

Asymptotics of polynomials and eigenfunctions

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Abstract

We review some recent results on asymptotic properties of polynomials of large degree, of general holomorphic sections of high powers of positive line bundles over Kähler manifolds, and of Laplace eigenfunctions of large eigenvalue on compact Riemannian manifolds. We describe statistical patterns in the zeros, critical points and L^p norms of random polynomials and holomorphic sections, and the influence of the Newton polytope on these patterns. For eigenfunctions, we discuss L^p norms and mass concentration of individual eigenfunctions and their relation to dynamics of the geodesic flow.

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1. Introduction

In many measures of ‘complexity’, eigenfunctions $\sqrt{\Delta}\varphi_\lambda = \lambda\varphi_\lambda$ of first order elliptic operators behave like polynomials $p(x) = \sum_{|\alpha| \leq x} c_\alpha x^\alpha$ of degree $N \sim \lambda$ [6]. The basic example we have in mind is the Laplacian Δ on a compact Riemannian manifold (M, g) , but the same is true of Schroedinger operators. The comparison is more than an analogy, since polynomials of degree N are eigenfunctions of a first order elliptic system.

The comparison between eigenfunctions and polynomials is an essentially local one, most accurate on small balls $B(x_0, \frac{1}{\lambda})$. Globally, eigenfunctions reflect the dynamics of the geodesic flow G^t on the unit (co-)tangent bundle S^*M . This is one of the principal themes of quantum chaos.

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In this article, we review some recent results on the asymptotics of polynomials and eigenfunctions, concentrating on our work in collaboration with P. Bleher, A. Hassell, B. Shiffman, C. Sogge, J. Toth and M. Zworski. A unifying feature is the asymptotic properties of reproducing kernels, namely Szegő kernels $\Pi_N(z, w)$ in the case of polynomials, and spectral projections $E_\lambda(x, y)$ for intervals $[\lambda, \lambda + 1]$ in the case of eigenfunctions of $\sqrt{\Delta}$. For other recent expository articles, see [9, 26].

2. Polynomials

There are several sources of interest in random polynomials. One is the desire to understand typical properties of real and complex algebraic varieties, and how they depend on the coefficients of the defining equations. Another is their use as a model for the local behavior of more general eigenfunctions. A third is that they may be viewed as the eigenvectors of random matrices. Just as random matrices model the spectra of ‘quantum chaotic’ systems, so random polynomials model their eigenfunctions.

2.1. $SU(m+1)$ polynomials on $\mathbb{C}\mathbb{P}^m$ and holomorphic sections

Complex polynomials of degree $\leq p$ in m variables form the vector space

$$\mathcal{P}_p^m := \{f(z_1, \dots, z_m) = \sum_{\alpha \in \mathbb{N}^m: |\alpha| \leq p} c_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m}, c_\alpha \in \mathbb{C}\}.$$

To put a probability measure on \mathcal{P}_p^m is to regard the coefficients c_α as random variables. The simplest measures are Gaussian measures corresponding to inner products on \mathcal{P}_p^m . By homogenizing f to $F(z_0, z_1, \dots, z_m)$ of degree p , we may identify \mathcal{P}_p^m with the space $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p))$ of holomorphic sections of the p th power of the hyperplane bundle. It carries the standard $SU(m+1)$ -invariant Fubini-Study inner product $\langle F_1, \bar{F}_2 \rangle_{FS} = \int_{S^{2m+1}} F_1 \bar{F}_2 d\sigma$, where $d\sigma$ is Haar measure on the $(2m+1)$ -sphere S^{2m+1} . An orthonormal basis of $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p))$ is given by $\{\frac{z^\alpha}{\|z^\alpha\|_{FS}}\}$. The corresponding $SU(m+1)$ -invariant Gaussian measure γ_δ is defined by

$$d\gamma_\delta(s) = \frac{1}{\pi^{k_p}} e^{-|\lambda|^2} d\lambda, \quad s = \sum_{|\alpha| \leq p} \lambda_\alpha \frac{z^\alpha}{\|z^\alpha\|_{FS}}.$$

Thus, the coefficients λ_α are independent complex Gaussian random variables with mean zero and variance one.

More generally, we can define Gaussian ensembles of holomorphic sections $H^0(M, L^N)$ of powers of a positive line bundle over any Kähler manifold (M, ω) . Endowing L with the unique hermitian metric h of curvature form ω , we induce an inner product $\langle \cdot, \cdot \rangle$ on $H^0(M, L^N)$ and a Gaussian measure γ_N . We denote the unit sphere in $H^0(M, L^N)$ relative to this inner product by $SH^0(M, L^N)$. The Haar measure on $SH^0(M, L^N)$ will be denoted μ_N . It is closely related to the Gaussian measure.

2.2. Zeros

The problems we discuss in this section involve the geometry of zeros of sections $s \in H^0(M, L^N)$ of general positive line bundles. There is a similar story for critical points.

- **Problem 1:** How are the simultaneous zeros $Z_s = \{z : s_1(z) = \cdots = s_k(z) = 0\}$ of a k -tuple $s = (s_1, \dots, s_k)$ of typical holomorphic sections distributed?
- **Problem 2** How are the zeros correlated? When $k = m$, the simultaneous zeros form a discrete set. Do zeros repel each other like charged particles? Or behave independently like particles of an ideal gas? Or attract like gravitating particles?

By the distribution of zeros we mean either the current of integration over Z_s or more simply the Riemannian $(2m-2k)$ -volume measure $(|Z_s|, \varphi) = \int_{Z_s} \varphi d\text{Vol}_{2m-2k}$. By the n -point zero correlation functions, we mean the generalized functions

$$K_{nk}^N(z^1, \dots, z^n) dz = \mathbf{E}|Z_s|^n,$$

where $|Z_s|^n$ denotes the product of the measures $|Z_s|$ on the punctured product $M_n = \{(z^1, \dots, z^n) \in M \times \cdots \times M : z^p \neq z^q \text{ for } p \neq q\}$ and where dz denotes the product volume form on M_n .

The answer to Problem 1 is that zeros almost surely become uniformly distributed relative to the curvature ω of the line bundle [18]. Curvature causes sections to oscillate more rapidly and hence to vanish more often. More precisely, we consider the space $\mathcal{S} = \prod_{N=1}^{\infty} SH^0(M, L^N)$ of random sequences, equipped with the product measure $\mu = \prod_{N=1}^{\infty} \mu_N$. An element in \mathcal{S} will be denoted $\mathbf{s} = \{s_N\}$. Then, $\frac{1}{N}Z_s \rightarrow \omega$, as $N \rightarrow \infty$ for almost every \mathbf{s} .

The answer to Problem 2 is more subtle: it depends on the dimension. We assume $k = m$ so that almost surely the simultaneous zeros of the k -tuple of sections form a discrete set. We find that these zeros behave almost independently if they are of distance $\geq \frac{D}{\sqrt{N}}$ apart for $D \gg 1$. So they only interact on distance scales of size $\frac{1}{\sqrt{N}}$. Since also the density of zeros in a unit ball $B_1(z_0)$ around z_0 grows like N^m , we rescale the zeros in the $1/\sqrt{N}$ -ball $B_{1/\sqrt{N}}(z_0)$ by a factor of \sqrt{N} to get configurations of zeros with a constant density as $N \rightarrow \infty$. We thus rescale the correlation functions and take the *scaling limits*

$$\tilde{K}_{nkm}^{\infty}(z^1, \dots, z^n) = \lim_{N \rightarrow \infty} K_{1k}^N(z_0)^{-n} K_{nk}^N(z_0 + \frac{z^1}{\sqrt{N}}, \dots, z_0 + \frac{z^n}{\sqrt{N}}). \quad (2.1)$$

In [1], we proved that the scaling limits of these correlation functions were universal, i.e. independent of M, L, ω, h . They depend only on the dimension m of the manifold and the codimension k of the zero set.

In [?], we found explicit formulae for these universal scaling limits. In the case $n = 2$, $\tilde{K}_{2km}^{\infty}(z^1, z^2)$, depends only on the distance between the points z^1, z^2 , since it is universal and hence invariant under rigid motions. Hence it may be written as:

$$\tilde{K}_{2km}^{\infty}(z^1, z^2) = \kappa_{km}(|z^1 - z^2|). \quad (2.2)$$

We refer to [1] for details.

Theorem 1 [?] *The pair correlation functions of zeros when $k = m$ are given by*

$$\kappa_{mm}(r) = \begin{cases} \frac{m+1}{4}r^{4-2m} + O(r^{8-2m}), & \text{as } r \rightarrow 0 \\ 1 + O(e^{-Cr^2}), \quad (C > 0) & \text{as } r \rightarrow \infty. \end{cases} \quad (2.3)$$

When $m = 1$, $\kappa_{mm}(r) \rightarrow 0$ as $r \rightarrow 0$ and one has ‘‘zero repulsion.’’ When $m = 2$, $\kappa_{mm}(r) \rightarrow 3/4$ as $r \rightarrow 0$ and one has a kind of neutrality. With $m \geq 3$, $\kappa_{mm}(r) \nearrow \infty$ as $r \rightarrow 0$ and there is some kind of attraction between zeros. More precisely, in dimensions greater than 2, one is more likely to find a zero at a small distance r from another zero than at a small distance r from a given point; i.e., zeros tend to clump together in high dimensions.

One can understand this dimensional dependence heuristically in terms of the geometry of the discriminant varieties $\mathcal{D}_N^m \subset H^0(M, L^N)^m$ of systems $S = (s_1, \dots, s_m)$ of m sections with a ‘double zero’. The ‘separation number’ $sep(F)$ of a system is the minimal distance between a pair of its zeros. Since the nearest element of \mathcal{D}_N^m to F is likely to have a simple double zero, one expects: $sep(F) \sim \sqrt{dist(F, \mathcal{D}_N^m)}$. Now, the degree of \mathcal{D}_N^m is approximately N^m . Hence, the tube $(\mathcal{D}_N^m)_\epsilon$ of radius ϵ contains a volume $\sim \epsilon^2 N^m$. When $\epsilon \sim N^{-m/2}$, the tube should cover $PH^0(M, L^N)$. Hence, any section should have a pair of zeros whose separation is $\sim N^{-m/4}$ apart. It is clear that this separation is larger than, equal to or less than $N^{-1/2}$ accordingly as $m = 1, m = 2, m \geq 3$.

2.3. Bergman-Szegö kernels

A key object in the proof of these results is the Bergman-Szegö kernel $\Pi_N(x, y)$, i.e. the kernel of the orthogonal projection onto $H^0(M, L^N)$ with respect to the Kähler form ω . For instance, the expected distribution of zeros is given by $\mathbf{E}_N(Z_f) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \Pi_N(z, z) + \omega$. Of even greater use is the joint probability distribution (JPD) $D_N(x^1, \dots, x^n; \xi^1, \dots, \xi^n; z^1, \dots, z^n)$ of the random variables $x^j(s) = s(z^j)$, $\xi^j(s) = \nabla s(z^j)$, which may be expressed in terms of Π_N and its derivatives. In turn, the correlation functions may be expressed in terms of the JPD by $K^N(z^1, \dots, z^n) = \int D_N(0, \xi, z) \prod_{j=1}^n (\|\xi^j\|^2 d\xi^j) d\xi$ [1].

The scaling asymptotics of the correlation functions then reduce to scaling asymptotics of the Bergman-Szegö kernel: In normal coordinates $\{z_j\}$ at $P_0 \in M$ and in a ‘preferred’ local frame for L , we have [1]:

$$\frac{\pi^m}{2^m} \Pi_N(P_0 + \frac{u}{\sqrt{N}}, \frac{\theta}{N}; P_0 + \frac{v}{\sqrt{N}}, \frac{\varphi}{N}) \sim e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)} \left[1 + b_1(u, v)N^{-\frac{1}{2}} + \dots \right].$$

To be precise, Π_N is the natural lift of the kernel as an equivariant kernel on the boundary ∂D^* of the unit (co-) disc bundle of L^* . Note that $e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)}$ is the Bergman-Szegö kernel of the Heisenberg group. These asymptotics use the Boutet de Monvel -Sjostrand parametrix for the Bergman-Szegö kernel [4], as applied in [29] to the Fourier coefficients of the kernel on powers of positive line bundles.

2.4. Polynomials with fixed Newton polytope

The well-known Bernstein-Kouchnirenko theorem states that the number of simultaneous zeros of (a generic family of) m polynomials with Newton polytope P equals $m!Vol(P)$. Recall that the Newton polytope P_f of a polynomial is the convex hull of its support $S_f = \{\alpha \in \mathbb{Z}^m : c_\alpha \neq 0\}$. Using the homogenization map $f \rightarrow F$, the space of polynomials f whose Newton polytope P_f contained in P may be identified with a subspace

$$H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p), P) = \{F \in H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p)) : P_f \subset P\} \quad (2.4)$$

of $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p))$.

The problem we address in this section is:

- **Problem 3:** How does the Newton polytope influence on the distribution of zeros of polynomials?

Again, one could ask the same question about L^2 mass, critical points and so on and obtain a similar story. In [19] we explore this influence in a statistical and asymptotic sense. The main theme is that for each property of polynomials under study, P gives rise to *classically allowed regions* where the behavior is the same as if no condition were placed on the polynomials, and *classically forbidden regions* where the behavior is exotic.

Let us define these terms. If $P \subset \mathbb{R}_+^m$ is a convex integral polytope, then the *classically allowed region* for polynomials in $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p), P)$ is the set

$$\mathcal{A}_P := \mu_\Sigma^{-1} \left(\frac{1}{p} P^\circ \right) \subset \mathbb{C}^{*m}$$

(where P° denotes the interior of P), and the *classically forbidden region* is its complement $\mathbb{C}^{*m} \setminus \mathcal{A}_P$. Here, $\mu_\Sigma(z) = \left(\frac{|z_1|^2}{1+\|z\|^2}, \dots, \frac{|z_m|^2}{1+\|z\|^2} \right)$ is the moment map of $\mathbb{C}\mathbb{P}^m$.

The result alluded to above is statistical. Since we view the polytope P of degree p as placing a condition on the Gaussian ensemble of $SU(P)$ polynomials of degree p , we endow $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p), P)$ with the *conditional probability measure* $\gamma_\delta|_P$:

$$d\gamma_\delta|_P(s) = \frac{1}{\pi^{\#P}} e^{-|\lambda|^2} d\lambda, \quad s = \sum_{\alpha \in P} \lambda_\alpha \frac{z^\alpha}{\|z^\alpha\|}, \quad (2.5)$$

where the coefficients λ_α are again independent complex Gaussian random variables with mean zero and variance one.

Our simplest result concerns the the expected density $\mathbf{E}|_P(Z_{f_1, \dots, f_m})$ of the simultaneous zeros of (f_1, \dots, f_m) chosen independently from $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p), P)$. It is the measure on \mathbb{C}^{*m} given by

$$\mathbf{E}|_P(Z_{f_1, \dots, f_m})(U) = \int d\gamma_{p|P}(f_1) \cdots \int d\gamma_{p|P}(f_m) [\#\{z \in U : f_1(z) = \cdots = f_m(z) = 0\}], \quad (2.6)$$

for $U \subset \mathbb{C}^{*m}$, where the integrals are over $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(p), P)$. We will determine the asymptotics of the expected density as the polytope is dilated $P \rightarrow NP, N \in \mathbb{N}$.

Theorem 2 [19] *Suppose that P is a simple polytope in \mathbb{R}^m . Then, as P is dilated to NP ,*

$$\frac{1}{(N\delta)^m} \mathbf{E}_{|NP}(Z_{f_1, \dots, f_m}) \rightarrow \begin{cases} \omega_{\text{FS}}^m & \text{on } \mathcal{A}_P \\ 0 & \text{on } \mathbb{C}^{*m} \setminus \mathcal{A}_P \end{cases},$$

*in the distribution sense; i.e., for any open $U \subset \mathbb{C}^{*m}$, we have*

$$\frac{1}{(N\delta)^m} \mathbf{E}_{|NP}(\#\{z \in U : f_1(z) = \dots = f_m(z) = 0\}) \rightarrow m! \text{Vol}_{\mathbb{C}\mathbb{P}^m}(U \cap \mathcal{A}_P).$$

There are also results for $k < m$ polynomials. The distribution of zeros is ω_{FS}^k in \mathcal{A}_P as if there were no constraint, while there is an exotic distribution in $\mathbb{C}^{*m} \setminus \mathcal{A}_P$ which depends on the exponentially decaying asymptotics of the conditional Bergman- Szegö kernel

$$\Pi_{|NP}(z, w) = \sum_{\alpha \in NP} \frac{z^\alpha \bar{w}^\alpha}{\|z^\alpha\|_{FS} \|w^\alpha\|_{FS}},$$

i.e. the orthogonal projection onto the subspace (2.4). It is obtained by sifting out terms in the (elementary) Szegö projector of $H^0(\mathbb{C}\mathbb{P}^m, \mathcal{O}(pN))$ using the polytope character $\chi_{NP}(e^{i\varphi}) = \sum_{\alpha \in NP} e^{i\langle \alpha, \varphi \rangle}$. To obtain asymptotics in the forbidden region, we write $\chi_{NP}(e^{i\varphi}) = \int_{M_P} \Pi_N^{M_P}(e^{i\varphi} w, w) dV(w)$, where Π^{M_P} is Bergman-Szegö kernel of the toric variety M_P associated to P . We then make an explicit construction of $\Pi_N^{M_P}$ as a complex oscillatory integral. An alternative is to express χ_{NP} as a Todd derivative of an exponential integral over P (following works of Khovanskii-Pukhlikov, Brion-Vergne and Guillemin). We thus obtain a complex oscillatory integral formula for $\Pi_{|NP}(z, w)$. To obtain asymptotics in the forbidden region we carefully deform the contour into the complex and apply a complex stationary phase method.

Although we only discuss expected behavior of zeros here, the distribution of zeros is *self-averaging*: i.e., almost all polynomials exhibit the expected behavior in an asymptotic sense. We also expect similar results for critical points.

3. Eigenfunctions

We now turn to the eigenvalue problem $\Delta_g \varphi_\nu = \lambda_\nu^2 \varphi_\nu$, $\langle \varphi_\nu, \varphi_{\nu'} \rangle = \delta_{\nu\nu'}$ on a compact Riemannian manifold (M, g) . We denote the λ -eigenspace by V_λ . The role of the Szegö kernel is now played by the kernel $E_\lambda(x, y) = \sum_{\lambda_\nu \leq \lambda} \varphi_\nu(x) \overline{\varphi_\nu(y)}$ of the spectral projections.

3.1. L^p bounds

Our first concern is with L^p norms of L^2 -normalized eigenfunctions. We measure the growth rate of L^p norms by $L^p(\lambda, g) = \sup_{\varphi \in V_\lambda: \|\varphi\|_{L^2}=1} \|\varphi\|_{L^p}$. By the local Weyl law, $E_\lambda(x, x) = \sum_{\lambda_\nu \leq \lambda} |\varphi_\nu(x)|^2 = (2\pi)^{-n} \int_{|\xi| \leq \lambda} d\xi + O(\lambda^{n-1})$, it follows that

$L^\infty(\lambda, g) = o(\lambda^{\frac{n-1}{2}})$ on any compact Riemannian manifold. This bound, which is based entirely on a local analysis, is sharp in the case of the standard round sphere, S^n or on any rotationally invariant metric on S^2 , but is far off in the case of flat tori. This motivates:

- **Problem 5** For which (M, g) is this estimate sharp? Which (M, g) are extremal for growth rates of $\|\varphi_\lambda\|_p$, both maximal and minimal? What if M has a boundary? What is the expected L^p norm of a ‘random’ L^2 -normalized polynomial or eigenfunction?

In [20], we give a necessary condition for *maximal* eigenfunction growth: there must exist a point $x \in M$ for which the set $\mathcal{L}_x = \{\xi \in S_x^*M : \exists T : \exp_x T\xi = x\}$ of directions of geodesic loops at x has positive surface measure.

Theorem 3 [20] *If \mathcal{L}_x has measure 0 in S_x^*M for every $x \in M$ then*

$$L^p(\lambda, g) = o(\lambda^{\delta(p)}), \quad p > \frac{2(n+1)}{n-1} \quad \delta(p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases} \quad (3.1)$$

The L^p -bounds $O(\lambda^{\delta(p)})$ were proved by Sogge to hold for all (M, g) . We further prove:

Theorem 4 [20] (see also [17]) *Suppose that (M, g) is:*

- *Real analytic and that $L^\infty(\lambda, g) = \Omega(\lambda^{(n-1)/2})$. Then (M, g) is a Y_ℓ^m -manifold, i.e. $\exists m$ such that all geodesics issuing from the point m return to m at time ℓ . In particular, if $\dim M = 2$, then M is topologically a 2-sphere S^2 or a real projective plane $\mathbb{R}P^2$.*
- *Generic. Then $L^\infty(\lambda, g) = o(\lambda^{(n-1)/2})$.*

Here, $\Omega(\lambda^{\frac{n-1}{2}})$ means $O(\lambda^{\frac{n-1}{2}})$ but not $o(\lambda^{\frac{n-1}{2}})$. The generic result holds because \mathcal{L}_x has measure 0 in S_x^*M for all $x \in M$ for a residual set of metrics.

In the case of random polynomials, or random combinations of eigenfunctions in short spectral intervals, the almost sure growth of L^∞ norms is $O(\sqrt{\log N})$ while the L^p norms for $p < \infty$ are bounded. This was proved by J. Vanderkam [24] for S^m , Nonnenmacher-Voros [14] for elliptic curves and Shiffman-Zelditch (to appear) for the general case using Levy concentration of measure estimates.

3.2. Integrable case

Results on *minimal growth* have been obtained by J. A. Toth and the author in the *quantum completely integrable* case, where $\sqrt{\Delta} = P_1$ commutes with $n-1$ first order pseudodifferential operators $P_2, \dots, P_n \in \Psi^1(M)$ ($n = \dim M$) satisfying $[P_i, P_j] = 0$ and whose symbols define a moment map $\mathcal{P} := (p_1, \dots, p_n)$ satisfying $dp_1 \wedge dp_2 \wedge \dots \wedge dp_n \neq 0$ on a dense open set $\Omega \subset T^*M - 0$. Since $\{p_i, p_j\} = 0$, the functions p_1, \dots, p_n generate a homogeneous Hamiltonian \mathbb{R}^n -action whose orbits foliate $T^*M - 0$. We refer to this foliation as the *Liouville foliation*.

We consider the L^p norms of the L^2 -normalized joint eigenfunctions $P_j \varphi_\lambda = \lambda_j \varphi_\lambda$. The spectrum of Δ often has bounded multiplicity, so the behaviour of joint eigenfunctions has implications for all eigenfunctions.

Theorem 5 [22, 23] *Suppose that the Laplacian Δ_g of (M, g) is quantum completely integrable and that the joint eigenfunctions have uniformly bounded L^∞ norms. Then (M, g) is a flat torus.*

This is a kind of quantum analogue of the ‘Hopf conjecture’ (proved by Burago-Ivanov) that metrics on tori without conjugate points are flat. In [23], a quantitative improvement is given under a further non-degeneracy assumption. Unless (M, g) is a flat torus, the Liouville foliation must possess a singular leaf of dimension $< n$. Let ℓ denote the minimum dimension of the leaves. We then construct a sequence of eigenfunctions satisfying:

$$\|\varphi_k\|_{L^\infty} \geq C(\epsilon) \lambda_k^{\frac{n-\ell}{4}-\epsilon}, \quad \|\varphi_k\|_{L^p} \geq C(\epsilon) \lambda_k^{\frac{(n-\ell)(p-2)}{4p}-\epsilon}, \quad (2 < p)$$

for any $\epsilon > 0$. It is easy to construct examples where $\ell = n - 1$, but it seems plausible that in ‘many’ cases $\ell = 1$. To investigate this, one would study the boundary faces of the image $\mathcal{P}(T^*M - 0)$ of $T^*M - 0$ under a homogeneous moment map. For a related study in the case of torus actions, see Lerman-Shirokova [12].

3.3. Quantum ergodicity

Quantum ergodicity is concerned with the sums ($A \in \Psi^0(M)$):

$$S_p(\lambda) = \sum_{\nu: \lambda_\nu \leq \lambda} |\langle A \varphi_\nu, \varphi_\nu \rangle - \omega(A)|^p, \quad \omega(A) = \frac{1}{\text{Vol}(S^*M)} \int_{S^*M} \sigma_A d\mu. \quad (3.2)$$

In work of A.I. Schnirelman [11], Colin de Verdiere and the author [27], it is shown that $S_p(\lambda) = o(N(\lambda))$ if G^t is ergodic. In the author’s view [27], this is best viewed as a convexity theorem. We mention briefly some new results.

In work of Gerard-Leichtnam [7] and Zelditch-Zworski [30], the ergodicity result was extended to domains $\partial\Omega$ with piecewise smooth boundary and ergodic billiard flow. Since the billiard map on $B^*\partial\Omega$ is ergodic whenever the billiard flow is, suitable boundary values of ergodic eigenfunctions (e.g. $\varphi_k|_{\partial\Omega}$ in the Neumann case or $\partial_\nu \varphi_k|_{\partial\Omega}$ in the Dirichlet case) should also have the ergodic property. This was conjectured by S. Ozawa in 1993. A proof is given in our work with A. Hassell [8] for convex piecewise smooth domains with ergodic billiards (in the case of domains with Lipschitz normal and with Dirichlet boundary conditions, this had earlier been proved in [7] by a different method).

Little is known about the rate of decay. For sufficiently chaotic systems (satisfying the central limit theorem) one can get the tiny improvement $S_p(\lambda) = O(N(\lambda)/(\log \lambda)^p)$ [28]. The asymptotics $S_2(\lambda) \sim B(A)\lambda$ have recently been obtained by Luo-Sarnak [13] for Hecke eigenfunctions of the modular group, exploiting the connections with L -functions. These asymptotics (though not the coefficient)

are predicted by the random polynomial model. Other strong bounds in the arithmetic case were obtained by Kurlberg-Rudnick for eigenfunctions of certain quantized torus automorphisms [10]. Bourgain-Lindenstrauss [3] and Wolpert [25] have developed the 'non-scarring' result of [16] to give entropy estimates of possible quantum limit measures in arithmetic cases.

A natural problem is the converse:

- **Problem 6:** What can be said of the dynamics if $S_p(\lambda) = o(N(\lambda))$? Does quantum ergodicity imply classical ergodicity?

It is known that classical ergodicity is equivalent to this bound plus estimates on off-diagonal terms [21]. The existence of KAM quasimodes (due to Lazutkin [11], Colin de Verdiere [5], and Popov [15]) makes it very likely that KAM systems are *not* quantum ergodic, nor are (M, g) which have stable elliptic orbits.

A further problem which may be accessible is:

- **Problem 7:** How are the nodal sets $\{\varphi_\nu = 0\}$ distributed in the limit $\nu \rightarrow \infty$?

In [14] (for elliptic curves) and [19] (general Kähler manifolds) it is proved that the *complex zeros* of quantum ergodic eigenfunctions become uniformly distributed relative to the volume form. Can one prove an analogue for the real zeros?

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