Bergman Kernels and Asymptotics of Polynomials II

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Our Topic

A fundamental invariant associated to a polynomial f is its Newton polytope P_f. This talk is about the impact of P_f on

- The distribution of zeros of f or of several polynomials with the same polytope;
- The distribution of mass $|f(z)|^2 dV$;
- The distribution of critical points of f;
- The positions of the 'tentacles' of the 'amoeba' associated to f.

We begin by recalling some basic definitions and results about polynomials and polytopes.

Polynomials and their Newton polytopes

- Monomials: $\chi_\alpha(z) = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$, $\alpha \in \mathbb{N}^m$.

- Polynomial of degree p (complex, holomorphic, not necessarily homogeneous):

$$f(z_1, \ldots, z_m) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| \leq p} c_\alpha \chi_\alpha(z_1, \ldots, z_m).$$

- *Support* of f

$$S_f = \{\alpha \in \mathbb{N}^m : c_\alpha \neq 0\},$$

- *Newton polytope* P_f

  $P_f := \widehat{S}_f =$ the convex hull in $\mathbb{R}^m$ of S_f.

- We denote by $\mathcal{P}_P^m$ the space of polynomials f in m variables with $P_f \subset P$.

Convex integral polytopes

By a convex integral polytope we mean the convex hull of a finite number of lattice points $\alpha_1, \ldots, \alpha_d \in \mathbb{N}^m$.

A convex integral polytope P with n *facets* (i.e. codimension-one faces) can be defined by linear equations

$$\ell_i(x) := \langle x, u_i \rangle + a_i \geq 0, \quad (i = 1, \ldots, n),$$

where $u_i \in \mathbb{Z}^m$ is the primitive interior normal to the i-th facet. The polytope P is called *Delzant* if each vertex is the intersection of exactly m facets whose primitive normal vectors generate the lattice $\mathbb{Z}^m$.

The Newton polynomial of a typical polynomial of degree p is the simplex $p\Sigma$ in $\mathbb{R}^m$ with vertices at $(0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0), \ldots, (0, \ldots, 0, 1)$.

By Vol(P) we denote the Euclidean volume of P.

Counting zeros of polynomials: Bezout and Bernstein-Kouchnirenko theorems

- **Bezout's theorem**: m generic homogeneous polynomials $F_1, \ldots, F_m$ of degree p have exactly p^m simultaneous zeros; these zeros all lie in $\mathbb{C}^{*m}$, for generic F_j.

- **Bernstein-Kouchnirenko Theorem** The number of joint zeros in $\mathbb{C}^{*m}$ of m generic polynomials $\{f_1, \ldots, f_m\}$ with given Newton polytope P equals $m!\text{Vol}(P)$.

- More generally, the f_j may have different Newton polytopes P_j; then, the number of zeros equals the 'mixed volume' of the P_j.

Consistency: If $P = p\Sigma$, where Σ is the standard unit simplex in $\mathbb{R}^m$, then $\text{Vol}(p\Sigma) = p^m\text{Vol}(\Sigma) = \frac{p^m}{m!}$, and we get Bézout's theorem.

Theme of talk

The Newton polytope of a polynomial f also has a crucial influence on its *mass density* $|f(z)|^2 dV$, and on the *spatial distribution* of *zeros* $\{f = 0\}$ and critical points $\{df = 0\}$.

- (i) There is a *classically allowed region*

$$\mathcal{A}_P = \mu_\Sigma^{-1}(\frac{1}{p}P)$$

  region where the mass almost surely concentrates and a *classically forbidden region* where it almost surely is exponentially decaying. Here, μ_Σ is the standard moment map of $\mathbb{CP}^m$, restricted to $(\mathbb{C}^*)^m$.

- (ii) The simultaneous zeros of m generic polynomials $f_1, \ldots, f_m$ in $\mathbb{C}^{*m}$ almost surely concentrates (in the limit of high degrees) in the classically allowed region (a quantitative Bernstein-Kouchnirenko theorem).

Theme of talk (cont.)

- (iii) The Newton polytope has an equally strong (though different) impact for $k < m$ polynomials $f_1, \ldots, f_k$. For instance, when $k = 1$, the image $\mu_\Sigma(Z_f)$ of the zero set of one polynomial f under the moment map is (up to a logarithmic re-parametrization) known as an *amoeba*. Our results show that the 'free tentacles' of typical amoebas have their ends in the classically allowed region.

- (iv) (Work in progress) The critical points of f with Newton polytope P almost surely concentrate in the classically allowed region $\mathcal{A}_P$.

Allowed and Forbidden regions

$$\mu_\Sigma(z) = \left(\frac{|z_1|^2}{1+\|z\|^2}, \dots, \frac{|z_m|^2}{1+\|z\|^2}\right) .$$

Via this moment map we define:

Definition: *Let* $P \subset \mathbb{R}^m_+$ *be an integral polytope. The classically allowed region for polynomials in* $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$ *is the set*

$$\mathcal{A}_P := \mu_\Sigma^{-1}\left(\frac{1}{p}P^\circ\right) \subset \mathbb{C}^{*m}$$

(where P° *denotes the interior of* P*), and the classically forbidden region is its complement* $\mathbb{C}^{*m} \setminus \mathcal{A}_P$*.*

Asymptotic and Statistical

Our results are:

- Statistical: Not *all* polynomials $f \in \mathcal{P}_P^m$ have this behaviour; but typical ones do in the probabilistic sense. We will endow $\mathcal{P}_P^m$ with a Gaussian probability measure, and show that the above patterns form the expected behaviour of random polynomials. With more work, we can also show that the patterns are true of 'almost every' sequence of polynomials $f_N \in \mathcal{P}_{NP}^m$.

- Asymptotic: the results become more and more accurate as the degree $\to \infty$. More precisely, we consider statistics of $f \in \mathcal{P}_{NP}^m$ as $N \to \infty$, where NP is the dilate of P.

Motivation to study asymptotic statistical patterns

- Polynomials exhibit a wide variety of behaviour vis a vis zeros, critical points, etc. Some patterns in their zeros and critical points are deterministic (e.g.one knows that certain tentacles of amoebas must land at vertices), but others are random and one wants to know the pattern of typical behaviour.

- Roughly, the degree of a polynomial is a measure of its 'complexity'. (More precisely, complexity is measured by the number of its monomials (Khovanskii)). High degree asymptotics are those in which the complexity of the polynomials grows. What becomes of zeros, critical points, topology of level sets, L^p norms (etc.) as the degree grows? Asymptotics with $P \to NP$ is a controlled increase in complexity.

$\mathcal{P}^m$ and $\mathcal{P}_P^m$

We now define probability measure on $\mathcal{P}^m$ and $\mathcal{P}_P^m$. The simplest are the Gaussian measures, which only require an inner product. To define them, we homogenize the polynomials on $(\mathbb{C}^*)^m$ so that homogeneous polynomials F of degree p in $m+1$ complex variables. This identifies:

- $\mathcal{P}^m = H^0(\mathbb{CP}^m, \mathcal{O}(p))$, the space of holomorphic sections of the pth power of the hyperplane line bundle $\mathcal{O}_{\mathbb{CP}^m}(1)$. Equivalently, they are CR functions on S^{2m+1} satisfying $F(e^{i\theta}x) = e^{ip\theta}F(x)$.

- $\mathcal{P}_P^m = H^0(\mathbb{CP}^m, \mathcal{O}(p), P) = \{F \in H^0(\mathbb{CP}^m, \mathcal{O}(p)) : P_f \subset P\}$, the space of homogeneous polynomials F of degree d whose associated inhomogeneous form $f(z_1, \ldots, z_m) = F(1, z_1, \ldots, z_m)$ has Newton polytope P_f contained in P.

Random $SU(m+1)$ polynomials

$H^0(\mathbb{CP}^m, \mathcal{O}(p))$ carries the SU($m+1$)-invariant inner product

$$\langle F_1, \bar{F}_2 \rangle = \int_{S^{2m+1}} F_1 \bar{F}_2 \, d\sigma \ ,$$

where $d\sigma$ is Haar measure on the $(2m+1)$-sphere S^{2m+1}. An orthonormal basis of $H^0(\mathbb{CP}^m, \mathcal{O}(p))$ is given by $\left\{ \|\chi_\alpha\|^{-1} \chi_\alpha \right\}_{|\alpha| \leq p}$, where $\| \cdot \|$ denotes the norm in $H^0(\mathbb{CP}^m, \mathcal{O}(p))$. (Note that $\|\chi_\alpha\|$ depends on p.) The corresponding SU($m+1$)-invariant Gaussian measure γ_p is defined by

$$(1) \quad d\gamma_p(s) = \frac{1}{\pi^{k_p}} e^{-|\lambda|^2} d\lambda, \quad s = \sum_{|\alpha| \leq p} \lambda_\alpha \frac{\chi_\alpha}{\|\chi_\alpha\|} \ ,$$

where $k_p = \#\{\alpha : |\alpha| \leq p\} = \binom{m+p}{p}$. Thus, the coefficients λ_α are independent complex Gaussian random variables with mean zero and variance one.

Random polynomials with prescribed Newton polytope

We then endow the space $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$ with the associated *conditional probability measure* $\gamma_p|_P$:
(2)

$$d\gamma_{p|P}(s) = \frac{1}{\pi^{\#P}} e^{-|\lambda|^2} d\lambda, \quad s = \sum_{\alpha \in P} \lambda_\alpha \frac{\chi\alpha}{\|\chi\alpha\|} ,$$

where the coefficients λ_α are again independent complex Gaussian random variables with mean zero and variance one. ($\#P$ denotes the cardinality of $P \cap \mathbb{Z}^m$.) As a subspace of $H^0(\mathbb{CP}^m, \mathcal{O}(p))$, $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$ inherits the inner product $\langle s_1, s_2 \rangle$ and $\gamma_{|P}$ is the induced Gaussian measure. Probabilities (or expectations) relative to $\gamma|_P$ can be considered as conditional probabilities; i.e. for any event E,

$$\text{Prob}_\gamma\{f \in E | P_f = P\} = \text{Prob}_{\gamma|_P}(E).$$

Expected Mass density

Theorem Suppose that P is a Delzant polytope in $\mathbb{R}^m$. Then the expected mass density of random $\mathcal{L}^2$ normalized polynomials with Newton polytope NP has $\mathcal{C}^\infty$ asymptotic expansions of the form: $\mathbf{E}_{\nu_{NP}}\left(|f(z)|^2_{\mathsf{FS}}\right)$

$$\sim \begin{cases} \frac{\prod_{j=1}^m (pN+j)}{\#(NP)}, & \text{for} \quad z \in \mathcal{A}_P = \mu_\Sigma^{-1}(\frac{1}{p}P^\circ) \\ N^{-s/2} e^{-Nb(z)} c_N^F(z), & \text{for } z \in \mathcal{R}_F^\circ \end{cases}$$

where $b|_{\mathcal{R}_F^\circ} > 0$, $c_N^F(z) = c_0 + c_1 N^{-1} + c_2 N^{-2} + \cdots$, and $s = \operatorname{codim} F$ (for each face $F \subset \Sigma^\circ$).

Ground state geometry

Ground states of Schrodinger operators $H_h = -\hbar^2 \Delta + V$ of energy E_0 concentrate in the allowed region $\{x : V(x) \leq E_0\}$ and satisfy $|\varphi(x)|^2 = O(e^{-d(x,C_E)/\hbar})$ in the complement. Here, $d(x, C_E)$ is the action to the allowed region.

In our setting, the Hamiltonian is $\bar{\partial}^* \bar{\partial}$ on $\mathcal{L}^2$-sections of powers $\mathcal{O}(Np)$ of the hyperplane section bundle, $\hbar = 1/N$, and the ground states are the holomorphic sections $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$.

Action to allowed region

For $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$, b is $\mathcal{C}^1$ (but not $\mathcal{C}^2$) on all of $\mathbb{C}^{*m}$. Our formula for b is, for $z \in \mathbb{C}^{*m} \setminus \mathcal{A}_P$:

(3)

$$b(z) = \int_0^{\tau_z} \left[q(e^{\sigma/2} \cdot z) - p\mu_\Sigma(e^{\sigma/2} \cdot z) \right] \cdot d\sigma$$

(for any path in $\mathbb{R}^m$ from 0 to τ_z). Here, we associate to $z \in \mathbb{C}^{*m} \setminus \mathcal{A}_P$ a unique point $\xi \in \partial\mathcal{A}_P$ of the form $\xi = e^{\tau/2} \cdot z$, where $-\tau$ is in the (real) normal cone to the convex set P at the point $p\mu_\Sigma(\xi) \in \partial P$. We write $\tau_z = \tau$, $q(z) = p\mu_\Sigma(\xi)$.

Distribution of zeros

We first define 'delta' functions on zero sets of one or several polynomials.

Given $f_1, \ldots, f_k$, $k \leq m$, put $Z_{f_1,\ldots,f_k} = \{z \in (C^*)^m : f_1(z) = \cdots = f_k(z) = 0\}$. $Z_{f_1,\ldots,f_k}$ defines a (k,k) current of integration:

$$\langle \psi, Z_{f_1,\ldots,f_k} \rangle = \int_{Z_f} \psi.$$

By Wirtinger's formula, the integral of a scalar function φ over Z_f can be defined as

$$\int_{Z_{f_1,\ldots,f_k}} \varphi \frac{\omega_{FS}^{n-k}}{(n-k)!}.$$

Expected zero current: Definition

Now consider m independent random polynomials with Newton polytope P, using the conditional probability $d\gamma_{p|P}$. We let $\mathbf{E}_{|P}(Z_{f_1,\dots,f_m})$ denote the expected density of their simultaneous zeros. It is the measure on $\mathbb{C}^{*m}$ given by

$$\mathbf{E}_{|P}(Z_{f_1,\dots,f_m})(U) = \int d\gamma_{p|P}(f_1) \cdots \int d\gamma_{p|P}(f_m)$$

$$\times \Big[\#\{z \in U : f_1(z) = \cdots = f_m(z) = 0\} \Big] \,,$$

for $U \subset \mathbb{C}^{*m}$, where the integrals are over $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$.

Expected distribution of zeros

Theorem 1 Suppose that P is a Delzant polytope in $\mathbb{R}^m$. Then

$$\frac{1}{(Np)^m}\mathbf{E}_{|NP}(Z_{f_1,\dots,f_m}) \to \begin{cases} \omega_{\mathsf{FS}}^m & \text{on } \ \mathcal{A}_P \\ 0 & \text{on } \ \mathbb{C}^{*m} \setminus \mathcal{A}_P \end{cases} ,$$

in the distribution sense; i.e., for any open $U \subset \mathbb{C}^{*m}$, we have

$$\frac{1}{(Np)^m}\mathbf{E}_{|NP}\Big(\#\{z \in U : f_1(z) = \dots = f_m(z) = 0\}\Big)$$

$$\to m!\mathrm{Vol}_{\mathbb{CP}^m}(U \cap \mathcal{A}_P) \ .$$

Convergence on the classically allowed region is exponentially fast in the sense that

$$\mathbf{E}_{|NP}(Z_{f_1,\dots,f_m}) = (Np)^m\omega_{\mathsf{FS}}^m + O\left(e^{-\lambda N}\right) \ \text{ on } \ \mathcal{A}_P \ ,$$

for some positive continuous function λ on $\mathcal{A}_P$.

Expected zeros of $k \le m$ polynomials

Theorem Let P be a Delzant polytope. Then there exists a closed semipositive $(1,1)$-form ψ_P on $\mathbb{C}^{*m}$ with piecewise $\mathcal{C}^\infty$ coefficients such that:

i) $N^{-1}\mathbf{E}_{|NP}(Z_f) \to \psi_P$ in $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{C}^{*m})$.

ii) $\psi_P = p\omega_{\mathrm{FS}}$ on the classically allowed region $\mu_\Sigma^{-1}(\frac{1}{p}P^\circ)$.

iii) On each region $\mathcal{R}_F^\circ$, the $(1,1)$-form ψ_P is $\mathcal{C}^\infty$ and has constant rank equal to $\dim F$; in particular, if $v \in \Sigma^\circ$ is a vertex of $\frac{1}{p}P$, then $\psi_P|_{\mathcal{R}_v^\circ} = 0$.

Theorem Let $P_1, \ldots, P_k$ be Delzant polytopes. Then

$$N^{-k}\mathbf{E}_{|NP_1,\ldots,NP_k}(Z_{f_1,\ldots,f_k}) \to \psi_{P_1}\wedge\cdots\wedge\psi_{P_k} \quad \text{in} \quad \mathcal{L}^1_{\mathrm{loc}}(\mathbb{C}^{*}$$

We see that $|Z_f|$ for a polynomial with polytope NP almost surely creeps into the classically forbidden region $\mu_\Sigma^{-1}(\Sigma \setminus \frac{1}{p}P)$ in the semiclassical limit $N \to \infty$. Indeed the expected volume of the zero set, or more generally the simultaneous zero set of k polynomials, has the following exotic distribution law:

Corollary **1** *Let $P_1, \ldots, P_k$ be Delzant polytopes. Then for any open set $U \subset\subset \mathbb{C}^{*m}$,*

$$\frac{1}{N^k}\mathbf{E}_{|NP_1,\ldots,NP_k}\mathrm{Vol}(|Z_{f_1,\ldots,f_k}|\cap U) \to \frac{1}{(m-k)!}\int_U \psi_{P_1}\wedge\cdots$$

Amoebas

Let $f(z_1, z_2) \in \mathcal{P}^2_P$. One defines:

Compact Amoeba of $f = \mu_\Sigma(Z_f)$;

Amoeba of $f = \mathrm{Log}\,(Z_f)$, where Log : $\mathbb{C}^{*m} \to \mathbb{R}^m$, is $(z_1, \ldots, z_m) \mapsto (\log|z_1|, \ldots, \log|z_m|)$.

Tentacles of Amoeba: its ends on $\partial\Sigma$. For a generic 2-dimensional amoeba with polytope P, lattice points in $\partial P \iff$ tentacle. Hence # tentacles of $A = \#\partial P \cap \mathbb{Z} =$ *length* of ∂P. We can decompose ∂P into two pieces: $\partial^\circ P = \partial P \cap p\Sigma^\circ$ and $\partial^e P = P \cap \partial(p\Sigma)$.

Each tentacle corresponds to a segment connecting 2 adjacent lattice points on ∂P. They correspond to segments of $\partial^\circ P$ end (in the compact picture Σ) at a vertex of Σ, and tentacles corresponding to segments of $\partial^e P$ are free to end anywhere on the face of Σ containing the segment. We call the latter *free tentacles*, and we say that a free tentacle is a *classically allowed tentacle* if its end is in the classically allowed region $\mathcal{A}_P$. For an amoeba A, we let $\nu_{\mathsf{AT}}(A)$ denote the number of classically allowed tentacles of A. It is clear from the above that

$$\nu_{\mathsf{AT}}(A) \leq \#\{\text{free tentacles}\} = \text{Length}(\partial^e P)$$

and that this bound can be attained for any polytope P. Here, 'Length' means the length in the above sense; i.e., the diagonal face of $p\Sigma$ is scaled to have length p. Our result is that the maximum is asymptotically the average:

Corollary **1** *For a Delzant polytope P, we have*

$$\frac{1}{N}\mathbf{E}_{|NP}\Big(\nu_{\mathsf{AT}}\Big(\text{Log}\,(Z_f)\Big)\Big) \to \text{Length}(\partial^e P)\;.$$

Ideas of Proof

A key object is the conditional Szegö projector

$$\Pi_{|P}(x,y) = \sum_{\alpha \in P} \frac{\widehat{\chi}_\alpha(x)\overline{\widehat{\chi}_\alpha(y)}}{||m_\alpha||^2}$$

of $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$. Its importance stems from:

- $\mathbf{E}_{|P}\left(|f(z)|^2_{\mathsf{FS}}\right) = \frac{1}{\#P}\Pi_{|P}(z,z)$;

- $\mathbf{E}_{|P}(Z_f) = \frac{1}{\#P}\bar{\partial}\partial \log \Pi_{|P}(z,z)$;

- $\mathbf{E}_{|P}(Z_{f_1,\ldots,f_k}) = [\frac{1}{\#P}\bar{\partial}\partial \log \Pi_{|P}(z,z)]^{\wedge k}$;

Szegö kernels, line bundles and circle bundles

The 'Szegö kernel' of a space $\mathcal{S}$ of holomorphic sections of a line bundle $L \to M$ refers to the kernel for the orthogonal projection to $\mathcal{S}$ from the space of L^2 sections.

For asymptotic analysis, it is best to lift sections, Szegö kernels etc. to the lined bundle $X = \partial D \to M$ associated to L, i.e. boundary of the associated unit disk bundle D relative to a hermitian metric. Then Π is orthogonal projection from space $\mathcal{L}^2(\partial D)$ of sections to lifts of sections in $\mathcal{S}$. It is of the form $\Pi(x, y) = \sum_j s_j(x)\overline{s_j(y)}$, where $\{s_j\}$ is an orthonormal basis of $\mathcal{S}$.

In the case $M = \mathbb{CP}^m, L = \mathcal{O}(p),\ X = S^{2m+1}$ and sections are just homogeneous polynomials restricted to S^{2m+1}. In this case,

$$\Pi_p^{\mathbb{CP}^m}(x, y) = C_m^p \langle x, y\rangle^p.$$

Conditional Szegö projector

We need the asymptotics of $\Pi_{|NP}(z,z)$ as $N \to \infty$.

Put:
(4)
$$\chi_{NP}(e^{i\varphi}) = \sum_{\alpha \in NP} e^{i\langle\alpha,\varphi\rangle}, \qquad e^{i\varphi} = (e^{i\varphi_1}, \ldots, e^{i\varphi_m}) .$$

It is the character of the torus $\mathbf{T}^m = \{(e^{i\varphi_1}, \ldots, e^{i\varphi_m})\}$ on $H^0(\mathbb{CP}^m, \mathcal{O}(pN), NP)$. Therefore:

$$(5) \quad \Pi_{|NP}(z,z) = \int_{\mathbf{T}^m} \Pi_{Np}^{\mathbb{CP}^m}(t \cdot z, z)\overline{\chi_{NP}(t)}dt.$$

We have: $\Pi_{Np}^{\mathbb{CP}^m}(x,y) = C_m^{Np}\langle x,y\rangle^{Np}$. To obtain asymptotics, we would like an oscillatory integral formulae for $\chi_{NP}(e^{i\varphi})$.

Polytope character

There are several different approaches to χ_{NP}.

- Via the toric variety M_P of P. It carries a holomorhic line bundle L_P such that $H^0(\mathbb{CP}^m, \mathcal{O}(Np), NP) \equiv H^0(M_P, L_P^N)$. We define $\Pi_N^{M_P}$ as the Szegö projector for $H^0(M_P, L_P^N)$. We then have:

  $$\chi_{NP}(e^{i\varphi}) = \int_{M_P} \Pi_N^{M_P}(e^{i\varphi} \cdot w, w)\, d\mathrm{Vol}_{M_P}(w).$$

- Lattice sum to integral approach. One has (Khovanskii-Pukhlikov, Brion-Vergne, Guillemin):

  $$\chi_P(e^{i\varphi}) = \mathrm{Todd}(\partial/\partial h) \left(\int_{P(h)} e^{i\langle x, \varphi \rangle} dx \right) \Big|_{h=0},$$

  where $P(h) = \{x : \langle u_j, x \rangle + a_j + h_j \geq 0,\ 1 \leq j \leq n\}$, and $\mathrm{Todd}(\partial/\partial h)$ is a certain infinite order differential operator known as a *Todd operator*.

Exact formula for Szegö kernel $\Pi_N^{M_P}$ of a toric variety

We will follow the Szegö kernel approach here. We need a formula for $\Pi_N^{M_P}$ from which we can obtain exponentially small asymptotics. The key formula is:

$$\Pi_N^{M_P} = (\mathcal{P}\mathcal{Q})_N^{-1} \Pi_1^N.$$

Here, $\Pi_1(x, y) := \langle \iota_P(x), \overline{\iota_P(y)} \rangle$, where $\iota_P : X_P \to S^{2\delta+1}$ is the lift of the monomial embedding $M_P \hookrightarrow \mathbb{CP}^\delta$ which defines M_P. Also, $(\mathcal{P}\mathcal{Q})$ is a 'Fourier multiplier' on M_P defined by:

(i) The 'partition function' $\mathcal{P}_N(\alpha) = \#\{(\beta_1, \ldots, \beta_N) : \beta_j \in P, \beta_1 + \cdots + \beta_N = \alpha\}$, where $\alpha \in NP$.

(ii) The norming function: $\mathcal{Q}_N(\alpha) := \int_{X_P} |\widehat{\chi}_\alpha^P(x)|^2 d\mathrm{Vol}_{X_P}(x)$.

'Fourier multiplier' means an operator commuting with the $\mathbf{T}^m$ action.

Complex oscillatory integrals

To use the exact formula, one proves:

- $(\mathcal{PQ})$ is an elliptic Toeplitz operator on M_P. Its inverse can be written $\Pi_N^{M_P} \sigma_N \Pi_N^{M_P}$ for a symbol σ_N.

- Putting things together (and using the Boutet de Monvel-Sjostrand parametrix): $\Pi_{|NP}(z,z)$

  $$= \frac{1}{(2\pi)^m} \int_{M_P} \int_{\mathbf{T}^m} e^{N\Psi(\varphi,w;z)} a_N(w)\, d\varphi\, d\mathrm{Vol}_{M_P}(w)$$

  where the phase $\Psi(\varphi, w; z)$

  $$= \log \sum_{\alpha \in P} e^{-i\langle \alpha, \varphi\rangle} |\widehat{m}_\alpha^P(w)|^2 + \log \sum_{|\alpha| \leq p} e^{i\langle \alpha, \varphi\rangle} |\widehat{m}_\alpha^{p\Sigma}(z)|$$

  $a_N(w) = \frac{(Np+m)!}{(Np)!} \sigma_N(w)$ is a symbol of order $2m$.

Asymptotics

Using this oscillatory integral expression, we find:

- The critical points occur only at $\varphi = 0, \mu_P(w) = \mu_{p\Sigma}(z)$, i.e. only in the allowed region. By stationary phase (etc.) one derives mass asymptotics.

- When $z \notin \mathcal{A}_P$ one must deform the $\mathbf{T}^m$ contour until one picks up the critical points with maximum $\Re$ part on the contour. The critical point equations are tricky and sometimes degenerate.

- Then methods of complex analysis (Poincare-Lelong, Bedford-Taylor) give asymptotics of zeros. For critical points one needs to use a different approach via Federer's coarea formula.

Final Remarks and Open problems

- If you are solving sparse polynomial systems, it pays most to hunt for zeros in the allowed region.

- How many forbidden zeros of m polynomials in m variables are there? How large are the 'holes' in the forbidden zero set?

- How are zeros, critical points distributed for polynomials whose monomials lie only on the boundary of NP? Or at its vertices? (These are fewnomials).

- There is a similar story for real polynomials with fixed Newton polytope (in progress). Besides the distribution of zeros, one may ask about the topology of the zero sets (how many components? what are their genera?)