Gaussian random polynomials and analytic functions

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Focus of talk

- Geometric aspects of random polynomials, sections, analytic functions of one variable : Gaussian measure induced from Hermitian metrics on a line bundle. Curvature influence on zeros. Bergman metrics and conformal (bi-holomorphic) invariance of zero point processes.

- Large N: large degree, fewnonmial number, inverse distance to boundary... Asymptotics problem: how do zeros and critical points behave as the "complexity" N of a random function increases?

- Szegö and Bergman kernels and their asymptotics.

Random polynomials of one variable

A polynomial of degree N in one complex variable is:

$$f(z) = \sum_{j=1}^{N} c_j z^j, \quad c_j \in \mathbb{C}$$

is specified by its coefficients $\{c_j\}$.

A 'random' polynomial is short for a probability measure P on the coefficients. Let

$$\begin{aligned} \mathcal{P}_N^{(1)} &= \{\textstyle\sum_{j=1}^{N} c_j z^j, \ (c_1, \ldots, c_N) \in \mathbb{C}^N\} \\ &\simeq \mathbb{C}^N. \end{aligned}$$

Endow $\mathbb{C}^N$ with probability measure dP.

We call $(\mathcal{P}_N^{(1)}, P)$ an 'ensemble' of random polynomials.

Kac polynomials

The simplest complex random polynomial is the 'Kac polynomial'

$$f(z) = \sum_{j=1}^{N} c_j z^j$$

where the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. Complex Gaussian:

$$\mathbf{E}\,(c_j) = 0 = E(c_j c_k), \quad E(c_j \bar{c}_k) = \delta_{jk}.$$

This defines a Gaussian measure γ_{KAC} on $\mathcal{P}_N^{(1)}$:

$$d\gamma_{KAC}(f) = e^{-|c|^2/2} dc.$$

Expected distribution of zeros

The distribution of zeros of a polynomial of degree N is the probability measure on $\mathbb{C}$ defined by

$$Z_f = \frac{1}{N} \sum_{z: f(z)=0} \delta_z,$$

where δ_z is the Dirac delta-function at z.

Definition: The expected distribution of zeros of random polynomials of degree N with measure P is the probability measure $\mathbf{E}_P Z_f$ on $\mathbb{C}$ defined by

$$\langle \mathbf{E}_P Z_f, \varphi \rangle = \int_{\mathcal{P}_N^{(1)}} \{ \frac{1}{N} \sum_{z: f(z)=0} \varphi(z) \} dP(f),$$

for $\varphi \in C_c(\mathbb{C})$.

How are zeros of complex Kac polynomials distributed?

Complex zeros concentrate in small annuli around the unit circle S^1. In the limit as the degree $N \to \infty$, the zeros asymptotically concentrate exactly on S^1:

Theorem **1 (Kac-Hammersley-Shepp-Vanderbei)** *The expected distribution of zeros of polynomials of degree N in the Kac ensemble has the asymptotics:*

$$\mathbf{E}^N_{KAC}(Z^N_f) \to \delta_{S^1} \quad \textit{as} \quad N \to \infty \ ,$$

$$\textit{where} \qquad (\delta_{S^1}, \varphi) := \frac{1}{2\pi} \int_{S^1} \varphi(e^{i\theta})\, d\theta.$$

Why the unit circle?

Do zeros of polynomials *really* tend to concentrate on S^1?

Answer: yes, for the polynomials which dominate the Kac measure $d\gamma^N_{KAC}$. (Obviously no for general polynomials)

The Kac-Hammersley-Shepp-Vanderbei measure γ^N_{KCA} weights polynomials with zeros near S^1 more than other polynomials.

It did this by an implicit choice of inner product on $\mathcal{P}_N^{(1)}$.

Gaussian measure and inner product

Choice of Gaussian measure on a vector space $\mathcal{H}$ = choice of inner product on $\mathcal{H}$.

The inner product induces an orthonormal basis $\{S_j\}$. The associated Gaussian measure dγ corresponds to random orthogonal sums

$$S = \sum_{j=1}^{d} c_j S_j,$$

where $\{c_j\}$ are independent complex normal random variables.

The inner product underlying the Kac measure on $\mathcal{P}_N^{(1)}$ makes the basis $\{z^j\}$ orthonormal. Namely, they were orthonormalized on S^1. And that is where the zeros concentrated.

Gaussian random polynomials adapted to domains

If we orthonormalize polynomials on the boundary $\partial\Omega$ of any simply connected, bounded domain $\Omega \subset \mathbb{C}$, the zeros of the associated random polynomials concentrate on $\partial\Omega$.

I.e. define the inner product on $\mathcal{P}_N^{(1)}$ by

$$\langle f, \bar{g} \rangle_{\partial\Omega} := \int_{\partial\Omega} f(z)\overline{g(z)}\, |dz| \ .$$

Let $\gamma_{\partial\Omega}^N =$ the Gaussian measure induced by $\langle f, \bar{g} \rangle_{\partial\Omega}$ and say that the Gaussian measure is adapted to Ω.

How do zeros of random polynomials adapted to Ω concentrate?

Equilibrium distribution of zeros

Denote the expectation relative to the ensemble $(\mathcal{P}_N, \gamma^N_{\partial\Omega})$ by $\mathbf{E}^N_{\partial\Omega}$.

Theorem **2**

$$\mathbf{E}^N_{\partial\Omega}(Z^N_f) = \nu_\Omega + O\left(1/N\right) \ ,$$

where ν_Ω is the equilibrium measure of $\bar{\Omega}$.

The equilibrium measure of a compact set K is the unique probability measure $d\nu_K$ which minimizes the energy

$$E(\mu) = -\int_K \int_K \log|z-w|\, d\mu(z)\, d\mu(w).$$

Thus, in the limit as the degree $N \to \infty$, random polynomials adapted to Ω act like electric charges in Ω.

Szegö kernels

Let $\Omega \subset \mathbb{C}$ be a smooth bounded domain. The Szegö kernel of Ω with respect to a measure ρds on $\partial\Omega$ is the orthogonal projection

$$(1) \qquad S : \mathcal{L}^2(\partial\Omega, \rho ds) \to \mathcal{H}^2(\partial\Omega, \rho ds)$$

onto the Hardy space of boundary values of holomorphic functions in Ω which belong to $\mathcal{L}^2(\partial\Omega, ds)$. The Schwartz kernel of S is denoted $S(z, w)$.

Szegö kernels in terms of orthonormal basis Let

$$\{P_j(z) = a_{j0} + a_{j1}z + \cdots + a_{jj}z^j\}$$

be the orthonormal basis of orthogonal polynomials for $\mathcal{L}^2(\partial\Omega, \rho ds)$ obtained by applying Gram-Schmidt to $\{1, z, z^2, \ldots, z^j, \ldots\}$. We have

$$(2) \quad S(z,w) = \sum_{k=0}^{\infty} P_k(z)\overline{P_k(w)}, \quad (z,w) \in \overline{\Omega} \times \overline{\Omega}$$

$S(z,z) < \infty$ for $z \in \Omega$, and thus $P_N \to 0$ on Ω. Hence,

$S_N(z,z) \to S(z,z)$, uniformly on compact subsets of

where $S_N(z,w) = \sum_{k=0}^{N} P_k(z)\overline{P_k(w)}$ is the partial Szegö kernel

Szegö kernel of unit disc

In the case of the unit disc U (with the weight $\rho \equiv 1$), one has:

$$S^U(z,w) = \frac{1}{2\pi(1 - z\bar{w})}.$$

The orthogonal polynomials are $P_k(z) = z^k$, hence
(3)

$$|P_N(z)|^2 = |z|^{2N}, \quad S_N^U(z,z) = \sum_{k=0}^{N-1} |z|^{2k} = \frac{1-|z|^{2N}}{1-|z|^2}.$$

Clearly, $S_N(z,z) \to \infty$ at an exponential rate in the exterior of Ω.

Bergman kernels

When the inner product is chosen to be $\langle f, \bar{g}\rangle_\Omega = \int_\Omega f\bar{g}\,dx\,dy$, the role of the Szegö kernel is played by the Bergman kernel, i.e. the orthogonal projection from $\mathcal{L}^2(\Omega)$ onto the subspace $\mathcal{H}^2(\Omega)$ spanned by the $\mathcal{L}^2$ holomorphic functions. We denote by

$$\{P_j(z) = a_{j0} + a_{j1}z + \cdots + a_{jj}z^j\}$$

the orthonormal basis of orthogonal polynomials for $\mathcal{L}^2(\Omega, dxdy)$ with positive leading coefficient. The Bergman kernel may be expressed in terms of the orthogonal polynomials by:
(4)

$$B(z,w) = \sum_{k=0}^{\infty} P_k(z)\overline{P_k(w)}\,, \qquad (z,w) \in \Omega \times \Omega.$$

We let $B_N(z,w) = \sum_{k=0}^{N} P_k(z)\overline{P_k(w)}$; as in the case of the Szegö kernel, we have

$$B_n(z,w) \to B(z,w)\,, \qquad B(z,z) > 0\,.$$

Gaussian random analytic functions adapted to domains

We can use the same inner product

$$\langle f, \bar{g} \rangle_{\partial\Omega} := \int_{\partial\Omega} f(z)\overline{g(z)}\, |dz|$$

on $H^2(\Omega)$ to define an infinite dimensional Gaussian ensemble of random analytic functions. We denote the Gaussian measure by $\gamma_{\partial\Omega}$.

We will come back to this later.

Zero distribution and Szegö /Bergman kernels

For convenience, we let

$$Z_f^N = \sum_{f(z)=0} \delta_z$$

denote the zero distribution of a polynomial f.

Normalize to $\widetilde{Z}_f = \frac{1}{N} Z_f$.

Proposition **3** *We have*

$$\mathbf{E}^N_{\partial\Omega,\rho}(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log S_N(z,z).$$

Proof

Since

$$Z_f = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log |f|^2,$$

we have

$$\mathbf{E}^N_{\partial\Omega,\rho}(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \mathbf{E}^N_{\partial\Omega,\rho}\left(\log |f|^2\right)\ .$$

Write f in terms of ONB basis $\{P_j\}$ of $\mathcal{P}_N$:

$$f(z) = \sum_{j=0}^{N} a_j P_j(z) = \langle a, p(z)\rangle\ ,$$

where $a = (a_0, \ldots, a_N)$, $P = (P_0, \ldots, P_N)$. Then,

$$\mathbf{E}^N_{\partial\Omega,\rho}(Z_f) = \partial\bar\partial \int_{\mathbb{C}^{N+1}} \log |\langle a, P(z)\rangle| \frac{1}{\pi^{N+1}} e^{-\|a\|^2}\, da.$$

We write

$$P(z) = \|P(z)\| u(z), \quad \|P(z)\|^2 = \textstyle\sum_{j=0}^N |P_j(z)|^2$$

$$= S_N(z,z), \quad \|u(z)\| = 1\ .$$

Proof

Then,

$$\log |\langle a, P(z)\rangle| = \log \|P(z)\| + \log |\langle a, u(z)\rangle| \,.$$

We observe that

$$\int_{\mathbb{C}^{N+1}} \log |\langle a, u(z)\rangle| \, e^{-\|a\|^2} \, da = \text{constant}$$

since for each z we may apply a unitary coordinate change so that $u(z) = (1, 0, \ldots, 0)$. Hence the derivative equals zero, and we have

$$\mathbf{E}^N_{\partial\Omega,\rho}(Z_f) = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial} \log \|P(z)\| = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log S_N(z, z)$$

By exactly the same argument, we also have:

Proposition **4** *We have*

$$\mathbf{E}\,^N_{\Omega}(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log B_N(z, z).$$

Sketch of proof of equilibrium distribution of zeros

The main point of the proof is to gain control over asymptotics of the partial Szegö and Bergman kernels. This is often the key technical step in geometric asymptotics of zeros/critical points.

In the present problem on planar domains, asymptotics of Szegö kernels is very different from the case of line bundles. It is mainly due to Szegö and Carleman, and involves orthogonal polynomials, Faber polynomials, Riemann mapping functions... We only sketch a bit of it.

Equilibrium measure and exterior Riemann mapping function

Let $\Omega \subset \mathbb{C}$ denote a simply connected bounded plane domain with C^∞ boundary $\partial\Omega$. The exterior domain $\widehat{\mathbb{C}} \setminus \overline{\Omega}$ is also a simply connected domain in $\widehat{\mathbb{C}}$ and we denote by

(5)

$$\Phi : \widehat{\mathbb{C}} \backslash \Omega \to \widehat{\mathbb{C}} \backslash U \,, \qquad \Phi(z) = cz + c_0 + c_1 z^{-1} + \cdots$$

the (unique) exterior Riemann mapping function with $\Phi(\infty) = \infty$, $\Phi'(\infty) \in \mathbb{R}^+$. We recall that the equilibrium measure ν_Ω of Ω is given

$$\nu_\Omega = \Phi^*(\frac{d\theta}{2\pi}).$$

Exterior Riemann mapping function and Szegö kernel Recalling that

$$\Phi_N^* S_N^U(z,z) = \sum_{n=0}^{N} |\Phi(z)|^{2n} ,$$

it follows that

$$S_N(z,z) = A_N(z)\,\Phi_N^* S_N^U(z,z) ,$$

where

$$0 < x < \inf_{z\in W_\varepsilon} A_N(z) < \sup_{z\in W_\varepsilon} A_N(z) \le C < +\infty$$

with

$$W_\varepsilon := (\widehat{\mathbb{C}}\backslash\Omega) \cup T_\varepsilon(\partial\Omega) ,$$

The full Szegö kernel has a nice transformation law with respect to the interior Riemann mapping function. The truncated (degree N) has only a partial transformation law with respect to the exterior Riemann mapping function.

Exterior Riemann mapping function and Szegö kernel

We now prove that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log S_N(z,z) = N\nu_\Omega + O(1)\ .$$

In the interior of Ω, $S_N(z,z) \to S(z,z)$ uniformly in compact subsets of Ω. Furthermore, $S(z,z) > 0$ on Ω and hence $\log S_N(z,z) \to \log S(z,z)$. Thus

$$\partial\bar{\partial}\log S_N(z,z) = \partial\bar{\partial}\log S(z,z) + o(1) = O(1) \quad \text{in } \mathcal{D}'($$

Using $\nu_\Omega = \Phi^*(\frac{d\theta}{2\pi})$ and the transformation law for the interior Szegö kernel reduces to the case of $\Omega = U$, where it is easy to check.

Exterior of Ω

On

$$W_\varepsilon := (\widehat{\mathbb{C}} \setminus \Omega) \cup T_\varepsilon(\partial\Omega) \ ,$$

for small ε we use that

$$\tfrac{i}{2\pi}(\Phi^*\partial\bar\partial \log S_N^U, \varphi) = \tfrac{i}{2\pi}(\partial\bar\partial \log S_N^U, \Phi^*\varphi)$$

$$= (\nu, \Phi^*\varphi) + O(1) = (\nu_\Omega, \varphi) + O(1) \ ,$$

and it suffices to show that $(u_N, \partial\bar\partial\varphi) = O(1)$ where

$$u_N := \log S_N - \Phi^* \log S_N^U = \log S_N - \log \frac{1 - |\Phi|^{2N+2}}{1 - |\Phi|^2} \ .$$

Exterior of Ω

Recalling that

$$\Phi_N^* S_N^U(z,z) = \sum_{n=0}^{N} |\Phi(z)|^{2n} \,,$$

and

$$S_N(z,z) = A_N(z)\, \Phi_N^* S_N^U(z,z) \,, \qquad \sup_{z\in W_\varepsilon} A_N(z) \leq C <$$

we find that the functions $u_N = -\log A_N$ are uniformly bounded on W_ε and hence $(u_N, \partial\bar{\partial}\varphi) = O(1)$. This completes the proof.

$SU(2)$ polynomials

Is there an inner product in which the expected distribution of zeros is 'uniform' on $\mathbb{C}$, i.e. doesn't concentrate anywhere? Yes, if we take 'uniform' to mean uniform on $\mathbb{CP}^1$ w.r.t. Fubini-Study area form ω_{FS}.

We define an inner product on $\mathcal{P}_N^{(1)}$ which depends on N:

$$\langle z^j, z^k \rangle_N = \frac{1}{\binom{N}{j}} \delta_{jk}.$$

Thus, a random $SU(2)$ polynomial has the form

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \overline{\lambda}_\beta) = \delta_{\alpha\beta}.$$

Proposition **5** *In the* $SU(2)$ *ensemble,* $\mathbf{E}\,(Z_f) = \omega_{FS}$*, the Fubini-Study area form on* $\mathbb{CP}^1$*.*

$SU(2)$ and holomorphic line bundles

Proof that $\mathbf{E}(Z_f) = \omega_{FS}$ is trivial if we make right identifications:

- $\mathcal{P}_N^{(1)} \simeq H^0(\mathbb{CP}^1, \mathcal{O}(N))$ where $\mathcal{O}(N) = N$th power of the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{CP}^1$.Indeed, $\mathcal{P}_N^{(1)} \iff$ homogeneous polynomials $F(z_0, z_1)$ of degree N: homogenize $f(z) \in \mathcal{P}_N^{(1)}$ to $F(z_0, z_1) = z_0^N f(z_1/z_0)$. Also $H^0(\mathbb{CP}^1, \mathcal{O}(N)) \iff$ homogeneous polynomials $F(z_0, z_1)$ of degree N.

- Fubini-Study inner product on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ $=$ inner product $\int_{S^3} |F(z_0, z_1)|^2 dV$ on the homogeneous polynomials.

- The inner product and Gaussian ensemble are thus $SU(2)$ invariant. Hence, $\mathbf{E}\, Z_f$ is $SU(2)$-invariant.

Gaussian random holomorphic sections of line bundles

The $SU(2)$ ensemble generalizes to all dimensions, and moreover to any positive holomorphic line bundle $L \to M$ over any Kähler manifold.

We endow L with a Hermitian metric h and M with a volume form dV. We define an inner product

$$\langle s_1, s_2 \rangle = \int_M h(s_1(z), s_2(z)) dV(z).$$

We let $\{S_j\}$ denote an orthonormla basis of the space $H^0(M, L)$ of holomorphic sections of L.

Then define Gaussian holomorphic sections $s \in H^0(M, L)$ by

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$.

Gaussian analytic functions

Before looking at general line bundles $L \to M$ let us consider Gaussian analytic functions on domains $\Omega \subset \mathbb{C}$ from a geometric viewpoint.

One goal is to understand when the zero point process of a random analytic function is conformally invariant.

Under a holomorphic map $T : M \to N$, $T(Z_s)$ becomes a point process on N. The process is said to be holomorphically invariant (or conformally invariant in dimension two) if the distribution of $T(Z_s)$ is the same as the distribution of Z_s.

Bergman metric

In the language of line bundles and hermitian metrics, to obtain a conformally (bi-holomorphically) invariant ensemble, we need to introduce line bundles and hermitian metrics which are canonically determined by the complex structure.

All the canonical metrics and bundles are derived from the Bergman metric.

Bergman metric cont.

The Bergman space of a domain $\Omega \subset \mathbb{C}$ is the natural Hilbert space $B^2(\Omega)$ of holomorphic square integrable $H^{1,0}$ forms $f dz$ with the inner product $\langle f dz, g dz \rangle = \int_\Omega f(z)\overline{g(z)} dz \wedge d\bar{z}$. The orthogonal projection from L^2 $(1,0)$ forms to $H^2(\Omega)$ is the double one form Bergman kernel

$$B(z,w) dz \otimes d\bar{w} = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(w)} dz \otimes d\bar{w}.$$

Let B be the unit ball in $\mathbb{C}$. Then

$$B_D(z,w) = \frac{1}{\pi}(1 - z\bar{w})^{-2} dz d\bar{w}.$$

Bergman Gaussian analytic forms We let $\{\varphi_n\}$ denote an orthonormal basis of $B^2(\Omega)$ and define the Bergman Gaussian analytic function by

$$f(z) = \sum_{n=1}^{\infty} a_n \varphi_n(z)$$

where the coefficients are independent Gaussian random variables with zero mean and unit variance.

In a standard way, the covariance kernel of such L^2 holomorphic functions is given by the diagonal part of the Bergman kernel:

$$B(z,w) = \mathbf{E}(f(z)\overline{f(w)} = \sum_{n=1}^{\infty} \varphi_n(z)\overline{\varphi_n(w)}.$$

Bergman Gaussian analytic forms

Let $\mathcal{C}^+$ denote the cone of positive currents on a complex manifold M. Then we have a map

$$f \to Z_f := \partial\bar{\partial} \log |f|^2, \;\; H^2(\Omega) \to \mathcal{C}^+(\Omega).$$

We can push forward the Gaussian measure to obtain a probability measure on the space of positive currents.

Let $F : \Omega_1 \to \Omega_2$ be a bi-holomorphic map. Then F induces a map $F^* : \mathcal{C}(\Omega_2) \to \mathcal{C}(\Omega_1)$ be pullback. This map preserves positivity (holomorphic maps preserve pluri-subharmonic functions).

Definition: The Bergman zero point process is the probability measure on $\mathcal{C}^+(\Omega)$ induced by the Bergman Gaussian measure on $H^{m,0}(\Omega)$.

Conformal invariance of Bergman zero point process

Definition: *We say that a current process is bi-holomorphically invariant if the probability measure is invariant under bi-holomorphic maps.*

Proposition **6** *The Bergman point process for $\Omega \subset \mathbb{C}$ is conformally invariant.*

Let $T : \Omega_1 \to \Omega_2$ be a conformal diffeomorphism or biholomorphic map. Then $T^* : H^2(\Omega_2) \to H^2(\Omega_1)$ is an isometry of Hilbert spaces. That is,

$$\begin{aligned} \langle T^* f, T^* g \rangle &= \int_{\Omega_1} T^*(f dz) \wedge \overline{T^* g dz} \\ &= \int_{\Omega_2} f(w) dw \wedge \overline{g(w) dw}. \end{aligned}$$

Therefore, T is a measure preserving transformation between the Gaussian spaces of random analytic functions.

Since $T^*(\partial\bar{\partial} \log |f|^2) = \partial\bar{\partial} \log |T^* f|^2$ it follows that T^* induces a measure preserving transformation on the corresponding spaces of currents.

The only point to check here is that in defining the map from the Bergman space to currents, we used the special Euclidean coordinate system and dropped the coordinate differentials. Since the change of variables has nowhere vanishing Jacobian, it will not change the current $\partial\bar{\partial} \log |f|^2$ and hence the map from holomorphic forms to currents is coordinate independent. Similarly for all other maps.

Sodin-Tsirelson-Peres-Virag point processes

We now consider the Sodin-Tsirelson-Peres-Virag point processes. They denote by $Z_{U,\rho}$ the zero set of

$$f = \sum_{n=0}^{\infty} \binom{-\rho}{n}^{\frac{1}{2}} a_n z^n$$

on the unit disc where $\{a_n\}$ are i.i.d. standard complex Gaussians. It is observed that $Z_{U,\rho}$ is invariant under Möbius transformations preserving U and that they are the only zero sets of Gaussian analytic functions with this invariance property. When $\rho = 1$ Peres-Virag proved that the point process is determinantal. The expected distribution of zeros is $\frac{\rho}{\pi(1-|z|^2)^2}$.

Sodin-Tsirelson-Peres-Virag point processes

They define Szegö random functions with parameter ρ as above and find that its covariance kernel in the case of the unit disc is $[S_U(z,w)]^\rho$.

The random power series with $\rho = 1$ is

$$f_U(z) = \sum_{n=0}^{\infty} a_n z^n$$

with $\{a_n\}$ i.i.d. standard normal. The two point function is $e^{-\frac{|z|^2}{\pi}}$. The radius of converge is almost surely one and one obtains a zero point process in the unit disc.

Conformal invariance of Sodin-Tsirelson Peres-Virag point processes

It is obvious when we reinterpret $Z_{U,\rho}$ as zeros of Gaussian random holomorphic sections of powers of a hermitian line bundle $L \to U$ where L and the hermitian metric are determined by the Bergman metric, viewed as a Kähler metric on U.

Proposition **7** *Let $T : \Omega_1 \to \Omega_2$ be bi-holomorphic. Then T^* induces a measure-preserving transformation between the zero current processes of the Gaussian Hilbert spaces $L^2 H^0(\Omega, L^\rho)$ above.*

Definitions

The Bergman Kähler form $\partial\bar{\partial}\log B(z,z)$ defines a Hermitian line bundle $(L,h) \to \Omega$ with associated Ricci curvature $\Theta_h = \partial\bar{\partial}\log B(z,z)$. The line bundle is trivial, so we may regard sections as functions; only the Hermitian metric is non-trivial. It equals $e^{\rho \log B(z,z)}$ on L^ρ.

Szegö kernels Π_N for $L^2 H^0(\Omega, L^N)$ have the form $C_N e^{N\psi(z,\bar{w})}$ where C_N is a normalizing constant and $\psi = \log K(z,\bar{w})$.

Example

In the case of the unit disc $D \subset \mathbb{C}$, the Bergman is $\frac{-i}{2}\partial\bar{\partial}\log(1-|z|^2)$, the space $L^2H^0(D, L^N)$ may be identified with the holomorphic functions square integrable with respect to the inner product

$$||f||_N^2 = \int_D |f(z)|^2(1-|z|^2)^{N-2}dz.$$

The factor $e^{N\log(1-|z|^2)}$ comes from the Hermitian metric. An orthonormal basis for the holomorphic sections of L^N is then given by the monomials $\sqrt{\binom{N+n-1}{n}}\, z^n$ $(n = 0, 1, 2, \dots)$. The Szegö kernels are given by $\Pi_N(z, w) = (1 - z\bar{w})^N$.

Notation

Sodin-Tsirelson define

$$\sum_{k=0}^{\infty} a_k \sqrt{\frac{L(L+1)\cdots(L+k-1)}{k!}} z^k$$

for the hyperbolic case and

$$\sum_{k=0}^{\infty} a_k \sqrt{\frac{L(L-1)\cdots(L-k+1)}{k!}} z^k$$

for the elliptic case.

Our ensemble above is the hyperbolic case and $N = L$. We note that Peres-Virag's $\binom{-\rho}{n}$ is the elliptic case with negative L. But also if one pulls out the factors of -1 it is the hyperbolic case with $N = \rho$. So the case $\rho = 1$ is just the first power of L.

Proof of conformal invariance

Because the unit disc is Kähler -Einstein, the curvature $(1,1)$ form of the Bergman Kähler metric is the Bergman metric again.

Proposition **8** *Let $T : \Omega_1 \to \Omega_2$ be bi-holomorphic. Then T^* induces a measure-preserving transformation between the zero current processes of the Gaussian Hilbert spaces $L^2H^0(\Omega, L^\rho)$ above.*

Proof:

T is an isometry of the Bergman metrics and therefore is isometric between the Hermitian line bundles. Hence it preserves the induced inner product on sections of these line bundles. By the previous argument it induces a measure preserving isomorphism on the zero currents of sections.

Observation about $\rho = 1$

If $\rho = 1$ corresponds to L then $\rho = 2$ corresponds to $H^{1,0}$. Thus, $L = \sqrt{H^{1,0}}$, i.e. is the bundle of forms $f\sqrt{dz}$. These indeed have a natural inner product over the boundary. This is the bundle of half-forms (spinors). Choosing a square root is choosing a spin structure. It is again natural that this bundle and its inner product are conformally invariant.

It is not clear why the pair correlation function of zeros of random sections of this bundle should be the same as the two-point function of Bergman random forms, nor why the point process should be determinantal.

Distribution of zeros of Gaussian random $s \in H^0(M, L^N)$

We now consider holomorphic sections of positive line bundles over Riemann surfaces. We have already seen that polynomials of degree N are the same as holomorphic sections of $\mathcal{O}(N) \to \mathbb{CP}^1$. Further,

- $M = T^2 = \mathbb{C}/\mathbb{Z}^2$: the line bundle L with curvature $dz \wedge d\bar{z}$ is the bundle for which $H^0(T^2, L^N)$ is the space of theta functions of level N;

- $M = \mathcal{H}/\Gamma$, a hyperbolic surface. Then with the hyperbolic area form, L^N is the bundle of differentials of type dz^N, i.e. $H^0(M, L^N)$ is the space of holomorphic differentials $f dz^N$.

Expected distribution of zeros

The expected distribution of zeros of a random holomorphic section is given by the following

Theorem 1 (Shiffman-Z) We have:

$$\frac{1}{(N)}\mathbf{E}_N(Z_f \to \omega$$

in the sense of weak convergence; i.e., for any open $U \subset \mathbb{C}^{*m}$, we have

$$\tfrac{1}{(N)}\mathbf{E}_N\Big(\#\{z \in U : f(z) = 0\}\Big)$$

$$\to \mathrm{Vol}_\omega(U)\ .$$

Zeros concentrate in curved regions. Curvature causes sections to oscillate and hence zeros to occur.

Almost sure distribution of zeros

The distribution of zeros is 'self-averaging': typical sections behave in the expected way. To prove this, we define the space of sequences of sections as the Cartesian product probability space

$$\Pi_{N=1}^{\infty} H^0(M, L^N), \quad \gamma_\infty := \Pi_{N=1}^{\infty} d\gamma_N.$$

THEOREM. (S–Zelditch, 1998) Consider a *random sequence* $\{f_N\}$ of sections of L^N (or polynomials of degree N), $N = 1, 2, 3, \ldots$. Then

$$\frac{1}{N} Z_{f_N} \to \omega \qquad \textit{almost surely}\, w.r.t. \gamma_\infty.$$

Critical points of Gaussian random holomorphic sections

We now turn to metric critical points, where geometry dominates even more. The setting is

- A holomorphic line bundle $L \to M$;
- A hermitian metric h on L;
- The Chern connection ∇_h of h;
- The curvature Θ_h of ∇_h.
- An inner product $\langle , \rangle$ on the space $H^0(M, L)$ of holomorphic sections (or on a subspace).
- The Gaussian measure γ associated to $\langle , \rangle$.

Metrics, connections, curvature

A Hermitian metric on L is a family of h_z of hermitian inner products on the lines L_z over $z \in M$. In a local frame $e(z)$, h_z is specified by the positive function $h(z) = ||e(z)||_h$.

Definition: the metric (Chern) connection $\nabla = \nabla_h$ of h is the unique connection preserving the metric h and satisfying $\nabla'' s = 0$ for any holomorphic section s. Here, $\nabla = \nabla' + \nabla''$ is the splitting of the connection into its $L \otimes T^{*1,0}$ resp. $L \otimes T^{*0,1}$ parts.

We denote by Θ_h the curvature of h:

$$\Theta_h = \partial\bar{\partial}K, \qquad K = -\log h.$$

Positive/negative line bundles

In a local frame e, the hermitian metric is a positive function $h(z) = ||e||_z$.

The curvature form is defined locally by

$$\Theta_h = \partial\bar{\partial} K, \qquad K = -\log h.$$

The bundle is called positive (resp. negative) if Θ_h is a positive (resp. negative) $(1,1)$ form.

Given one positive metric h_0 on L, the other metrics have the form $h_\varphi = e^\varphi h$ and $\Theta_h = \Theta_{h_0} - \partial\bar{\partial}\varphi$, with $\varphi \in C^\infty(M)$.

Critical point

Definition: *Let* $(L,h) \to M$ *be a Hermitian holomorphic line bundle over a complex manifold* M*, and let* $\nabla = \nabla_h$ *be its Chern connection.*

A critical point of a holomorphic section $s \in H^0(M,L)$ *is defined to be a point* $z \in M$ *where* $\nabla s(z) = 0$*, or equivalently,* $\nabla' s(z) = 0$*.*

We denote the set of critical points of s *with respect to the Chern connection* ∇ *of a Hermitian metric* h *by* $Crit(s,h)$*.*

Critical points depend on the metric

The set of critical points $Crit(s, h)$ of s, and even its number $\#Crit(s, h)$, depends on ∇_h or equivalently on the metric h.

In a local frame e critical point equation for $s = fe$ reads:

$$\partial f + f \partial K = 0.$$

Recall that $K = -\log h$.

The critical point equation is only C^∞ and not holomorphic since K is not holomorphic.

Hence $Crit(s, h)$ is not a topological invariant of L, unlike (say) the number of zeros in dimension one. It is a non-trivial random variable.

An equivalent definition of critical point

An essentially equivalent definition: $w \in Crit(s, h)$ if

$$d|s(w)|_h^2 = 0. \tag{6}$$

Since

$$d|s(w)|_h^2 = 0 \iff 0 = \partial|s(w)|_h^2 = h_w(\nabla' s(w), s(w))$$

it follows that $\nabla' s(w) = 0$ as long as $s(w) \neq 0$. So this notion of critical point is the union of the zeros and critical points.

The Morse theory of connection critical points $\nabla s(w) = 0$ is equivalent to the Morse theory of $|s(w)|_h^2$.

Hermitian Gaussian random holomorphic sections

A Gaussian measure γ on $H^0(M, L)$ is induced by an inner product on $H^0(M, L)$. The simplest are the *Hermitian Gaussian measures* induced by a hermitian metric h on L:

$$(7) \qquad \langle s_1, s_2 \rangle_h = \int_M h(s_1(z), s_2(z)) dV(z)$$

on $H^0(M, L)$, where $dV = \frac{\Theta_h^m}{m!}$. By definition,

$$(8) \qquad d\gamma(s) = \frac{1}{\pi^d} e^{-\|c\|^2} dc\,, \qquad s = \sum_{j=1}^{d} c_j e_j,$$

where dc is Lebesgue measure and $\{e_j\}$ is an orthonormal basis basis. We denote the expected value of a random variable X on with respect to γ by $\mathbf{E}_\gamma$.

Statistics of critical points I: density

The distribution of critical points of a fixed section s with respect to h (or ∇_h) is the measure

$$C_s^h := \sum_{z \in Crit(s,h)} \delta_z, \tag{9}$$

where δ_z is the Dirac point mass at z.

Definition: The (expected) density of critical points of $s \in \mathcal{S} \subset H^0(M, L)$ with respect to h and a Gaussian measure γ is defined by

$$K^{\text{crit}}(z)\, dV(z) = \mathbf{E}_\gamma C_s^h\,,$$

i.e.,

$$\int_M \varphi(z) K^{\text{crit}}(z)\, dV(z) = \int_{\mathcal{S}} \left[\sum_{z : \nabla_h s(z) = 0} \varphi(z) \right] d\gamma(s).$$

Expected number of critical points

Definition: The expected number of critical points of a Gaussian random section is defined by

$$\begin{aligned} \mathcal{N}^{crit}(h,\gamma) &= \int_M K^{\mathrm{crit}}(z)\, dV(z) \\ &= \int_{\mathcal{S}} \# Crit(s,h) d\gamma(s). \end{aligned}$$

For Hermitian Gaussian measures, where γ comes from the inner product $\langle , \rangle_h$, $\mathcal{N}^{crit}(h,\gamma)$ is a purely metric invariant of a line bundle.

Positive/Negative line bundles

Corollary **9** *Let* $(L, h) \to M$ *denote a positive or negative holomorphic line bundle. Give* M *the volume form* $dV = \frac{1}{m!}\left(\pm\frac{i}{2}\Theta_h\right)^m$ *induced from the curvature of* L*. Then*

$$K^{\mathrm{crit}}_{h,\mathcal{S}}(z) = \frac{1}{\det A \det \Lambda} \int_{\mathrm{Sym}(m,\mathbb{C})\times\mathbb{C}} \left|\det(H'H'^* - |x|^2 I)\right.$$

$$\left.\right| e^{-\langle \Lambda(z)^{-1}(H',x),(H',x)\rangle}\, dH'\, dx\,.$$

Here, $H' \in Sym(m, \mathbb{C})$ is a complex symmetric matrix, and the matrix Λ is a Hermitian operator on the complex vector space $Sym(m, \mathbb{C}) \times \mathbb{C}$.

Formulae for $A(z)$ and $\Lambda(z)$

$A(z)$ and $\Lambda(z)$ depend only on ∇ and on the Szegö kernel, i.e. orthogonal projection

$$\Pi_{\mathcal{S}} : L^2(M,L) \to \mathcal{S} \subset H^0(M,L),$$

for $\mathcal{S}$ and for the inner product. Let $F_{\mathcal{S}}(z,w)$ be the local expression for $\Pi_{\mathcal{S}}(z,w)$ in the frame e_L. Then $\Lambda = C - B^* A^{-1} B$, where

$$\begin{aligned}
&A = \left(\frac{\partial^2}{\partial z_j \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w)|_{z=w} \right), \\
&B = \left[\left(\frac{\partial^3}{\partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}}) F_{\mathcal{S}}|_{z=w} \right) \quad \left((\frac{\partial}{\partial z_j} F_{\mathcal{S}}|_{z=w} \right) \right], \\
&C = \begin{bmatrix} \left(\frac{\partial^4}{\partial z_q \partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}|_{z=w} \right) & \left(\frac{\partial^2}{\partial z_j \partial z_q} F_{\mathcal{S}} \right) \\ \left(\frac{\partial^2}{\partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}} \right) |_{z=w} & F_{\mathcal{S}}(z,z) \end{bmatrix}, \\
&1 \le j \le m\,, 1 \le j \le q \le m\,, 1 \le j' \le q' \le m\,.
\end{aligned}$$

In the above, A, B, C are $m \times m$, $m \times n$, $n \times n$ matrices, respectively, where $n = \frac{1}{2}(m^2 + m + 2)$.

Geometric problem

In the case of a positive line bundle, we now have an explicit formula for the expected number $\mathcal{N}^{\text{crit}}(h)$ of critical points of a Gaussian random holomorphic section relative to the Hermitian Gaussian measure and the full space $H^0(M, L)$:

$$\mathcal{N}^{\text{crit}}(h) = \int_M \{ \frac{1}{\det A \det \Lambda} \int_{\text{Sym}(m,\mathbb{C})\times\mathbb{C}}$$

$$\left| \det(H'H'^* - |x|^2 I) \right| e^{-\langle \Lambda(z)^{-1}(H',x),(H',x)\rangle} \, dH' \, dx \} dV_h \, .$$

Here, Λ, A depend only on the Szegö kernel for $H^0(M, L)$.

$\mathcal{N}^{\text{crit}}(h)$ is a *purely metric* invariant of (L, h). Is it a topological invariant, or does it depend on the metric? If the latter, how?

Universal limit theorem

We now give an asymptotic expansion for $K_N^{\text{crit}}(z)$:

Theorem **10** *For any positive Hermitian line bundle $(L,h) \to (M,\omega)$ over any compact Kähler manifold, the critical point density relative to the curvature volume form has an asymptotic expansion of the form*

$$N^{-m} K_N^{\text{crit}}(z) \sim \Gamma_m^{\text{crit}} + a_1(z) N^{-1} + a_2(z) N^{-2} + \cdots ,$$

where Γ_m^{crit} is a universal constant depending only on the dimension m of M. Hence the expected total number of critical points on M is

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!} \Gamma_m^{\text{crit}}\, c_1(L)^m\, N^m + O(N^{m-1}) .$$

The leading constant in the expansion is given by the integral formula

$$\Gamma_m^{\text{crit}} = \left(2\pi^{\frac{m+3}{2}}\right)^{-m} \int_0^{+\infty} \int_{\text{Sym}(m,\mathbb{C})} |\det(SS^* - tI)|$$

$$e^{-\frac{1}{2}\|S\|^2 - t}\, dS\, dt ,$$

Comments

- The leading order term being constant, critical points are uniformly distributed relative to the curvature volume form in the $N \to \infty$ limit. Curvature causes sections to oscillate more rapidly, so critical points concentrate where the curvature concentrates.

- Universality of the leading term is not so surprising, since it is a local calculation.

- As we will see, the leading order constant is larger than 1, so positive curvature causes polynomials of degree N to have substantially more critical points than in the classical flat sense of $dF = 0$.

Riemann surfaces In the case of Riemann surfaces, we can explicitly evaluate the leading coefficient:

Corollary **11** *For the case where M is a Riemann surface, we have $\Gamma_1^{\text{crit}} = \frac{5}{3\pi}$, and hence the expected number of critical points is $\mathcal{N}(h^N) = \frac{5}{3}c_1(L)N + O(\sqrt{N})$. The expected number of saddle points is $\frac{4}{3}N$ while the expected number of local maxima is $\frac{1}{3}N$.*

There are $\sim N$ critical points of a polynomial of degree N in the classical sense, all of which are saddle points. There are an extra $\frac{1}{3}N$ saddles cancelled by an extra $\frac{1}{3}N$ local maxima.

Density of critical points on Riemann surfaces

For Riemann surfaces, we can give a simple explicit formula for the density of critical points. We measure the critical point density with respect to the volume form $\pm\frac{i}{2}\Theta_h$. Put:

$$Q = \begin{pmatrix} -1 & 0 \\ \\ 0 & 1 \end{pmatrix}$$

and denote the eigenvalues of $\Lambda(z)Q$ by μ_1, μ_2. We observe that μ_1, μ_2 have opposite signs since $\det Q\Lambda = -\det\Lambda < 0$. Let $\mu_2 < 0 < \mu_1$.

Theorem **12** *let* $(L, h) \to M$ *be a positive or negative Hermitian line bundle on a (possibly non-compact) Riemann surface* M *with volume form* $dV = \pm\frac{i}{2}\Theta_h$*. Then:*

$$K_h^{\text{crit}}(z) = \frac{1}{\pi A(z)} \frac{\mu_1^2 + \mu_2^2}{|\mu_1| + |\mu_2|}\ ,.$$

Exact formula on $\mathbb{CP}^1$

Theorem **13** *The expected number of critical points of a random section* $s_N \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ *(with respect to the Gaussian measure on* $H^0(\mathbb{CP}^1, \mathcal{O}$ *induced from the Fubini-Study metrics on* $\mathcal{O}(N)$ *and* $\mathbb{CP}^1$*) is*

$$\frac{5N^2 - 8N + 4}{3N - 2} = \frac{5}{3}N - \frac{14}{9} + \frac{8}{27}N^{-1} \cdots .$$

Of course, relative to the flat connection d/dz the number is $N - 1$.

Asymptotic expansion for the expected number of critical points

Theorem **14** *Let* (L, h) *be a positive hermitian line bundle. Let* $\mathcal{N}^{\text{crit}}(h^N)$ *denote the expected number of critical points of random* $s \in H^0(M, L^N)$ *with respect to the Hermitian Gaussian measure. Then,*

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\; c_1(L)^m\, N^m$$

$$+ \int_M \rho dV_\omega N^{m-1}$$

$$+[C_m \int_M \rho^2 dV_\Omega + \mathit{top}] N^{m-2} + O(N^{m-3})\,.$$

Here, ρ is the scalar curvature of ω_h, the curvature of h.

$\Gamma_m^{\text{crit}}\, c_1(L)^m$ is larger than for a flat connection.

To what degree is the expected number of critical points a topological invariant?

The first two terms are topological invariants of a positive line bundle, i.e. independent of the metric! (Both are Chern numbers of L). But the non-topological part of the third term

$$C_m \int_M \rho^2 dV_\Omega N^{m-2}$$

is a non-topological invariant, as long as $C_m \neq 0$. It is a multiple of the Calabi functional.

(These calculations are based on the Tian-Yau-Zelditch (and Catlin) expansion of the Szegö kernel and on Zhiqin Lu's calculation of the coefficients in that expansion.)

Asymptotic expansion for number of critical points

We calculate the first three terms in the expansion of the number of critical points for (L^N, h^N):

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\, c_1(L)^m\, N^m$$

$$+ \int_M \rho dV_\omega N^{m-1}$$

$$+ C_m \int_M \rho^2 dV_\Omega N^{m-2} + O(N^{m-3})\,.$$

The first two terms are topological invariants of a positive line bundle, i.e. independent of the metric! (Both are Chern numbers of L).

As long as $C_m \neq 0$ the third term is not a topological invariant.

Asymptotically minimal number of critical points

The average number of critical points can grow without bound, but has a lower bound (the Euler characteristic). Which hermitian metrics minimize the expected number of critical points? These would be ideal for vacuum selection.

To put the question precisely, let $L \to (M, [\omega])$ be a holomorphic line bundle over any compact Kähler manifold with $c_1(L) = [\omega]$, and consider the space of Hermitian metrics h on L for which the curvature form is a positive $(1,1)$ form:

$$P(M, [\omega]) = \{h : \frac{i}{2}\Theta(h) \text{ is a positive } (1,1)-\text{ form }\}.$$

Definition: We say that $h \in P([\omega])$ is asymptotically minimal if

(10)

$$\exists N_0 : \forall N \geq N_0, \; \mathcal{N}(h^N) \leq \mathcal{N}(h_1^N), \quad \forall h_1 \in P([\omega]).$$

Calabi extremal metrics are asymptotic minimizers

From the expansion

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\text{crit}}\; c_1(L)^m\, N^m + \int_M \rho dV_\omega N^{m-1}$$

$$+C_m \int_M \rho^2 dV_\Omega N^{m-2} + O(N^{m-3})\,,$$

we see that if $C_m > 0$ the metric with asymptotically minimal $\mathcal{N}(h^N)$ is the one with minimal $\int_M \rho^2 dV_\omega$. This is the Calabi extremal metric! Thus, Calabi extremal metrics are minimizers of the metric invariant = average number of critical points.

E.g. for the canonical bundle, Kähler -Einstein metrics are asymptotic minimizers of the functional $\mathcal{N}(h^N)$.

When is $C_m > 0$?

We have shown that $C_m > 0$ in dimensions ≤ 5. We conjecture that $C_m > 5$ in all dimensions.

Ben Baugher has recently shown that this is equivalent to a random matrix integral identity. The identity is true in dimensions ≤ 5 and is almost certainly true in higher dimensions.

Complex oscillatory integrals

This is obtained from the Boutet de Monvel - Sjostrand parametrix for the full Bergman kernel: $\Pi(x,y) = \sum_{N=1}^{\infty} \Pi_N(x,y)$:

$$\Pi(x,y) \sim \int_0^{\infty} e^{\psi(x,y)} s(x,y,\lambda) d\lambda,$$

where

- $\psi(x,y) = (1 - \lambda \bar{\mu} \frac{\sqrt{a(x,y)}}{\sqrt{a(x,x) a(y,y)}}$.

- $s(x,y,\lambda) \sim \lambda^m \sum_{j=0}^{\infty} a_j(x) \lambda^{-j}$.

Asymptotics of Π_N

Π_N is a Fourier coefficient of Π:

$$\Pi_N(x,y) = \int_{S^1} \Pi(x, e^{i\theta}y) e^{iN\theta} d\theta.$$

Hence, we get

$$\Pi_N(z,z) = \int_0^\infty \int_{S^1} e^{N\Psi(z,\theta,\lambda)} a_N(z,\theta,\lambda,$$

with

- $\Psi(x,\theta) = (1 - \psi(x, e^{i\theta}x))$.
- $a_N(x,\theta) = s(x, e^{i\theta}x, \lambda)$.

Applying stationary phase: (The critical points occur only at $\varphi = 0$.)

$$\Pi_N(z,z) \sim N^m \sum_{j=0}^{\infty} a_j(z) N^{-j}.$$

Correlations between zeros

Although the *expected* distribution of zeros is uniform, the zeros do not behave as if they are thrown down independently. The zeros are "correlated." To make this precise, we introduce the n *point correlation functions sections of degree* N

$$K_n^N(z^1, \dots, z^n) = E(|Z_s|^n),$$

= *the probability density of zeros at points* $z^1, \dots, z^n$. *Here,*

$$|Z_s|^n = \Big(\underbrace{|Z_s}\,| \times \cdots \times |Z_s|_n\Big)$$

is product measure on

$$M_n = \{(z^1, \dots, z^n) \in M^n : z^p \neq z^q \text{ for } p \neq q\}.$$

Correlations functions (cont.)

Examples

- When $n = 1$, we get the one-point correlation function $K_1^N(z)$ = the expected density of simultaneous zeros of k sections at z. We have seen: $K_1^N(z) = c_{mk}N + O(N^{-1})$, for any positive line bundle.

- When $n = 2$ we get the *pair correlation function*. $K_2^N(z^1, z^2)$ = the probabality of finding a pair of simeltaneous zeros of k sections at (z_1, z_2).

Scaling limit of correlation functions

For any positive line bundle $L^N \to M$ over any Kähler manifold, the density of zeros increases with N. If we scale by a factor $\sqrt{N}$, the expected density of zeros stays constant. We fix a point z_0 and consider the pattern of zeros in a small ball $B(z_0, \frac{1}{\sqrt{N}})$. We fix local coordinates z for which $z^0 = 0$ and rescale. In the limit we obtain the *n-point scaling limit zero correlation function*

(11)

$$K^{\infty}_{nkm}(z^1, \ldots, z^n)$$

$$= \lim_{N\to\infty} \left(c_{mk} N^k\right)^{-n} K^N_{nk}\left(\frac{z^1}{\sqrt{N}}, \ldots, \frac{z^n}{\sqrt{N}}\right).$$

Universality of the scaling limit

Theorem **15** *(Bleher-Shiffman-Z) The scaling limit correlation functions* $K^{\infty}_{nkm}(z^1, \dots, z^n)$ *are universal, i.e. independent of* M, L, ω*. They are given by a universal rational function, homogeneous of degree* 0*, in the values of the function* $e^{i\Im(z\cdot\bar{w})-\frac{1}{2}|z-w|^2}$ *and its first and second derivatives at the points* $(z, w) = (z^p, z^{p'})$*,* $1 \leq p, p' \leq n$*. Alternately it is a rational function in* $z^p_q, \bar{z}^p_q, e^{z^p \cdot \bar{z}^{p'}}$

The function $e^{i\Im(z\cdot\bar{w})-\frac{1}{2}|z-w|^2}$ which appears in the universal scaling limit is (up to a constant factor) the Bergman kernel $\Pi^{\mathbf{H}}_1(z, w)$ of level one for the reduced Heisenberg group $\mathbf{H}^n_{\mathrm{red}}$.

Decay of correlations

We have explicit formulas for $\widetilde{K}^\infty_{nkm}$ in all dimensions and codimensions.

The universal scaling limit pair correlation function is a function only of distance between the points:

$$K^\infty_{2km}(z^1, z^2) = \kappa_{km}(|z_1 - z_2|).$$

Theorem **16** *(BSZ)* $\kappa_{km}(r) = 1 + O(r^4 e^{-r^2}), \ \ r \to +\infty$.

Thus, even the scale $\frac{r}{\sqrt{N}}$, correlations are extremely short range: they differ from the case of independent random points by an exponentially decaying term.

Discrete case

Pair correlation between zeros of m independent sections in m variables in dimension m:

Theorem **17** *(Bleher-Shiffman-Z, 2001):*

$$\kappa_{mm}(r) = \begin{cases} \frac{m+1}{4} r^{4-2m} + O(r^{8-2m}), & \textit{as} \quad r \to 0, \\ \\ 1 + O(r^4 e^{-r^2}), & r \to +\infty. \end{cases}$$

- When $m = 1, \kappa_{mm}(r) \to 0$ as $r \to 0$ and one has "zero repulsion."

- When $m = 2$, $\kappa_{mm}(r) \to 3/4$ as $r \to 0$ and one has a kind of neutrality.

- With $m \geq 3$, $\kappa_{mm}(r) \nearrow \infty$ as $r \to 0$ and zeros attract (or 'clump together'): One is more likely to find a zero at a small distance r from another zero than at a small distance r from a given point.

Ideas of Proofs

A key object is the Bergman-Szegö projector

$$\Pi_N(x,y) = \sum_{j=1}^{d_N} S_j(x)\overline{S}_j(y)$$

of $H^0(M, L^N)$. Here, $\{S_j\}$ is an ONB. Its importance stems from:

- $\mathbf{E}_N\left(|f(z)|^2\right) = \Pi_N(z,z)$;
- $\mathbf{E}_N(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Pi_N(z,z)$;
- $\mathbf{E}_N(Z_{f_1,\ldots,f_k}) = [\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Pi_N(z,z)]^{\wedge k}$;

Here, we use the Poincare-Lelong formula:

$$Z_s = \frac{\sqrt{-1}}{2\pi}\left(\partial\bar{\partial}\log\|s\|_h^2 + \partial\bar{\partial}\log h\right)$$

Co-area formula for correlations

A second key ingredient is the denote the joint probability distrbution (JPD)

$$D_N(x^1, \ldots, x^n; \xi^1, \ldots, \xi^n; z^1, \ldots, z^n)$$

of the random variables

$$x^j(s) = s(z^j),\ \xi^j(s) = \nabla s(z^j), \quad j = 1, \ldots, n.$$

The zero correlations may be expressed in terms of the JPD by a formula generalizing the Kac-Rice formula: $K^N(z^1, \ldots, z^n)$

$$= \int D_N(0, \xi, z) \prod_{j=1}^{n} \left(\|\xi^j\|^2 d\xi^j \right) d\xi$$

Bergman kernels and JPD

The JPD is a Gaussian measure.

$$D_n(x, \xi; z) = \frac{\exp\langle -\Delta_n^{-1} v, v\rangle}{\pi^{kn(1+m)} \det \Delta_n}\,, \qquad v = (x\xi)\,,$$

where

$$\Delta_n = \begin{pmatrix} A_n & B_n \\ B_n^* & C_n \end{pmatrix}$$

where

$$A_n = \Big(\mathbf{E}\, x_j^p \bar{x}_{j'}^{p'}\Big), \qquad B_n = \Big(\mathbf{E}\, x_j^p \bar{\xi}_{j'q'}^{p'}\Big),$$

$$C_n = \Big(\mathbf{E}\, \xi_{jq}^p \bar{\xi}_{j'q'}^{p'}\Big);$$

$j, j' = 1, \ldots, k; \quad p, p' = 1, \ldots, n; \quad q, q' = 1, \ldots, m.$

All the expected values can be expressed in terms of Π_N and its covariant derivatives.

Scaling asymptotics of the Bergman kernel

The scaling asymptotics of the correlation functions then reduce to scaling asymptotics of the Bergman kernel.

Theorem **18** *(Bleher-Shiffman-Z, 1999): In normal coordinates* $\{z_j\}$ *at* $P_0 \in M$ *and in a 'preferred' local frame for* L*, we have:*

$$N^{-m}\Pi_N(P_0 + \tfrac{u}{\sqrt{N}}, \tfrac{\theta}{N}; P_0 + \tfrac{v}{\sqrt{N}}, \tfrac{\varphi}{N})$$

$$\sim \tfrac{1}{\pi^m} e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)} \cdot \left[1 + \textstyle\sum_{r=1}^{K} N^{-\frac{r}{2}} b_r(P_0,u,v) + \cdots\right],$$

Note that $e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)}$ is the Bergman-Szegö kernel of the Heisenberg group. These asymptotics use the Boutet de Monvel -Sjostrand parametrix for the Bergman kernel.