Survey of positive results on the inverse spectral problem

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Purpose of talk

Our purpose is to survey positive results on the basic inverse spectral problems:

- Determine as much as possible of the geometry of a Riemannian manifold (M, g) from the spectrum of its Laplacian.
- Determine as much as possible of the geometry of a domain $(\Omega, g) \subset (M, g)$, in particular a Euclidean domain in $\mathbb{R}^n$, from its spectrum with fixed (e.g. Dirichlet or Neumann) boundary conditions.

Positive = results which say that some feature can be determined or is rigid, at least in a restricted setting.

Basic strategy for positive results

The basic strategy is:

(A) Define a lot of spectral invariants;

(B) Calculate them in terms of geometric or dynamical invariants;

(C) Try to determine the metric or domain from the invariants.

We focus on *wave invariants*, which are related to dynamics of the geodesic flow. We do not survey results on λ_1 or $\det \Delta$ or on compactness.

Dimensional issues

The inverse spectral problem becomes more difficult as the dimension increases.

- Inverse spectral theory for $1D$ Sturm-Liouville operators on an interval is still an active research area (B. Simon et al), though much was done by Gelfand-Levitan, Marchenko, Trubowitz and many others.

- Inverse spectral theory in dimension 2 is 'infinitely more complex' than inverse spectral theory in dimension 1. It is still wide open.

- There are few positive results in dimensions ≥ 3 except for rigidity results, compactness results, and results for special extremal metrics.

Focus on dynamics

Wave invariants contain dynamical invariants. So many of the results are based on the reduction: Inverse spectral problem $\to$ dynamical inverse problem. The latter is often very hard to solve, and is of independent interest.

The true dynamics is that of the wave group, quantum dynamics. At the 'principal symbol level' it is the dynamics of the geodesic flow.

Since wave invariants are 'quantum', it is possible to calculate and use them without considering the geodesic flow at all, and for bounded domains this is to date the most successful approach.

Ingredients from dynamics

- Geodesic flow (boundaryless case), billiard flow and billiard map (boundary case);
- Closed geodesics: length spectrum, Poincare map, type (elliptic, hyperbolic, etc.);
- Birkhoff normal form of metric or Poincare map, Aubry-Mather theory.
- Zeta functions of flows;
- Livsic cohomology problem;
- Symplectic conjugacy problem;

Classical-Quantum heuristics

Quantum: $\Delta, U(t) = e^{it\sqrt{\Delta}}$.
Classical: $|\xi|_g^2 = \sum_{i,j} g^{ij}\xi_i\xi_j$, geodesic flow G^t.

Two Laplacians are isospectral if

$$\Delta_{g_1} = U \Delta_{g_2} U^*, \tag{1}$$

where $U : L^2(M_1, g_1) \to L^2(M_2, g_2)$ is a unitary operator. The classical analogue of this similarity is the symplectic conjugacy

$$|\xi|_{g_1} = \chi^* |\xi|_{g_2} \iff G^t_{g_1} = \chi \circ G^t_{g_2} \circ \chi^{-1}$$

of the corresponding geodesic flows. Here, $\chi : T^*M_1 \backslash 0 \to T^*M_2 \backslash 0$ is a homogeneous symplectic diffeomorphism.) This analogy should not be taken literally, but it is useful in suggesting conjectures.

Caveats on the heuristics

- The heuristics are literally true if there exists an intertwiner U which is a global FIO (Fourier integral operator) associated to a canonical transformation. But this is rarely if ever the case. In the Sunada examples, there exists a U which is an FIO but only associated to a multi-valued correspondence. There is no apriori reason why any intertwiner should be an FIO.

- It is very difficult to relate the classical and quantum mechanics when the length spectrum is multiple. We will almost always assume there is just one closed geodesic (or component) of a given length. Are there any known counterexamples with simple length spectrum?

Heuristic/rigorous results

The heuristics are quite correct on a local (formal) level.

- (i) Isospectrality (with a simple length spectrum assumption) implies the *formal local* symplectic equivalence of the geodesic flows around corresponding pairs of closed geodesics, i.e. it implies the equality of their *Birkhoff normal forms*. (Guillemin, with some inputs/additions by S. Z.).

- (ii) This implies local smooth symplectic equivalence around hyperbolic orbits, although not around elliptic orbits.

Wave invariants:

We now introduce the main invariants: take the wave group $U(t) = e^{it\sqrt{\Delta}}$ of (M, g) and consider its (distribution) trace

$$(2) \qquad TrU(t) = \sum_{\lambda_j \in Sp(\sqrt{\Delta})} e^{it\lambda_j}.$$

It is a tempered distribution on $\mathbb{R}$. We denote its singular support (the complement of the set where it is a smooth function) by Sing Supp $TrU(t)$.

Poisson relation

$$(3) \qquad \text{Sing Supp} TrU(t) \subset \quad Lsp(M, g),$$

Here, we denote the length of a closed geodesic γ by L_γ and the set of lengths (the length spectrum) by $Lsp(M, g)$.

Is entropy a spectral invariant?

The Poisson relation raises an unsolved problem:

The topological entropy of the geodesic flow of (M, g) is the exponential growth rate of the length spectrum:

$$h_{top} := \liminf_{L \to \infty} \log \#\{\gamma : L_\gamma \leq L\}. \tag{4}$$

Here, lengths are counted with multipicity. In the case where geodesics come in families, we count components of the fixed point sets.

Problem **1** *Is h_{top} a spectral invariant?*

Discussion

The question is only non-trivial when multiplicities in the length spectrum grow as fast as the length spectrum, as occurs for compact hyperbolic manifolds in dim ≥ 3 and for arithmetic hyperbolic quotients in dimension 2 . As observed by Besson-Courtois-Gallot, an affirmative answer would show that that hyperbolic metrics are spectrally determined among other negatively curved metrics, since they are the unique minimizers of h_{top}.

Problem **2** *Suppose that* M *with* $\dim M \geq 3$ *possesses a hyperbolic metric, and let* $\mathcal{M}_-$ *denote the class of negatively curved metrics on* M*. Is the hyperbolic metric* g *determined by its spectrum among metrics in* $\mathcal{M}_-$*? I.e. can there exist another non-isometric metric in this class which is isospectral to the hyperbolic metric?*

Is real analyticity of g a spectral invariant

We propose a new problem suggested by the Poisson relation:

Problem **3** *Can one tell from $Spec(\Delta)$ if a metric (or the underlying manifold or domain) is real analytic?*

The analytic wave front set of $TrU(t)$

$$WF_a(\sum_{\lambda_j \in Sp(\sqrt{\Delta})} e^{it\lambda_j}).$$

is the complement of the set where the trace is real analytic. When (M, g) is a C^ω, $WF_a TrU(t) \subset Lsp(M, g)$. If (M, g) is C^∞ but not C^ω, it is plausible that $WF_a TrU(t)$ could contain an interval or be all of $\mathbb{R}$. Thus simply the discreteness of $WF_a TrU(t)$ would say that (M, g) is real analytic.

Singularity expansion of the wave trace

We now develop the theory further:

Theorem **1** $TrU(t)$ *has a singularity expansion:*

$$TrU(t) \equiv e_0(t) + \sum_{L \in Lsp(M,g)} e_L(t) \ \ mod \ \ C^\infty,$$

where $singsuppe_0 = \{0\}, singsuppe_L = \{L\}$. *When all closed geodesics are non-degenerate, the expansions take the form*

$$e_0(t) = a_{0,-n}(t+i0)^{-n} + a_{0,-n+1}(t+i0)^{-n+1} + \cdots ,$$

$$\begin{aligned} e_L(t) &= a_{L,-1}(t-L+i0)^{-1} + a_{L,0}\log(t-(L+i0)) \\ &+ a_{L,1}(t-L+i0)\log(t-(L+i0)) + \cdots \ , \end{aligned}$$

where $\cdots$ *refers to homogeneous terms of ever higher integral degrees.*

Wave trace invariants

The wave coefficients $a_{0,k}$ at $t = 0$ are essentially the same as the singular heat coefficients, hence are given by integrals over M of $\int_M P_j(R, \nabla R, ...)$dvol of homogeneous curvature polynomials. The wave invariants for $t \neq 0$ have the form:

$$a_{L,j} = \sum_{\gamma : L_\gamma = L} a_{\gamma,j}, \tag{5}$$

where $a_{\gamma,j}$ involves on the germ of the metric along γ. Here, $\{\gamma\}$ runs over the set of closed geodesics, and where L_γ, $L_\gamma^{\#}$, m_γ, resp. P_γ are the length, primitive length, Maslov index and linear Poincaré map of γ. For instance, the principal wave invariant at $t = L$ in the case of a non-degenerate closed geodesic is given by

$$a_{L,-1} = \sum_{\gamma : L_\gamma = L} \frac{e^{\frac{i\pi}{4} m_\gamma} L_\gamma^{\#}}{|\det(I - P_\gamma)|^{\frac{1}{2}}}. \tag{6}$$

Definition of Poincaré map

Nonlinear Poincaré map $\mathcal{P}_\gamma$: in S^*M one forms a symplectic transversal S_γ to γ at some point m_0. One then defines the first return map, or nonlinear Poincaré map,

$$\mathcal{P}_\gamma(\zeta) : S_\gamma \to S_\gamma$$

by setting $\mathcal{P}_\gamma(\zeta) = G^{T(\zeta)}(\zeta)$, where $T(\zeta)$ is the first return time of the trajectory to S_γ. This map is well-defined and symplectic from a small neighborhood of $\gamma(0) = m_0$ to a larger neighborhood. By definition, the linear Poincare map is its derivative, $P_\gamma = d\mathcal{P}_\gamma(m_0)$.

Appendix: Classification of closed orbits

Closed geodesics are classified by the spectral properties of the symplectic linear map P_γ. Its eigenvalues come in 4-tuples $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$. A closed geodesic γ is called *non-degenerate* if $\det(I - P_\gamma) \neq 0$, and

- *elliptic* if all of its eigenvalues are of modulus one, in which case they come in complex conjugate pairs $e^{i \pm \alpha_j}$.

- *hyperbollic* if all of its eigenvalues are real, in which case they come in inverse pairs $\lambda_j \lambda_j^{-1}$; *loxodromic* or *complex hyperbolic* in the case where the 4-tuple consists of distinct eigenvlaues as above.

Local problems on wave trace invariants

Thus, associated to any closed geodesic γ of (M,g) is the sequence $\{a_{\gamma^r,j}\}$ of wave invariants of γ and of its iterates γ^r. These invariants depend only on the germ of the metric at γ. The principal question of this survey may be stated as follows:

Problem **4** *How much of the local geometry of the metric g at γ is contained in the wave invariants $\{a_{\gamma^r,j}\}$? Can the germ of the metric g at γ be determined from the wave invariants? At least, can the symplectic equivalence class of its germ be determined?*

The principal wave invariant

It was proved by D. Fried many years ago that from

$$a_{\gamma^r,-1} = \frac{e^{\frac{i\pi}{4} m_\gamma} L_\gamma^{\#}}{|\,\mathrm{det}(I - P_\gamma^r)|^{\frac{1}{2}}}, \;\; r = 1, 2, \cdots$$

one can determine the eigenvalues of P_γ (in the non-degenerate case). Hence one can determine the type of γ.

Until very recently, all inverse spectral results using wave invariants only used the length spectrum and the spectrum of P_γ.

To go further one needs to calculate the lower wave invariants.

Calculation of wave invariants

For applications to inverse spectral theory it is important to calculate wave invariants. Results:

- There is a useful algorithm (quantum normal form + non-commutative residue) which calculates wave invariants for non-degenerate closed geodesics. The formulae are complicated except for special metrics such as surfaces of revolution.

- There is a better algorithm (Balian-Bloch) in the case of Euclidean domains with boundary. It gives explcit formula for all wave invariants in terms of the boundary defining funcion.

Global problems on wave trace invariants

One may divide the potential use of wave invariants into two classes: (i) those which use all of the closed geodesics, and (ii) those which involve one or a few closed geodesics.

Problem **5** *How much of the global geometry* (M, g) *is contained in the entire set of wave invariants* $\{a_{\gamma^r,j}\}$*?*

To the author's knowledge, the global problem of combining wave invariants of all closed geodesics has only been led to successful results on isospectral deformations. Otherwise it is hard to combine the information.

Inverse spectral results using wave invariants

The positive results based on wave invariant analysis are as follows.

- Negatively curved compact manifolds are spectrally rigid (Guillemin-Kazhdan, Croke)

- Simple real analytic surfaces of revolution (with one critical distance from the axis) are spectrally determined within the class of such surfaces. Any other metric on S^2 which is isospectral to a simple surface of revolution must be C^0-integrable. Smooth surfaces of revolution with a mirror symmetry thru the $x-y$ plane are spectrally determined among metrics of this kind.

Inverse spectral results (cont.)

- Simply connected analytic plane domains with two symmetry axes (i.e. with the symmetries of an ellipse) and with a bouncing ball orbit of fixed length L are spectrally determined within this class. Convex analytic domains with two symmetry axes are spectrally determined within this class. The shortest orbit is necessarily a bouncing ball orbit and of course its length is a spectral invariant (Ghomi).

- New result: Simply connected analytic plane domains with one axis of symmetry are spectrally determined within this class. (This implies the preceding result, but we state it separately since it is a new result based on different methods).

Inverse spectral results (cont.)

- There is a spectrally determined class of convex plane domains (ellipses?) for which each element is spectrally determined among all convex plane domains (Marvizi-Melrose).

- The *mean minimal action* of a convex billiard table is invariant under isospectral deformations (K. F. Siburg).

Wave invariants at $t = 0$

They are the same as the (singular) heat invariants. Some classical results:

- Spheres: Tanno used a_0, a_1, a_2, a_3 to prove that the round metric g_0 on S^n for $n \leq 6$ is determined among all Riemannian manifolds by its spectrum, i.e. any isospectral metric g is necessarily isometric to g_0. He also used a_3 to prove that canonical spheres are locally spectrally determined (hence spectrally rigid) in all dimensions. Patodi proved that round spheres are determined by the spectra $\text{Spec}^0(M, g)$ and $\text{Spec}^1(M, g)$ on zero and 1 forms.

- Complex projective space: Let (M, g, J) be a compact Kähler manifold and let $(CP^n(H), g_0, H_0$

be a complex n-dimensional projective space with the Fubini-Study metric of constant holomorphic sectional curvature H. Tanno proves that if the complex dimension $n \leq 6$ and if Spec$(M, g, J) =$ Spec$(CP^n(H), g_0, J_0)$, then (M, g, J) is holomorphically isometric to $(CP^n(H), g_0, J_0)$. He also proves that $(CP^n(H), g_0, H_0)$ is locally spectrally determined in all dimensions.

- Flat manifolds: Patodi and Tanno used the heat invariants to prove in dimension ≤ 5 that if (M, g) is isospectral to a flat manifold, then it is flat. More precisely, they showed that if $a_j = 0$ for $j \geq 1$, and if $n \leq 5$ then (M, g) is flat. The result is sharp, as Patodi (loc. cit.) showed that $a_j = 0$ for $j \geq 1$ for the product of a 3-dimensional sphere with a 3-dimensional space of constant negative curvature. In fact, Tanno showed that if $a_2 = a_3 = 0$, then (M, g) is either E^6/Γ_1, where Γ_1 is some discontinuous group of translations of the Euclidean space E^6, or (2) $[S^3(C) \times H^3(-C)]/\Gamma_2$, where $S^3(C)[H^3(-C)]$ is the 3-sphere [hyperbolic 3-space] with constant curvature $C > 0[-C < 0]$ and Γ_2 is some discontinuous group of isometries of $S^3(C)\times H^3(-C)$. Kuwabara used the invariants to prove that flat manifolds are locally spectrally determined, hence spectrally rigid.

Still open, after all these years

- Are (round) spheres spectrally determined? (Unknown in dim ≥ 7)?

- Are ellipses spectrally determined among (analytic, convex) plane domains?

- Can you tell from the spectrum if a metric is flat?

Two domains with the same wave invariants

A Penrose mushroom type example due to Michael Lifshits shows that wave invariants are not sufficient to discriminate between all pairs of smooth billiard tables. Indeed, Lifshitz constructs (many) pairs of smooth domains (Ω_1, Ω_2) which have the same length spectra and the same wave invariants at corresponding pairs of closed billiard orbits γ_j of Ω_j $(j = 1, 2)$. It follows that the Poincaré maps $\mathcal{P}_{\gamma_j}$ have the same Birkhoff normal forms as well. It is unknown (but probably false) if the domains are isospectral.

Construction: take an ellipse and divide the major axis into two parts, the segment between the foci and the other two segments.

Remove the segment between the foci and replace it with any simple curve with the foci as endpoints, e.g. a 'tongue' below the segment. Next, remove the outer segments of the axis between the foci and replace them with any 'bumps' below the segment. The trajectories which start in the outer bumps never intersect the segment between the foci and therefore never go into the tongue. Similarly trajectories which come into the elliptical part from the tongue pass through the segment between the foci and never go into the outer bumps. To obtain two domains, just reverse the orientation of the tongue relative to the bumps.

Inverse spectral problem for bounded analytic plane domains

Let us sketch the proof that a bounded simply connected plane domain with one symmetry is determined by its Dirichlet (or Neumann) spectrum among other such domains.

The key is to calculate all of the wave invariants at a bouncing ball orbit which is reversed by the symmetry. A bouncing ball orbit is a closed billiard trajectory formed by a segment which hits the boundary orthogonally at both endpoints.

We calculate the wave invariants by a new approach.

Balian-Bloch 30 ans après

We follow Balian-Bloch (1971) in using the asymptotics of the resolvent along horizontal lines in the upper half plane to define wave invariants.

We then use the exact (Fredholm-Neumann) formula:

$$R_\Omega(k+i\tau) = R_0(k+i\tau)$$

$$-\mathcal{D}\ell(k+i\tau)(I+N(k+i\tau))^{-1}\gamma\mathcal{S}\ell^{tr}(k+i\tau),$$

where $R_0(k+i\tau)$ is the free resolvent $-(\Delta_0 + (k+i\tau)^2)^{-1}$ on $\mathbb{R}^2$; γ is the restriction to the boundary, and $\mathcal{D}\ell(k+i\tau)$ (resp. $\mathcal{S}\ell(k+i\tau)$) is the double (resp. single) layer potential

The quantized billiard map

The key object in this is the boundary integral operator

(7)

$$N(k+i\tau)f(q) = 2\int_{\partial\Omega} \frac{\partial}{\partial\nu_y} G_0(k+i\tau, q, q')f(q')ds(q')$$

induced by $\mathcal{D}\ell$.

This is an elementary operator which is a hybrid (semiclassical) Fourier integral operator. It has phase $d_{\partial\Omega}(q,q')$, the distance between boundary points. Thus, it quantizes the billiard map. It also has homogeneous singularities on the diagonal.

Wave invariants and N

The relation to wave invariants is:

Proposition **2** *Suppose that L_γ is the only length in the support of $\widehat{\rho}$. Then,*

$$\int_{\mathbb{R}} \rho(k-\lambda) \frac{d}{d\lambda} \log \det(I+N(\lambda+i\tau))d\lambda \sim \sum_{j=0}^{\infty} B_{\gamma;j} k^{-j},$$

where $B_{\gamma;j}$ are the wave invariants of γ.

We use the explicit WKB formula for N, regularize it and apply stationary phase to compute the coefficients $B_{\gamma^r;j}$ when γ is a bouncing ball orbit.

Calculation of wave invariants at a bouncing ball orbit

In the case where the domain has a symmetry interchanging the top and bottom, so that $f_\pm$ are mirror images of the graph of $y = f(x)$, the formula simplies as follows:

Proposition **3** *Suppose that γ (as above) is invariant under an isometric involution σ. Then, modulo the error term $R_{2r}(j^{2j-2}f(0))$, we have:*

$$a_{\gamma^r,j-1} = r\{2(h^{11})^j f^{(2j)}(0)$$

$$+\{2(h^{11})^j \frac{1}{2-2\cos\alpha/2}$$

$$+(h^{11})^{j-2}\sum_{q=1}^{2r}(h^{1q})^3\} f^{(3)}(0) f^{(2j-1)}(0)\}\}.$$

Here, one sums over repeated indices.

Inverse results

If Ω has two symmetries, the odd derivatives vanish and we obtain all Taylor coefficients of f.

If the domain has just one symmetry, we need to obtain both even and odd coefficients.

The inverse spectral problem is then reduced to analyzing sums of powers of inverse Hessian coefficients. It turns out that different powers give independent coefficients as r varies. So we can separate out even and odd derivatives and find that all are spectral invariants. We again recover the domain.

Inverse Birkhoff normal form results

The inverse result for domains with two symmetries was first proved using Birkhoff normal forms (by S. Z., and then by Iantchenko-Sjostrand-Zworski using monodromy operators). We briefly sketch the ideas. First, the BNF on the (quasi-) eigenvalue level.

Let γ be a non-degenerate elliptic closed geodesic on an n-dimensional Riemannian manifold. For each transversal quantum number $q \in \mathbb{Z}^{n-1}$, there exists an approximate eigenvalue of the form

$$\lambda_{kq} \equiv r_{kq} + \frac{p_1(q)}{r_{kq}} + \frac{p_2(q)}{r_{kq}^2} + \ldots \tag{8}$$

where

$$r_{kq} = \frac{1}{L}(2\pi k + \sum_{j=1}^{n}(q_j + \frac{1}{2})\alpha_j).$$

BNF

The coefficients $p_j(q)$ are polynomials of specified degrees and parities. They are the quantum BNF invariants at γ.

Operator level:

$$W\sqrt{\Delta_\psi}W^{-1} \equiv \mathcal{D} + \frac{\tilde{p}_1(\hat{I}_1,\dots,\hat{I}_n)}{L\mathcal{D}} + \frac{\tilde{p}_2(\hat{I}_1,\dots,\hat{I}_n)}{(L\mathcal{D})^2}$$

$$+\cdots+\frac{\tilde{p}_{k+1}(\hat{I}_1,\dots,\hat{I}_n)}{(L\mathcal{D})^{k+1}}+\dots$$

where the numerators $p_j(\hat{I}_1,...,\hat{I}_n), \tilde{p}_j(\hat{I}_1,...,\hat{I}_n)$ are polynomials of degree j+1 in the variables $\hat{I}_1,...,\hat{I}_n$, where W^{-1} denotes a microlocal inverse to W. Here, $\mathcal{D} = D_s + \frac{1}{L}H_\alpha$.

BNF

Above,

$$I_j = I_j(y, D_y) := \frac{1}{2}(D_{y_j}^2 + y_j^2)$$

are the action operators (harmonic oscillators) and $e^{\pm i\alpha_k}$ are the eigenvalues of the Poincare map P_γ.

The inverse result of Guillemin (and partially the author) is:

Theorem *Let γ be a non-degenerate closed geodesic. Then the quantum Birkhoff normal form polynomials $p_j(\hat{I}_1, ..., \hat{I}_n)$ from the wave trace invariants of Δ at γ. around γ is a spectral invariant; in particular the classical Birhoff normal form is a spectral invariant.*

From BNF to inverse results

To obtain the domain, one then has to compute the quantum BNF coefficients. The algorithm is quite different for manifolds without boundary and for those with boundary. The latter has to date never been carried out. Rather, one argues indirectly: if one knows the quantum BNF, one knows the classical BNF of the Poincare map. Colin de Verdière showed that the latter determines the defining function f for domains with two symmetries, and that completes the proof.

It does not determine a domain with fewer symmetries. So one needs the full quantum BNF or else to use another method to compute wave invariants (a là Balian Bloch, e.g.).

Spectral rigidity

We move on to other results. One of the first uses of wave invariants and dynamics was the result of [Guillemin-Kazhdan, Croke]:

Theorem **4** *Negatively curved manifolds are spectrally rigid.*

Let us recall the idea. In the negatively curved case, the Morse indices are always zero and no cancellation takes place in the wave trace formula. Hence, $Lsp(M, g)$ is a spectral invariant of negatively curved manifolds. An isospectral deformation therefore preserves the length spectrum.

Spectral rigidity

If γ is an isolated, non-degenerate closed geodesic of g, then for any deformation g_t of g, γ deforms smoothly as a closed geodesic γ_t of g_t and one may define its variation

$$\dot{L_\gamma} = \frac{d}{dt}|_{t=0} L_{\gamma_t}. \tag{9}$$

It is not hard to compute that

$$\dot{L_\gamma} = \int_\gamma \dot{g} ds, \tag{10}$$

where $\dot{g}$ is viewed as a quadratic function on TM and γ is viewed as the curve $(\gamma(t), \gamma'(t))$ in TM.

It follows that whenever the closed geodesics are non-degenerate and of multiplicity one in the length spectrum, we have

$$\int_\gamma \dot{g} ds = 0, \quad \forall \gamma.$$

Livsic cohomology

This raises the Livsic cohomology problem. In

the case of negatively curved surfaces, the geodesic flow is Anosov and it is known that the cohomology is trivial, i.e. that

$$\int_\gamma \dot{g} ds = 0, \quad \forall \gamma \implies \dot{g} = \Xi(f)$$

for some smooth f.

The next step is to study harmonic analysis on the unit sphere bundle S^*M to determine if there actually can exist f when $\dot{g}$ is a quadratic form. For surfaces of negative curvature, there cannot exist a smooth solution when $\dot{g}$ is quadratic, and hence there exist no isospectral deformations of negatively curved surfaces.

More on Livsic cohomology

This is such a general argument that one is tempted to apply it to many other situations. The first question is whether the Livsic cohomology equation is solvable for other systems.

Hyperbolic billiard domains (e.g. Sinai billiards) might be candidates, but there are no results at this time due to the singularities always occurring in billiard problems.

One might also consider integrable systems such as surfaces of revolution. The general Livsic equation leads to small divisor problems, but one has more control due to known spectral invariants (see below).

Isospectral deformations and marked length spectrum

Isospectral deformations also lead to the inverse marked length spectral problem and to the symplectic equivalence of the geodesic flows. An isospectral deformation preserves lengths of closed geodesics in the generic case where the multiplicities all equal one, and therefore marks the length spectrum, i.e. gives a correspondence between closed geodesics and lengths, by the length spectrum of the initial metric. More precisely, as one deforms the metric or domain, each one-parameter family of closed geodesics γ_ϵ stays in a fixed free homotopy class of the fundamental group of M. Therefore, an isospectral deformation gives rise to a one-parameter family of metrics or domains with the same marked length spectrum.

Length spectrum and marked length spectrum

The set of lengths is unformatted in the sense that one does not know which lengths in the list correspond to which closed geodesics. A formatted notion in which lengths are assigned to topogically distinct types of closed geodesics is the *marked length spectrum* ML_g. When $\partial M = \emptyset$, it is the function assigning to a free homotopy class of geodesics the length of the shortest geodesic in the class.

On a convex plane domain with boundary, topologically distinct closed geodesics correspond to different rotation numbers $\frac{m}{n} = \frac{\text{winding number}}{\text{number of reflectio}}$
The marked length spectrum associates to each rational rotation number the *maximal length* of the closed geodesics having n reflection points and winding number m.

Spectral rigidity of Zoll manifolds?

To give an extreme example of an open isospectral deformation problem, consider the case of Zoll manifolds. It is simple to see that isospectral deformations of Zoll manifolds must be Zoll, but it is an open problem whether any non-trivial isospectral deformations exist even for $M = S^2$. One of the main problems is that all principal symbol level spectral invariants of Zoll manifolds are the same, so one has to dig further into the wave trace expansions to find obstructions to isospectral deformability.

When does the marked length spectrum determine (M, g) up to isometry?

Isospectral deformations preserve the MLS, so the question is:

Problem **6** *For which* (M, g) *does the marked length spectrum of* (M, g) *determine* (M, g) *up to isometry?*

Results:

- V. Bangert proved that the marked length spectrum of a flat two-torus determines the flat metric up to isometry.

- J. P. Otal and C. Croke independently proved that the answer is ‘yes’ under certain non-positive curvature assumptions.

- An example due to F. Bonahon shows that the answer is ‘no’ for metric structures more general than Riemannian metrics.

- U. Hamenstadt proved that a locally symmetric manifold is determined by its marked length spectrum.

Symplectic equivalence problem

Having the same MLS sometimes implies existence of a time-preserving conjugacy between the geodesic flows.

Problem **7** *When does this hold? When does the symplectic equivalence of two geodesic flows (in the sense of (11)) imply isometry of the metrics? Does symplectic equivalence of billiard maps of convex domains imply their isometry?*

Two geodesic flows are called symplectically equivalent or conjugate if there exists a homogeneous symplectic diffeomorphism $\chi : T^*M_1 \to T^*M_2$ satisfying

$$\chi \circ G_1^t \circ \chi^{-1} = G_2^t. \tag{11}$$

Results on the symplectic equivalence problem

- Weinstein showed that the geodesic flow of any Zoll surface is symplectically equivalent to that of the standard 2-sphere.

- J. P. Otal and by C. Croke proved independently that negatively curved surfaces with the same marked length spectrum must have conjugate geodesic flows and that such surfaces with conjugate flows must be isometric. The corresponding statement in higher dimensions still appears to be open in general.

- Hammenstädt constructed a time-preserving conjugacy between negatively curved manifolds with the same marked length spectrum. When this conjugacy is C^1 the work of Besson, Courtois and Gallot proves that the manifolds must be isometric. The method of Hammenstadt avoids this regularity issue.

- The conjugacy problem for nilmanifolds was studied by C. Gordon, D. Schueth and Y. Mao. They proved for special classes of 2-step nilmanifolds that conjugacy of the geodesic flow implies isometry. The relations between marked length spectral equivalence, conjugacy of geodesic flows and isometry does not seem to have been studied in other settings.

Equivalence problem for Birkhoff normal forms

A much related problem is the equivalence problem for Birkhoff normal form.

Problem **8** *To what extent can the germ of a metric g at γ be determined by its Birkhoff normal form at γ?*

Let us recall the definition and the relations to symplectic equivalence.

Birkhoff normal form

Birkhoff normal forms are approximations to

Hamiltonians (or symplectic maps) near equilibria by completely integrable Hamiltonians (or symplectic maps).

Let us first consider the Birkhoff normal form of the metric Hamiltonian near a non- degenerate elliptic closed geodesic γ. The normal form algorithm defines a sequence of canonical transformations χ_M at γ which conjugate H to the normal forms

(12)

$$\chi_M^* H \equiv \sigma + \frac{1}{L} \sum_{i,j=1}^{n} \alpha_j I_j$$

$$+ \frac{p_1(I_1,\ldots,I_n)}{\sigma} + \ldots + \frac{p_M(I_1,\ldots,I_n)}{\sigma^M} \quad mod \quad O^1_{M+1}$$

where p_k is homogeneous of order k+1 in $I_1, \ldots, I_n$, and where O^1_{M+1}= germs of functions homogeneous of degree 1 which vanish to order $M + 1$ along γ.

BNF

The Birkhoff normal form for the Poincaré map:
(13)

$$\mathcal{P}(I,\varphi) = (I, \varphi + \nabla_I G_M(I)), \quad mod \quad O^1_{M+1},$$

where $G_M(I)$ is a polynomial of degree M in the I variables. Thus, to order $M+1$, the Poincare map leaves invariant the level sets of the actions (ellipses, hyperbolas etc. according to the type of γ) and 'rotates' the angle along them.

Convergence of BNF

The question arises whether the full (infinite series) Birkhoff normal form converges and whether the formal symplectic map conjugating the Hamiltonian to its normal form converges. According to a recent article of Perez-Marco, there are no known examples of analytic Hamiltonians having divergent Birkhoff normal forms In the case of hyperbolic orbits of analytic symplectic maps in two degrees of freedom, it was proved by J. Moser that the Birkhoff normal form and transformation do converge. In Rouleux used results of de la Llave et al to prove the existence of a local smooth symplectic conjugacy $\kappa^* H = q(I)$ of the metric Hamiltonian H near a hyperbolic fixed point (or orbits) to a smooth normal form $q(I)$.

Local symplectic equivalence problem

Thus, in the hyperbolic case, equality of BNF's implies local symplectic conjugacy = symplectic conjugacy between Poincare maps:

(14)

$$\chi : S_{\gamma_1} \subset S^*_{g_1} M_1 \to S_{\gamma_2} \subset S^*_{g_2} M_2, \quad \chi \mathcal{P}_{\gamma_1} \chi^{-1} = \mathcal{P}_{\gamma_2}.$$

One could ask:

Problem **9** *When does local symplectic conjugacy of Poincare maps at a closed geodesic (or local symplectic conjugacy of geodesic flows) imply local isometry?*

Local symplectic equivalence problem: hyperbolic metrics

As a special case of the local equivalence problem, suppose that (M, g) is a hyperbolic manifold, and let γ be a closed geodesic. Let g' be a second real analytic metric on M. Suppose that there exists a closed geodesic for g' for which the Poincare maps $\mathcal{P}_\gamma$ are symplectically conjugate. Must g' be a hyperbolic metric?

Zeta functions

We now turn to a different spectral invariant connected to dynamics: the Ruelle zeta function

$$(15) \quad \log\ Z_{(M,g)}(s) = \sum_{\gamma \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{e^{-skL(\gamma)}}{k|det(I - P_\gamma^k)|},$$

Here, we are assuming that the metric is bumpy so that the fixed point sets of the geodesic flow consist of isolated, non-degenerate closed orbits. We denote by $\mathcal{P} = \{\gamma\}$ the set of primitive periodic orbits of the geodesic flow Φ^t and by P_γ the linear Poincare map of γ. It is a spectral invariant as long as the (extended) length spectrum is simple (multiplicity free) or if the metric has no conjugate points.

Zeta function and spectrum of the geodesic flow

The zeta function $\log Z_{(M,g)}(s)$ arises as the Laplace transform of the so-called flat trace of the unitary Koopman operator

$$W_t : L^2(S^*M) \to L^2(S^*M), \;\; W_t f(x,\xi) := f(G^t(x,\xi)).$$

This operator does not possess a distribution trace but it has a *flat trace* given by

$$(16) \quad TrW(t) = e_0(t) + \sum_{\gamma\in\mathcal{P}} \sum_{k=1}^{\infty} \frac{\delta(t - L_\gamma)}{|det(I - P_\gamma^k)|}.$$

The Laplace transform may be viewed as the flat trace of the resolvent of Ξ, so that

$$(17) \qquad \log \; Z_{(M,g)}(s) = Tr(\Xi - s)^{-1}.$$

Zeta function and spectrum of the geodesic flow

The inverse problem is:

Problem **10** *What are the analytical properties of the zeta function when the geodesic flow has some given dynamical signature. Can one determine apriori from $Z_{(M,g)}(s)$ whether the geodesic flow is ergodic, weak mixing or mixing?*

Discussion

Formally, the Ruelle zeta function is the trace of a spectral function of the geodesic flow and it is natural to wonder how much of the spectrum of W_t can be determined from the traces. In particular, ergodicity, weak mixing and mixing are spectral properties of geodesic flows, i.e. can be read off from the spectrum of W_t. For instance, ergodicity is equivalent to the statement that the multiplicity of the eigenvalue 1 for W_t equals one.

But the trace is not a distribution trace and it is far from clear that one can read off spectral properties of W_t from these traces. Almost nothing seems to be known about zeta functions except in the case of hyperbolic flows, in which case one knows many properties of the spectrum of the geodesic flow, e.g. that it is mixing.

Inverse problems for integrable systems

The geodesic flow of an n-dimensional Riemmanian manifold (M, g) is *completely integrable* if it commutes with a Hamiltonian action of $\mathbb{R}^n$ on $T^*M - 0$ ($n = \dim M$). That is, the metric Hamiltonian $|\xi|_g$ satisfies

$$\{|\xi|_g, p_j\} = 0 = \{p_i, p_j\}, \;\; i, j = 1, \ldots, n$$

where $p_j : T^*M - 0 \to \mathbb{R}$ are homogeneous of degree one and are independent in the sense that

$$dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \neq 0 \text{ on a dense open subset } U \subset T^*M$$

The orbits of an $\mathbb{R}^n$ action give a (usually singular) torus (or affine) foliation of $S^*_g M$. The orbits which contain periodic geodesics are sometimes called 'periodic tori', and they furnish the components $\mathcal{T}$ in the trace formula.

Trace formulae for integrable systems

For integrable systems, periodic orbits usually come in families filling out invariant tori. We assume that the closed geodesics components are clean (Bott-Morse).

The trace of its wave group is then:

$$Tre^{it\sqrt{\Delta_g}} = e_0(t) + \sum_{\mathcal{T}} e_{\mathcal{T}}(t) \tag{18}$$

where the singular term $e_0(t) = C_n\, Vol(M,g)\, (t+i0)^{-n} + \dots$ at $t=0$ is the same as in the non-degenerate case and $\{\mathcal{T}\}$ runs over the critical point components of L_g.

Wave coefficiens

One has:

$$e_{\mathcal{T}} = c_{\mathcal{T},d_T}(t - L_{\mathcal{T}} + i0)^{-d_T/2}$$

$$+c_{\mathcal{T},d_T-1}(t - L_{\mathcal{T}} + i0)^{(-d_T/2+1)} + \dots, \quad d_{\mathcal{T}} = \dim \mathcal{T}.$$

Here, d_T is the dimension of the symplectic cone formed by the family of closed geodesics within T_g^*M and $L_{\mathcal{T}}$ is the common length of the closed geodesic in $\mathcal{T}$. For instance, in the non-degenerate case, $\mathcal{T} = \mathbb{R}_+\gamma$ is the symplectic cone generated by $\gamma \subset S_g^*M$ and $d_{\mathcal{T}} = 2$.

Spectral determination of analytic simple surfaces of revolution

We first recall that such surfaces are spectrally determined among other simple analytic surfaces of revolution and then sketch a result observed independently by the author and G. Forni and by K.F. Siburg regarding their spectral determination among all metrics on S^2.

The precise class of metrics we consider are those metrics g on S^2 which belong to the class $\mathcal{R}^*$ of real analytic, rotationally invariant metrics on S^2 with simple length spectrum in the above sense and satisfying the following 'simplicity' condition

- $g = dr^2 + a(r)^2 d\theta^2$
- $\exists! r_0 : a'(r_0) = 0$;
- The Poincare map $\mathcal{P}_0$; is elliptic of twist type

Spectral determination of analytic simple surfaces of revolution

Theorem **5** *Suppose that* g_1, g_2 *are two real analytic metrics on* S^2 *such that* (S^2, g_i) *are simple surfaces of revolution with simple length spectra. Then* $Sp(\Delta_{g_1}) = Sp(\Delta_{g_2})$ *implies* $g_1 = g_2$.

The proof is based on quantum Birkhoff normal forms for the Laplacian Δ. There is a global normal form, and the spectrum determines it. One calculates the principal and subprincipal term to get the Taylor coefficients of the profile curve at the equator.

Spectral determination of surfaces of revolution?

We now ask whether we can remove the assumption that $g_2 \in \mathcal{R}^*$? The question is, if Spec$\Delta_g =$ SpecΔ_h and $g \in \mathcal{R}^*$, then is $h \in \mathcal{R}^*$? An affirmative answer would give a large class of metrics which are spectrally determined. To the author's knowledge, the only metric on S^2 known to be spectrally determined is the canonical round one.

Partial result (Forni-Z, Siburg): any isospectral metric has a C^0 integrable geodesic flow.

Aubrey-Mather and inverse spectral theory

Theorem **6** *Let $g \in \mathcal{R}^*$ and suppose that h is any metric on S^2 with simple clean length spectrum for which SpecΔ_g = SpecΔ_h. Then h has the following properties:*

- *(i) It has just one isolated non-degenerate closed geodesic γ_h (up to orientation); all other closed geodesics come in one-parameter families lying on invariant tori in $S_h^* S^2$;*
- *(ii) The Birkhoff normal form of G_h^t at γ_h is identical to that of G_g^t at its unique non-degenerate closed orbit. Hence it is convergent.*

- *(iii) the geodesic flow G_h^t of h is C^0- integrable. That is, $S_h^* S^2$ has a C^0-foliation by 2-tori invariant under G_h^t.*

If we knew in (ii) that the Birkhoff transformation conjugating G_h^t to its Birkhoff normal form was convergent, then it would follow that G_h^t is completely integrable with global action-angle variables, and that it would commute with a Hamiltonian torus action. We conjecture that this is the case. Statement (iii) shows that it is at least integrable in the C^0 sense. At the present time, metrics on S^2 whose geodesic flows commute with Hamiltonian torus actions have not been classified.

Wave invariants for $g \in \mathcal{R}^*$

The proof is based on a study of the wave trace formula in this setting.

Proposition **7** *Suppose that $g \in \mathcal{R}^*$. Then the trace of its wave group has the form:*

$$Tre^{it\sqrt{\Delta_g}} = e_0(t) + e_{\gamma_g}(t) + \sum_{\mathcal{T}} e_{\mathcal{T}}(t)$$

where

$$e_0(t) = C_n area(M, g)(t + i0)^{-n} + \dots$$

is singular only at $t = 0$, where

$$e_{\gamma_g}(t) = c_\gamma(t - L_\gamma + i0)^{-1} + a_{\gamma 0} \log(t - L_\gamma + i0) + \dots$$

and where

$$e_{\mathcal{T}} = c_{\mathcal{T}}(t - L_{\mathcal{T}} + i0)^{-3/2} + \dots.$$

Suppose now that h is any other metric with Spec $\Delta_h = \text{Spec}\Delta_g$ and with simple length spectrum. Then the wave trace of h has precisely the same singularities as the wave trace of g. Since there is only one singularity of the order $(t + i0)^{-1}$, there can exist only one non-degenerate closed geodesic, proving (i). By Guillemin's inverse result, the Birkhoff normal form of the metric and Poincaré map for γ_h is the same as for γ_g. In particular, the Poincare map $\mathcal{P}_h$ of γ_h is elliptic of twist type. From the fact that all other critical components have the singularity of a three-dimensional cone, it follows that in $S^*_h S^2$ the other closed geodesics come in one-parameter families. They weep out a surface foliated by circles, which can only be a two dimensional torus, proving (ii)

It is the third statement (iii) which requires a new idea. So far we only know that periodic orbits lie on invariant tori, but we do not know what lies between these tori. Aubry-Mather theory will now close the gaps.

Aubry-Mather

Aubry-Mather theory is concerned with an area-preserving diffeomorphism φ of an annulus $A = S^1 \times (a, b)$. Let $\tilde{\varphi}$ denote a lift to $\mathbb{R} \times (a, b)$ with $\tilde{\varphi}(x + 1, y) = \tilde{\varphi}(x, y) + (1, 0)$. The map φ is called a *monotone twist mapping* if it preserves the orientation of A, if it preserves the boundary components and if the lift $\tilde{\varphi}(x_0, y_0) = (x_1, y_1)$ satisfies the twist condition: $\frac{\partial x_1}{\partial y_0} > 0$;

If a, b are finite, $\tilde{\varphi}$ extends continuously to the boundary as a pair of 'rotations':

$$\tilde{\varphi}(x, a) = (x + \omega_-, a), ; \quad \tilde{\varphi}(x, b) = (x + \omega_+, b).$$

The interval (ω_-, ω_+) is called the *twist interval* of φ.

Aubry-Mather

Let $\{(x_i, y_i)\}$ be an orbit of $\tilde{\varphi}$. Its *rotation number* is defined to be

$$\lim_{|i|\to\infty} \frac{x_i - x_0}{i}.$$

A curve $C \subset A$ is called an *invariant circle* if it is an invariant set which is homeomorphic to the circle and which separates boundary components. According to Birkhoff's invariant circle theorem, an invariant circle is a Lipschitz graph over the factor S^1 of A. Any invariant circle has a well-defined rotation number (the common rotation number of orbits in the circle) and the rotation number belongs to the twist interval.

An orbit $\{(x_i, y_i)\}$ is determined by the sequence $\{x_i\}$ of its x-coordinates. It is called *minimal* if every finite segment is action-minimizing with fixed endpoints

Aubry-Mather

The corresponding orbit orbit (x_i, y_i) is called a *minimal orbit*. The Aubrey-Mather theorem states:

A monotone twist map possesses minimal orbits for each rotation number $\omega \in (\omega_1, \omega_+)$ in its twist interval. Every minimal orbit lies on a Lipschitz graph over the x-axis. For each rational rotation number $\omega = \frac{p}{q}$, there exists a periodic minimal orbit of rotation number $\frac{p}{q}$. When ω is irrational, there exists either an invariant circle with rotation number ω, or an invariant Cantor set E.

The theorem also describes three possible orbit types in both the rational or irrational case.

Applications to inverse spectral theory

We now return to our problem on simple surfaces of revolution. We fix Poincaré sections to the geodesic flows at the equators γ_g, resp. γ_h. The Poincaré maps $\mathcal{P}_g$ are area-preserving twist maps of Poincarè sections with a non-degenerate elliptic fixed point. The foliation of $S^*S_g\backslash\gamma_h$ by 2-tori intersects S_g in a foliation by invariant circles converging to the fixed point. Circles with rational rotation numbers contain only periodic orbits, and conversely all periodic orbits belong to invariant circles with rational rotation numbers.

Aubrey Mather and ISP

Since $\mathcal{P}_h$ is a twist map, the rotation number ω_I is a monotone increasing function of I.

It follows by the Aubry-Mather theorem that every rational number p/q in the twist interval is the rotation number of a periodic circle $C_I \subset S_h$.

Lemma **8** *S_h is foliated in the C^0 sense by invariant circles for $\mathcal{P}_h$.*

Proof: If not there exists an annulus $A \subset S_h$ with boundary consisting of two invariant circles and containing no invariant circles in its interior. But by the Aubrey-Mather theorem, there must exist a periodic point in A. Since the periodic points come in circles, there must exist a periodic circle, contradicting the non-existence of invariant circles in A. QED

Isospectral class of an ellipse

Instead of surfaces in $\mathcal{R}^*$, one can apply this reasoning to the Dirichlet (or Neumann) problem for an ellipse $E_{a,b} = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$. It is well-known that ellipses have integrable billiards.

The ellipse has three distinguished periodic billiard orbits:

- The bouncing ball orbit along the minor axis, which is a non-degenerate elliptic orbit;

- The bouncing ball orbit along the major axis, which is a non-degenerate hyperbolic orbit;

- Its boundary.

All other periodic orbits come in one-parameter families. The wave trace formula shows that any domain Ω with $Spec(\Omega) = Spec(E_{a,b})$ has precisely one isolated elliptic orbit, one isolated hyperbolic orbit. The accumulation points in the length spectrum must be multiples of the perimeter of the domain (a spectral invariant), so the boundary must be a closed geodesic as well.

One can apply the twist map theory either to the boundary orbit or to the unique non-degenerate elliptic orbit with isolated length in the length spectrum. The argument above shows that there exists a C^0 foliation by invariant circles at least near these two orbits.

Marked length spectral rigidity of domains

Finally, we present a result of Siburg on the MLS of bounded domains. The key invariant is the *mean minimal action*

$$\alpha : [\omega_-, \omega_+] \to \mathbb{R}, \tag{19}$$

of a twist map φ, which associates to a rotation number ω in the 'twist interval' the mean action

$$\alpha(\omega) = -\lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N} h(x_i, x_{i+1}) \tag{20}$$

of a minimal orbit (q_i, η_i) of φ of rotation number ω.

Properties of the mean minimal action

It is a strictly convex function which is differentiable at all irrational numbers. If $\omega = p/q$, then α is differentiable at ω if and only if there exists an invariant circle of rotation number p/q consisting entirely of periodic minimal orbits. If a monotone twist map possesses an invariant circle of rotation number ω, then every orbit on the circle is minimal. In the case of a bounded plane domain, $h(q, q') = -|q - q'|$.

K. F. Siburg:the marked length spectrum is essentially the same invariant as the mean minimial action. The mean minimal action is therefore an isospectral deformation invariant.

Inverse mean minimal action

The only explicitly known mean minimal action is that of the disc D: $\alpha(\omega) = \frac{-1}{\pi} \sin \pi\omega$ ([**?**]); α is only smooth when $\Omega = D$

For the ellipse? Perhaps ellipses characterized by their mean minimal actions.

Problem **11** *Is the map from curvature functions κ of convex plane domains to the mean minimal action α of the associated convex domain injective or finitely many to one? at least near ellipses or under some additional analyticity or discrete symmetry condition?*