempert and R. Szöke, The tangent bundle of an almost complex manifold, Canad. Math. Bull. 44 (2001), no. 1, 70–79.

[Lin] E. Lindenstrauss, Invariant measures and arithmetic quantum unique ergodicity (Annals Math., to appear).

[MS] A. Melin and J. Sjöstrand, Fourier integral operators with complex-valued phase functions. *Fourier integral operators and partial differential equations* (Colloq. Internat., Univ. Nice, Nice, 1974), pp. 120–223. Lecture Notes in Math., Vol. 459, Springer, Berlin, 1975.

[NPSo] F. Nazarov, L.Polterovich, and Mikhail Sodin, Sign and area in nodal geometry of Laplace eigenfunctions Authors: , to appear in Amer. J. Math. (math.AP/0402412)

[Ne] J. Neuheisel, The asymptotic distribution of nodal sets on spheres, Ph.D. thesis, Johns Hopkins University, 2000.

[NV] S. Nonnenmacher and A. Voros, Chaotic eigenfunctions in phase space, J. Statist. Phys. 92 (1998), no. 3-4, 431–518.

[PW] G. Patrizio and P.M. Wong, Stein manifolds with compact symmetric center. Math. Ann. 289 (1991), no. 3, 355–382.

[R] Z. Rudnick On the asymptotic distribution of zeros of modular forms, preprint math.NT/0506267 (2005).

[RS] Z. Rudnick and P. Sarnak, The behaviour of eigenstates of arithmetic hyperbolic manifolds. Comm. Math. Phys. 161 (1994), no. 1, 195–213.

[SZ] Shiffman, Bernard; Zelditch, Steve Distribution of zeros of random and quantum chaotic sections of positive line bundles. Comm. Math. Phys. 200 (1999), no. 3, 661–683.

[Shn] A. I. Shnirelman, Ergodic properties of eigenfunctions, *Usp. Mat. Nauk.* 29/6 (1974), 181–182.

[T] F. Tréves, *Introduction to pseudodifferential and Fourier integral operators. Vol. 2. Fourier integral operators.* The University Series in Mathematics. Plenum Press, New York-London, 1980.

[Z] S. Zelditch, Quantum ergodicity of C^* dynamical systems, Comm. Math. Phys. 177 (1996), 507–528.

[Z2] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces. Duke Math. J. 55 (1987), no. 4, 919–941.

[Z3] S. Zelditch, A random matrix model for quantum mixing, Int.Math. Res.Not. 3 (1996), 115-137.

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