Feynman diagrams in inverse spectral theory

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Atlanta AMS: Saturday, Jan. 8, 10: 00

Inverse spectral problem for bounded analytic plane domains

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected analytic plane domain. Let Δ_D, resp. Δ_N be its Laplacian with Dirichlet, resp. Neumann boundary conditions.

The well-known question is:

- If the Dirichlet (resp. Neumann) spectra of two analytic domains Ω_1, Ω_2 are the same, are the domains isometric?

The known isospectral, non-isometric plane domains (Gordon-Webb-Wolpert) have corners.

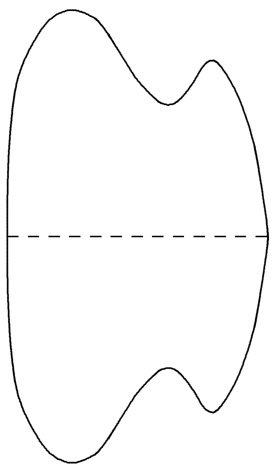
Simply connected analytic domains with one symmetry

The class $\mathcal{D}_{1,L}$ consists of simply connected real-analytic plane domains Ω satisfying:

- (i) There exists an isometric involution σ of Ω which 'reverses' a non-degenerate bouncing ball orbit $\gamma \to \gamma^{-1}$;

- (ii) Some standard generic assumptions on non-degeneracy and multiplicity of lengths and Poincaré eigenvalues of γ.

Figure 1



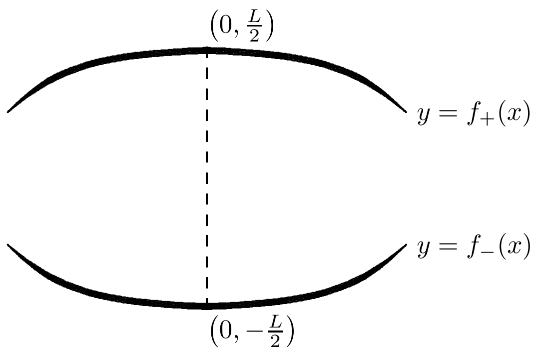
A domain in $\mathcal{D}_{1,L}$

Statement of results

Let Spec(Ω) denote the spectrum of the Laplacian Δ_Ω of the domain Ω with Dirichlet boundary conditions.

Theorem **1** *Spec*: $\mathcal{D}_{1,L} \mapsto \mathbb{R}_+^{\mathbf{N}}$ *is 1-1.*

Figure 2



$\partial\Omega$ as a pair of local graphs

Wave trace

Proof based on new approach to wave trace:

Let

$$E_B^\Omega(t,x,y) = \sum_j \cos t\lambda_j \varphi_j(x)\varphi_j(y)$$

conditions B.

We assume B is either Dirichlet $Bu = u|_{\partial\Omega}$ or Neumann $Bu = \partial_\nu u|_{\partial\Omega}$ boundary conditions.

Its distribution trace is defined by

$$Tr 1_\Omega E_B^\Omega(t) := \int_\Omega E_B^\Omega(t,x,x)dx = \sum_{j=1}^\infty \cos t\lambda_j.$$

Wave trace expansion

Let γ be a non-degenerate billiard trajectory whose length L_γ is isolated and of multiplicity one in $Lsp(\Omega)$. Then for t near L_γ, the trace of the even part of the wave group has the singularity expansion

$$Tr 1_\Omega E_B^\Omega(t)$$

$$\sim \Re\{a_\gamma (t - L_\gamma + i0)^{-1} + a_{\gamma 0} \log(t - L_\gamma + i0)$$

$$+ \sum_{k=1}^\infty a_{\gamma k} (t - L_\gamma + i0)^k \log(t - L_\gamma + i0)\}.$$

The coefficients $a_{\gamma k}$ = the wave trace invariants.

We calculate wave trace invariants and determine domain from them.

Lagrangian parametrix $\hat{E}$.

Outline of proof of inverse result

1. Find useful oscillatory integral expression for wave trace. Apply stationary phase to obtain wave coefficients.
2. Use Feynman diagrams to sift out terms with maximum number of derivatives of boundary defining function in each wave invariant $a_{\gamma^r,k}$.
3. Determine domain from these terms. Dedekind sums play basic role.

Wave trace $\to$ Resolvent trace

The wave trace expansion = the asymptotic expansion of the (regularized resolvent):

$$Tr 1_\Omega R^\Omega_{B\rho}(k+i\tau) \sim e^{(ik-\tau)L_\gamma} \sum_{j=1}^\infty B_{\gamma,j} k^{-j}, \quad k \to \infty.$$

Here: $\hat{\rho} \in C_0^\infty(L_\gamma - \epsilon, L_\gamma + \epsilon)$ be a cutoff, equal to one on an interval $(L_\gamma - \epsilon/2, L_\gamma + \epsilon/2)$ which contains no other lengths in Lsp(Ω) occur in its support, and define the smoothed (and localized) resolvent with a choice of boundary conditions by

$$R^\Omega_{B\rho}(k+i\tau) := \int_{\mathbb{R}} \rho(k-\mu)(\mu+i\tau) R^\Omega_B(\mu+i\tau) d\mu.$$

Reduction to the boundary

Define the boundary integral operator:

(1)

$$N(k+i\tau)f(q) = 2\int_{\partial\Omega} \frac{\partial}{\partial\nu_y} G_0(k+i\tau, q, q') f(q') ds(q'),$$

where $G_0(\lambda, x, y)$ is the free Green's function (resolvent kernel).

Proposition **2** *Suppose that L_γ is the only length in the support of $\hat{\rho}$. Then,*

$$\int_{\mathbb{R}} \rho(k-\lambda) \tfrac{d}{d\lambda} \log\det(I + N(\lambda + i\tau)) d\lambda$$

$$\sim \frac{e^{(ik-\tau)L_\gamma} e^{i\frac{\pi}{4} m_\gamma)}}{\sqrt{\det(I - P_\gamma)}} \sum_{j=0}^{\infty} [B_{\gamma,j} + B_{\gamma^{-1},j}] k^{-j},$$

where $B_{\gamma;j}$ are the Balian-Bloch invariants of the union of the periodic orbits γ of length L_γ of the interior and exterior problems.

The principal oscillatory integral

By expressing $(I + N(\lambda + i\tau))^{-1}$ as a a geometric series and simplifying, we find:

Theorem **3** *Modulo an error term* $R_{2r}(j^{2j-2}\kappa(a_j)$ *depending only on the* $(2j-2)$*-jet of curvature* κ *of* $\partial\Omega$ *at the reflection points* a_j *of* γ*, we have*

$$Tr 1_\Omega R^\Omega_{B\rho}(k + i\tau) \sim \sum_\pm \int_{[-\epsilon,\epsilon]^{2r}}$$

$$e^{ik\mathcal{L}_\pm(x_1,\ldots,x_{2r})}\widehat{\rho}(\mathcal{L}_\pm(x_1,\ldots,x_{2r}))$$

$$a^0_\pm(k, x_1, x_2, \ldots, x_{2r})dx_1 \cdots dx_{2r},$$

$$.$$

where the sum is over the two orientations of γ*, and where the phase is the length functional* $\mathcal{L}_\pm$ *of the polygon with endpoints* $(x_1, \ldots, x_{2r})$ *in the indicated orientation.*

Properties of phase/amplitude

1. The phase has the form

   $\mathcal{L}(x_1,\ldots,x_{2r}) = \sum_{p=1}^{2r-1}$

   $\sqrt{(x_p - x_{p+1})^2 + (f_{\epsilon_p}(x_p) - f_{\epsilon_{p+1}}(x_{p+1}))^2},$

   where $\epsilon_{2p} = 1, \epsilon_{2p+1} = -1.$
2. In its dependence on the boundary defining function f, the amplitude $a^0_{\pm}$ has the form $\mathcal{A}(k,x,y,f,f')$. It admits an expansion,

   $$\sum_{n=0}^{\infty} k^{-n}\mathcal{A}_n(x),$$

   where $\mathcal{A}_n$ depends only on the first $n+2$ derivatives of f_σ and

   $$\nabla a^0_{\pm}(k,x_1,\ldots,x_{2r})|_{x=0} = 0.$$

Stationary phase and Feynman diagrams

$$Z_k = \int_{\mathbb{R}^n} a(x) e^{ikS(x)} dx.$$

Then:

$$Z_k \sim \sum_{I=0}^{\infty} \sum_{V=0}^{\infty} \sum_{(\Gamma,\ell)\in\mathcal{G}_{V,I}} \frac{I_\ell(\Gamma)}{S(\Gamma)}$$

where:

- $\mathcal{G}_{V,I}$ = labelled graphs (Γ, ℓ) with V closed vertices of valency ≥ 3, one open vertex (corresponding to the amplitude), and with I edges and label ℓ
- $S(\Gamma)$ = order of symmetry of Γ;
- and $I_\ell(\Gamma)$ denotes the 'Feynman amplitude' associated to (Γ, ℓ).

Euler characteristic expansion

We note that the power of k in a given term with V vertices and I edges equals $k^{-\chi_{\Gamma'}}$, where $\chi_{\Gamma'} = V - I$ equals the Euler characteristic of the graph Γ' defined to be Γ minus the open vertex. We thus have;

$$Z_k^{h\ell} = \sum_{j=0}^{\infty} k^{-j} \{ \sum_{\Gamma : \chi_{\Gamma'} = j} \frac{I(\Gamma)}{S(\Gamma)} \}.$$

We note that there are only finitely many graphs for each χ because the valency condition forces $I \geq 3/2V$. Thus, $V \leq 2j, I \leq 3j$.

Application to wave trace

Our objective in diagrammatic language:

- We wish to enumerate the diagrams of each Euler characteristic whose amplitudes contain the**maximum number of derivatives of** $\partial\Omega$ among diagrams of the same Euler characteristic.
  τ)).

Wave invariants at a bouncing ball orbit of a $\mathbb{Z}_2$-symmetric domain

Theorem **4** *Suppose that γ (as above) is invariant under an isometric involution σ. Then, modulo the error term $R_{2r}(j^{2j-2}f(0))$, we have:*

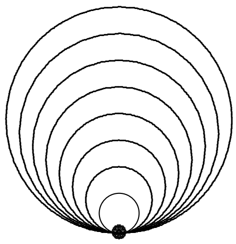
$$B_{\gamma^r,j-1} \equiv$$

$$\equiv r\{2(h_{2r}^{11})^j f^{(2j)}(0) + \{2(h_{2r}^{11})^j \frac{1}{2-2\cos\alpha/2}$$

$$+(h_{2r}^{11})^{j-2}\sum_{q=1}^{2r}(h_{2r}^{1q})^3\} f^{(3)}(0) f^{(2j-1)}(0)\}\}.$$

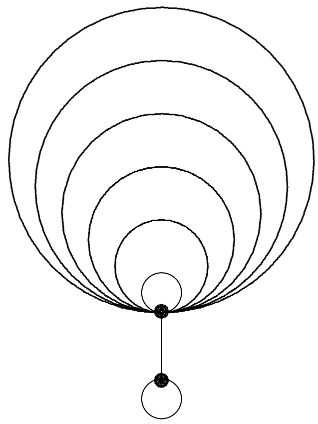
Here, h^{pq} are the matrix elements of the inverse Hessian H_{2r}^{-1} of the length function (phase) at the critical point γ.

Feynman diagram I

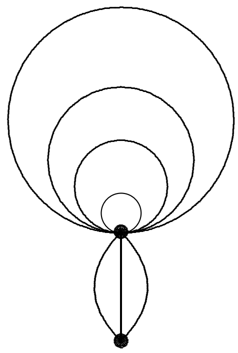


$\mathcal{G}^{2j,0}_{1,j}(-\chi = j-1, V=1, I=j)$, j loops at one closed vertex. All labels the same. Form of Feynman amplitude: $(h^{pp}_{+})^j D^{(2j)}_{x_p}\mathcal{L}_+ \equiv (h^{pp}_{+})^j f^{(2j)}(0)$

Figures



$(iv) : \mathcal{G}_{2,j+1}^{2j-1,3,0} \subset \mathcal{G}_{2,j+1}(-\chi = j-1; V = 2, I = j+1)$:



$(v) : \mathcal{G}_{2,j+1}^{2j-1,3,0} \subset \mathcal{G}_{2,j+1}(-\chi = j-1; V = 2, I = j+1)$:

Inverse spectral problem

We now return to the wave invariant formulae

$$B_{\gamma^r, j-1} \equiv$$

$$\equiv r\{2(h^{11}_{2r})^j f^{(2j)}(0) + \{2(h^{11}_{2r})^j \frac{1}{2-2\cos\alpha/2}$$

$$+(h^{11}_{2r})^{j-2} \sum_{q=1}^{2r} (h^{1q}_{2r})^3\} f^{(3)}(0) f^{(2j-1)}(0)\}\}.$$

and try to determine the domain from them.

Can we decouple the terms? We recall that the key issue is to decouple the two terms of { 2

$$h_{2r}^{11})^j f^{(2j)}(0) + 2(h_{2r}^{11})^j \frac{1}{2 - 2\cos\alpha/2}$$

and

$$+(h_{2r}^{11})^{j-2} \sum_{q=1}^{2r} (h_{2r}^{1q})^3 f^{(3)}(0) f^{(2j-1)}(0)$$

We must show they have different behavior in the iterate number $r = 1, 2, 3, \ldots$.

We need to study Hessian sums.

Sums of powers of h^{pq}

It is easy to calculate the sum of the matrix elements in the first row $[H_{2r}^{-1}]_1 = (h^{11}, \ldots, h^{1(2r)})$ (or column) of the inverse.

Proposition **5** *Suppose that* γ *is a* $\mathbb{Z}_2$*-symmetric bouncing ball orbit. Then, for any* p, $\sum_{q=1}^{2r} h^{pq} = \frac{1}{2+\cos\alpha/2}$.

This result is ‘disappointing’ in that the sum is constant in r, and hence does not help to decouple even and odd derivatives of f as one lets $r \to \infty$.

Sums of powers of $(h^{pq})^3$

We now consider sums $\sum_{q=1}^{2r}(h^{pq})^3$ of cubic powers. The sum is constant in p, so we put

$$F_3(r, \cos\alpha/2) = \sum_{q=1}^{2r}(h_{2r}^{1q})^3$$

and show that $F(r, \cos\alpha/2)$ is non-constant in r for all but a finite number of angles α. Our first calculation is based on the circulant approach.

Proposition **6** *In the elliptic case, we have:*

$$F_3(r, \cos\alpha/2) = \frac{1}{(2r)^2}\sum_{k_1,k_2=0}^{2r-1}$$

$$\frac{1}{(\cos\alpha/2+\cos\frac{k_1\pi}{r})(\cos\alpha/2+\cos\frac{k_2\pi}{r})(\cos\alpha/2+\cos\frac{(k_1+k_2)\pi}{r})}.$$

In the hyperbolic case, we obtain a similar result with cos *replaced by* cosh.

Dedekind sums

For each r, the sum $F_3(r,z)$ is a generalized Dedekind sum, i.e. the sum

$$F_3(r,z) = \sum_{\zeta \in D_r} I_3(\zeta, z)$$

of a function over the set D_{2r} of rth roots of unity $\frac{\pi k}{r}$mod $2\pi\mathbb{Z}^2$ with $k = (k_1, k_2) \in [0, 2r - 1] \times [0, 2r - 1]$ of the torus. The summand is

$$I_3(x; z) = [(z + \cos 2\pi x_1)(z + \cos 2\pi x_2)$$

$$(z + \cos 2\pi(x_1 + x_2))]^{-1},$$

which is a continuous periodic function of $(x_1, x_2) \in [0,1] \times [0,1]$ for $z \notin [-1,1]$.

Proof of inverse result

It suffices to decouple the terms

$$2(h_{2r}^{11})^2\{f^{(2j)}(0)+\frac{1}{2-2\cos\alpha/2}f^{(3)}(0)f^{(2j-1)}(0)\},$$

$$\{\sum_{q=1}^{2r}(h_{2r}^{1q})^3\}f^{(3)}(0)f^{(2j-1)}(0).$$

We use the simple observation:

Lemma **7** *If* $F_3(r,\cos\alpha/2)=\sum_{q=1}^{2r}(h_{2r}^{1q})^3$ *is non-constant in* $r=1,2,3,\ldots$ *then both terms can be determined from their sum as* r *ranges over* $\mathbf{N}$.